Rational approximations of $\ln 2$

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Abstract

We examine a sequence of rational numbers $\{r_n\}$ that yield approximations of $\ln 2$ that become more precise as n increases. The approximation improves in accuracy by roughly an order of magnitude as n increases by one unit.

1. Introduction

There are many ways to obtain rational approximations of irrational numbers. In this short note, I will exploit a very clever method that is based on examining a sequence of integrals that is suitably chosen. The method is probably best known in association with obtaining rational approximations of π [1]. Indeed, the famous approximation that we all learned in high school, $\pi \approx 22/7$, is naturally obtained as the first of a sequence of ever improving approximations derived from examining the sequence of integrals introduced below. After a lightning review of the method for approximating π, I then apply the method, following Ref. [2], to obtain rational approximations of $\ln 2$.

Rational approximations of π can be obtained by considering the following sequence of integrals,

$$\mathcal{J}_n \equiv 4(-1)^n \int_0^1 \frac{dx}{1+x^2}\left(\frac{x^2(1-x)^2}{2}\right)^{2n} . \tag{1}$$

As shown in Ref. [1],

$$\mathcal{J}_n = \pi - p_n \,, \tag{2}$$

where p_n is a positive rational number for positive integer values of n. Since $0 \leq x(1-x) \leq \frac{1}{4}$ for $0 \leq x \leq 1$, it follows that

$$0 < (-1)^n \mathcal{J}_n < \frac{1}{1024^n} \qquad \Longrightarrow \qquad \lim_{n\to\infty} p_n = \pi \,. \tag{3}$$

Hence for finite values of n, the sequence of rational numbers $\{p_n\}$ provides rational approximations of π with improving accuracy as n increases. Indeed, eq. (3) guarantees that the accuracy improves by more than three orders of magnitude as n is increased by one unit.

Remarkably, a closed form expression is given for p_n in Ref. [1],

$$p_n = \sum_{k=0}^{n-1} (-1)^k \frac{2^{4-2k}(4k)!(4k+3)!}{(8k+7)!}\left(820k^3 + 1533k^2 + 902k + 165\right) . \tag{4}$$

This result is derived by employing an identity given in Ref. [3] with no proof,

$$\frac{x^{4n}(1-x)^{4n}}{1+x^2} = \big(x^6-4x^5+5x^4-4x^2+4\big)\sum_{k=0}^{n-1}(-4)^{n-1-k}x^{4k}(1-x)^{4k} + \frac{(-4)^n}{1+x^2}\,. \tag{5}$$

It is not very difficult to provide a derivation of eq. (5). We begin with the identity,

$$\frac{x^4(1-x)^4}{1+x^2} = P(x) + \frac{R}{1+x^2}\,, \tag{6}$$

for some polynomial $P(x)$ and constant R to be determined. To obtain R, we extend the function $x^4(1-x)^n/(1+x^2)$ into the complex plane. We then demand that the residue at the poles $\pm i$ are the same on both sides of eq. (6). This yields $R=-4$. It then follows that

$$P(x) = \frac{x^4(1-x)^4+4}{1+x^2}\,. \tag{7}$$

We can then use Mathematica to factor the numerator of $P(x)$,

$$x^4(1-x)^4+4 = (1+x^2)(4-4x^2+5x^4-4x^5+x^6)\,. \tag{8}$$

Hence,

$$P(x) = x^6-4x^5+5x^4-4x^2+4\,. \tag{9}$$

Next, we multiply both sides of eq. (6) by $x^4(1-x)^4$. When carrying out the multiplication on the last term on the right hand side of eq. (6), we shall employ $x^4(1-x)^4=(1+x^2)P(x)-4$. The end result is

$$\frac{x^8(1-x)^8}{1+x^2} = \big[x^4(1-x)^4-4\big]P(x) + \frac{(-4)^2}{1+x^2}\,. \tag{10}$$

We again multiply both sides of eq. (10) by $x^4(1-x)^4$ and follow the same strategy as before to obtain,

$$\frac{x^{12}(1-x)^{12}}{1+x^2} = \big[x^8(1-x)^8-4x^4(1-x)^4+(-4)^2\big]P(x) + \frac{(-4)^3}{1+x^2}\,. \tag{11}$$

Continuing the process, it should be clear that after n steps we arrive at

$$\frac{x^{4n}(1-x)^{4n}}{1+x^2} = P(x)\sum_{k=0}^{n-1}(-4)^{n-1-k}x^{4k}(1-x)^{4k} + \frac{(-4)^n}{1+x^2}\,. \tag{12}$$

Having identified $P(x)$ in eq. (9), we have indeed established the result quoted in eq. (5).

The evaluation of $\mathcal{J}_n$ is now straightforward, as the expression for p_n is expressed as the sum of integrals, each of which is recognized as the integral representation of a Beta function. In light of $\lim_{n\to\infty}\mathcal{J}_n=0$, it follows that a rational approximation to π that monotonically improves in accuracy as n increases is given by $\pi\simeq p_n$, where p_n consists of the sum of the first n terms of the series given in eq. (4),

$$\pi = \frac{22}{7} - \frac{19}{15015} + \frac{543}{594914320} - \frac{77}{104187267600} + \cdots\,. \tag{13}$$

In the next section, we shall use a similar technique to obtain rational approximations of $\ln 2$.

2. Rational approximations of ln 2

Consider the following sequence of integrals [2],

$$\mathcal{I}_n \equiv (-1)^n \int_0^1 \frac{dx}{1+x}\left(\frac{x(1-x)}{2}\right)^n . \tag{14}$$

We will demonstrate below that

$$\mathcal{I}_n = \ln 2 - r_n \,, \tag{15}$$

where r_n is a positive rational number for positive integer values of n. Since $0 \leq x(1-x) \leq \frac{1}{4}$ for $0 \leq x \leq 1$, it follows that

$$0 < (-1)^n \mathcal{I}_n < \frac{1}{8^n} \quad \Longrightarrow \quad \lim_{n\to\infty} r_n = \ln 2 \,. \tag{16}$$

Hence for finite values of n, the sequence of rational numbers $\{r_n\}$ provides rational approximations of ln 2 with improving accuracy as n increases. Indeed, eq. (16) guarantees that the accuracy improves by roughly an order of magnitude as n is increased by one unit.

The relevant identity analogous to eq. (5) is derived using the same technique employed in Section 1. We first note that

$$\frac{x(1-x)}{1+x} = 2 - x - \frac{2}{1+x} \,. \tag{17}$$

Multiplying both sides of this equation by $x(1-x)$ and using $x(1-x) = (2-x)(1+x) - 2$ when multiplying the last term on the right hand side of eq. (17) yields,

$$\frac{x^2(1-x)^2}{1+x} = \big[x(1-x) - 2\big](2-x) + \frac{(-2)^2}{1+x} \,. \tag{18}$$

We again multiply by $x(1-x)$ and use $x(1-x) = (2-x)(1+x) - 2$ when multiplying the last term on the right hand side of eq. (17) to obtain

$$\frac{x^3(1-x)^3}{1+x} = \big[x^2(1-x)^2 - 2x(1-x) + (-2)^2\big](2-x) + \frac{(-2)^3}{1+x} \,. \tag{19}$$

Is should now be clear that after n steps, the end result is given by

$$\frac{x^n(1-x)^n}{1+x} = (2-x)\sum_{k=0}^{n-1}(-2)^{n-1-k}x^k(1-x)^k + \frac{(-2)^n}{1+x} \,. \tag{20}$$

Hence, it follows that

$$\mathcal{I}_n = \ln 2 + \sum_{k=0}^{n-1}(-2)^{-1-k}\int_0^1 x^k(1-x)^k(2-x)\,dx \,. \tag{21}$$

Using the integral expression of the Beta function,

$$B(r,s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)} = \frac{(r-1)!(s-1)!}{(r+s-1)!} = \int_0^1 x^{r-1}(1-x)^{s-1}\,dx \,, \tag{22}$$

it follows that

$$\mathcal{I}_n = \ln 2 + \sum_{k=0}^{n-1}(-2)^{-1-k}\left[2B(k+1,k+1) - B(k+2,k+1)\right] , \tag{23}$$

which can be simplified to obtain our final result,

$$\boxed{\mathcal{I}_n = \ln 2 - \frac{3}{4}\sum_{k=0}^{n-1}\frac{(-1)^k[k!]^2}{2^k(2k+1)!}} \tag{24}$$

Taking the limit as $n \to \infty$ and making use of $\lim_{n\to\infty}\mathcal{I}_n = 0$, we arrive at an interesting series expansion for $\ln 2$ (see Appendix A for an independent derivation),

$$\ln 2 = \frac{3}{4}\sum_{k=0}^{\infty}\frac{(-1)^k[k!]^2}{2^k(2k+1)!} = \frac{3}{4} - \frac{1}{16} + \frac{1}{160} - \frac{3}{4480} + \frac{1}{13440} - \frac{1}{118272} + \cdots . \tag{25}$$

It then follows that a good approximation to $\ln 2$ is given by the first n terms of eq. (25). That is, in eq. (15) r_n is given by,

$$r_n = \frac{3}{4}\sum_{k=0}^{n-1}\frac{(-1)^k[k!]^2}{2^k(2k+1)!} . \tag{26}$$

In Table 1 we list the individual terms, a_k, that appear in eq. (25) and the corresponding values of $r_n \equiv \sum_{k=0}^{n-1} a_k$ for positive integer values of $n \leq 11$. Ten digit accuracy is obtained for $n = 11$. Note that the accuracy of the approximation to $\ln 2$ improves by roughly an order of magnitude with each increase of n by one unit, as anticipated.

n	a_{n-1}	r_n	numerical value
1	$\frac{3}{4}$	$\frac{3}{4}$	0.75
2	$-\frac{1}{16}$	$\frac{11}{16}$	0.6875
3	$\frac{1}{160}$	$\frac{111}{160}$	0.69375
4	$-\frac{3}{4480}$	$\frac{621}{896}$	0.69308035714
5	$\frac{1}{13440}$	$\frac{2329}{3360}$	0.69315476190
6	$-\frac{1}{118272}$	$\frac{19519}{28160}$	0.69314630682
7	$\frac{1}{1025024}$	$\frac{3552463}{5125120}$	0.69314728241
8	$-\frac{1}{8785920}$	$\frac{42629549}{61501440}$	0.69314716859
9	$\frac{1}{74680320}$	$\frac{241567449}{348508160}$	0.69314718198
10	$-\frac{3}{1891901440}$	$\frac{834505731}{1203937280}$	0.69314718039
11	$\frac{1}{5297324032}$	$\frac{18359126087}{26486620160}$	0.69314718058

Table 1: Rational approximations, $r_n = \sum_{k=0}^{n-1} a_k$, of $\ln 2$ obtained using Mathematica. The approximations become more accurate as n increases. With eleven digit accuracy, $\ln 2 \simeq 0.69314718056$.

3. An alternative approach to evaluating the integrals $\mathcal{I}_n$

We return the sequence of integrals specified in eq. (14). In this section, we shall make use of the properties of the Gauss hypergeometric function taken from Ref. [5].

Using the integral expression for the Gauss hypergeometric function (cf. eq. (1) on p. 114 of Ref. [5]), it follows that

$$\mathcal{I}_n = \frac{(-1)^n [n!]^2}{2^n(2n+1)!}\,{}_2F_1(1, n+1; 2n+2; -1)\,. \tag{27}$$

Next, we use one of the quadratic transformations of the hypergeometric function (cf. eq. (28) on p. 112 of Ref. [5]), to obtain,

$${}_2F_1(1, n+1; 2n+2; z) = \frac{2}{2-z}\,{}_2F_1(\tfrac{1}{2}, 1; n+\tfrac{3}{2}; z^2/(2-z)^2)\,. \tag{28}$$

In light of eq. (24) on p. 102 of Ref. [5]), it follows that

$$\frac{d^n}{dw^n}\,{}_2F_1(\tfrac{1}{2}, 1, \tfrac{3}{2}; w) = \frac{n!}{2n+1}\frac{1}{(1-w)^n}\,{}_2F_1(\tfrac{1}{2}, 1; n+\tfrac{3}{2}; w)\,. \tag{29}$$

Finally, by employing eq. (16) on p. 102 of Ref. [5], it follows that for $w \geq 0$,

$${}_2F_1(\tfrac{1}{2}, 1, \tfrac{3}{2}; w) = \frac{1}{2\sqrt{w}} \ln\left(\frac{1+\sqrt{w}}{1-\sqrt{w}}\right)\,. \tag{30}$$

Combining eqs. (29) and (30) yields,

$${}_2F_1(\tfrac{1}{2}, 1; n+\tfrac{3}{2}; z^2/(2-z)^2) = \frac{2^{2n-1}(2n+1)}{n!}\left(\frac{1-z}{(2-z)^2}\right)^2 \frac{d^n}{dw^n}\left\{\frac{1}{\sqrt{w}}\ln\left(\frac{1+\sqrt{w}}{1-\sqrt{w}}\right)\right\}\Bigg|_{w=z^2/(2-z)^2}\,. \tag{31}$$

It then follows that

$${}_2F_1(1, n+1; 2n+2; z) = \frac{2^{2n}(2n+1)}{n!}\frac{(1-z)^n}{(2-z)^{2n+1}}\frac{d^n}{dw^n}\left\{\frac{1}{\sqrt{w}}\ln\left(\frac{1+\sqrt{w}}{1-\sqrt{w}}\right)\right\}\Bigg|_{w=z^2/(2-z)^2}\,. \tag{32}$$

Setting $z = -1$, we arrive at our final expression for $\mathcal{I}_n$,

$$\mathcal{I}_n = \frac{(-1)^n 2^{2n} n!}{3^{2n+1}(2n)!}\frac{d^n}{dw^n}\left\{\frac{1}{\sqrt{w}}\ln\left(\frac{1+\sqrt{w}}{1-\sqrt{w}}\right)\right\}\Bigg|_{w=1/9}\,. \tag{33}$$

It is immediately clear that

$$\mathcal{I}_n = a_n \ln 2 - r_n\,, \tag{34}$$

where r_n is a rational number and

$$a_n = \frac{(-1)^n 2^{2n} n!}{3^{2n+1}(2n)!}\frac{d^n}{dw^n}\left\{\frac{1}{\sqrt{w}}\right\}\Bigg|_{w=1/9}\,. \tag{35}$$

Using the result

$$\frac{d^n}{dw^n} w^p = \frac{\Gamma(p+1)}{\Gamma(p-n+1)} w^{p-n} \,, \tag{36}$$

and inserting $p = -\frac{1}{2}$, it follows that

$$\frac{d^n}{dw^n} \left\{ \frac{1}{\sqrt{w}} \right\} = \frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}-n)} w^{-n-\frac{1}{2}} \,, \tag{37}$$

After using the duplication and reflection formulae for the Gamma function,

$$\Gamma(\tfrac{1}{2}-n) = (-1)^n 2^{2n} \sqrt{\pi} \, \frac{\Gamma(n+1)}{\Gamma(2n+1)} \,. \tag{38}$$

Hence,

$$\frac{d^n}{dw^n} \left\{ \frac{1}{\sqrt{w}} \right\} = \frac{(-1)^n (2n)!}{2^{2n} n!} w^{-n-\frac{1}{2}} \,. \tag{39}$$

Plugging this last result into eq. (35) yields $a_n = 1$.

Hence, as advertised in eq. (15), we have demonstrated that

$$\mathcal{I}_n = \ln 2 - r_n \,, \tag{40}$$

where

$$r_n = \frac{(-1)^{n+1} 2^{2n} n!}{3^{2n+1} (2n)!} \frac{d^n}{dw^n} \left\{ \frac{1}{\sqrt{w}} \ln\left(\frac{1+\sqrt{w}}{1-\sqrt{w}} \right) - \frac{\ln 2}{\sqrt{w}} \right\} \bigg|_{w=1/9} \,. \tag{41}$$

To evaluate r_n, consider

$$s_n(w) \equiv \frac{d^n}{dw^n} \left\{ \frac{1}{\sqrt{w}} \ln\left(\frac{1+\sqrt{w}}{2[1-\sqrt{w}]} \right) \right\} \,. \tag{42}$$

Writing $D \equiv d/dw$ and employing eq. (68), it follows that for any nonnegative integer n,

$$\begin{aligned} s_{n+1} &= D^{n+1} \left\{ \frac{1}{\sqrt{w}} \ln\left(\frac{1+\sqrt{w}}{2[1-\sqrt{w}]} \right) \right\} \\ &= D^n \left\{ \frac{1}{w(1-w)} - \frac{1}{2w^{3/2}} \ln\left(\frac{1+\sqrt{w}}{2[1-\sqrt{w}]} \right) \right\} \\ &= (-1)^n n! \, w^{-1-n} + n! \, (1-w)^{-1-n} - \sum_{k=0}^{n} \binom{n}{k} D^{n-k} \left(\frac{1}{2w} \right) D^k \left\{ \frac{1}{\sqrt{w}} \ln\left(\frac{1+\sqrt{w}}{2[1-\sqrt{w}]} \right) \right\} \\ &= (-1)^n n! \left\{ \frac{1}{w^{n+1}} - \frac{1}{(w-1)^{n+1}} - \frac{1}{2} \sum_{k=0}^{n} \frac{(-1)^k s_k}{k! \, w^{1+n-k}} \right\} \,, \end{aligned} \tag{43}$$

after using the Leibniz rule for the nth derivative of a product of two functions and employing the relation,

$$D^n \left(\frac{1}{w-a} \right) = \frac{(-1)^n n!}{(w-a)^{n+1}} \,, \tag{44}$$

for any constant a.

It is more convenient to replace $n+1$ with n in eq. (43), which yields

$$s_n = (-1)^n (n-1)! \left\{ \frac{1}{(w-1)^n} - \frac{1}{w^n} + \frac{1}{2}\sum_{k=0}^{n-1} \frac{(-1)^k s_k}{k!\, w^{n-k}} \right\}, \quad \text{for } n = 1,2,3,\ldots \tag{45}$$

We now introduce the notation,

$$\overline{s}_n \equiv s_n\big|_{w=1/9}\,. \tag{46}$$

It then follows that

$$\overline{s}_n = (-1)^n 3^{2n} (n-1)! \left\{ (-2)^{-3n} - 1 + \frac{1}{2}\sum_{k=0}^{n-1} \frac{(-1)^k 3^{-2k} \overline{s}_k}{k!} \right\}. \tag{47}$$

Using eq. (41) then yields,

$$r_n = \frac{2^{2n-1}[(n-1)!]^2}{3(2n-1)!} \left[1 - (-2)^{-3n} + \frac{3}{2}\sum_{k=0}^{n-1} \frac{(2k)!\, r_k}{2^{2k}[k!]^2} \right]. \tag{48}$$

We can obtain a useful recursion relation by employing eq. (48) to derive,

$$r_n - \frac{2(n-1)}{2n-1} r_{n-1} = \frac{3[(n-1)!]^2}{(-2)^{n+1}(2n-1)!} + \frac{r_{n-1}}{2n-1}\,. \tag{49}$$

This result simplifies, and we end up with,

$$r_n - r_{n-1} = -\frac{3(-1)^n n!\,(n-1)!}{2^n (2n)!}\,, \quad \text{where } r_0 = 0 \text{ and } n = 1,2,3,\ldots \tag{50}$$

Indeed, eq. (50) provides the simplest method for quickly obtaining the results exhibited in Table 1. It then follows that

$$r_n = \sum_{k=0}^{n-1} (r_{k+1} - r_k) = \frac{3}{2}\sum_{k=0}^{n-1} \frac{(-1)^k k!(k+1)!}{2^k (2k+2)!}\,. \tag{51}$$

which is equivalent to the result obtained in eq. (26).

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APPENDIX A

In this Appendix, an independent derivation of eq. (25) is provided [4]. We first employ eq. (16) on p. 102 of Ref. [5],

$$ {}_2F_1(\tfrac{1}{2}, 1, \tfrac{3}{2}; z^2) = \frac{1}{2z} \ln\left(\frac{1+z}{1-z}\right) , \tag{52} $$

where ${}_2F_1$ is the Gauss hypergeometric function. We next use eq. (4) on p. 105 of Ref. [5],

$$ {}_2F_1(a, b; c; z) = (1-z)^{-b}\, {}_2F_1\big(c-a, b; c, z/(z-1)\big) , \tag{53} $$

to obtain,

$$ \ln\left(\frac{1+z}{1-z}\right) = \frac{2z}{1-z^2}\, {}_2F_1\big(1, 1; \tfrac{3}{2}, z^2/(z^2-1)\big) . \tag{54} $$

Setting $z = \frac{1}{3}$ yields,

$$ \ln 2 = \tfrac{3}{4}\, {}_2F_1\big(1, 1; \tfrac{3}{2}, -\tfrac{1}{8}\big) . \tag{55} $$

Using the series representation of the Gauss hypergeometric function given by eqs. (1) and (2) on p. 101 of Ref. [5],

$$ {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)k!}\, z^k , \tag{56} $$

it follows that

$$ \ln 2 = \tfrac{3}{4}\Gamma\big(\tfrac{3}{2}\big) \sum_{k=0}^{\infty} \frac{k!}{2^{3k}\Gamma\big(\frac{3}{2}+k\big)} . \tag{57} $$

Finally, using $\Gamma\big(\frac{3}{2}\big) = \frac{1}{2}\sqrt{\pi}$ and making use of the duplication formula (see, e.g., eq. (1.2.3) of Ref. [6]),

$$ \Gamma(n+\tfrac{1}{2}) = \frac{\sqrt{\pi}\,\Gamma(2n+1)}{2^{2n}\,\Gamma(n+1)} = \frac{\sqrt{\pi}(2n)!}{2^{2n}\, n!} , \tag{58} $$

we arrive at our final result,

$$ \ln 2 = \frac{3}{4} \sum_{k=0}^{\infty} \frac{(-1)^k [k!]^2}{2^k (2k+1)!} , \tag{59} $$

in agreement with eq. (25).

APPENDIX B

It is instructive to use the method presented in Section 3 to explicitly work out the cases of $n = 0$ and $n = 1$. For $n = 0$, eq. (15) on p. 102 of Ref. [5]) yields,

$$ {}_2F_1(1, 1; 2; z) = -\frac{1}{z} \ln(1-z) . \tag{60} $$

Setting $n=0$ in eq. (28), we obtain

$$ {}_2F_1(1,1;2;z) = {}_2F_1(\tfrac{1}{2},1;\tfrac{3}{2};z^2/(2-z)^2)\,. \tag{61} $$

In light of eq. (30), for values of $0<z<2$ we can identify $\sqrt{w}=z/(2-z)$, and it follows that

$$ {}_2F_1(\tfrac{1}{2},1;\tfrac{3}{2};z^2/(2-z)^2) = -\frac{2-z}{2z}\ln(1-z)\,. \tag{62} $$

and hence

$$ \frac{2}{2-z}{}_2F_1(\tfrac{1}{2},1;\tfrac{3}{2};z^2/(2-z)^2) = -\frac{1}{z}\ln(1-z)\,, \tag{63} $$

in agreement with eq. (60). Likewise, if $z<0$ or $z>2$ then can identify $\sqrt{w}=z/(z-2)$, and we again obtain eq. (63) in agreement with eq. (60). Finally, setting $z=-1$ yields ${}_2F_1(1,1;2;-1)=\ln 2$. Hence, eq. (27) yields

$$ \mathcal{I}_0 = \ln 2\,, \tag{64} $$

This result is consistent with eqs. (40) and (41) given that $r_0=0$.

Next, we use eq. (28) to obtain

$$ {}_2F_1(1,2;4;z) = \frac{2}{2-z}\,{}_2F_1(\tfrac{1}{2},1;\tfrac{5}{2};z^2/(2-z)^2)\,, \tag{65} $$

and eq. (29) to obtain

$$ \frac{d}{dw}{}_2F_1(\tfrac{1}{2},1;\tfrac{3}{2};w) = \frac{1}{3(1-w)}\,{}_2F_1(\tfrac{1}{2},1;\tfrac{5}{2};w)\,. \tag{66} $$

Making use of eq. (30),

$$ \frac{d}{dw}\left\{\frac{1}{\sqrt{w}}\ln\left(\frac{1+\sqrt{w}}{1-\sqrt{w}}\right)\right\} = \frac{2}{3(1-w)}\,{}_2F_1(\tfrac{1}{2},1;\tfrac{5}{2};w)\,. \tag{67} $$

Using the chain rule,

$$ \begin{aligned} \frac{d}{dw}\left\{\frac{1}{\sqrt{w}}\ln\left(\frac{1+\sqrt{w}}{1-\sqrt{w}}\right)\right\} &= \frac{d\sqrt{w}}{dw}\frac{d}{d\sqrt{w}}\left\{\frac{1}{\sqrt{w}}\ln\left(\frac{1+\sqrt{w}}{1-\sqrt{w}}\right)\right\} \\ &= \frac{1}{2\sqrt{w}}\left[-\frac{1}{w}\ln\left(\frac{1+\sqrt{w}}{1-\sqrt{w}}\right)+\frac{1}{\sqrt{w}}\frac{1-\sqrt{w}}{1+\sqrt{w}}\frac{(1-\sqrt{w})+(1+\sqrt{w})}{(1-\sqrt{w})^2}\right] \\ &= \frac{1}{2\sqrt{w}}\left[-\frac{1}{w}\ln\left(\frac{1+\sqrt{w}}{1-\sqrt{w}}\right)+\frac{2}{\sqrt{w}(1-w)}\right] \\ &= -\frac{1}{2w^{3/2}}\ln\left(\frac{1+\sqrt{w}}{1-\sqrt{w}}\right)+\frac{1}{w(1-w)}\,. \end{aligned} \tag{68} $$

Setting $w=z^2/(2-z)^2$ in eqs. (67) and (68) yields

$$ \frac{(2-z)^3}{2z^2}\left[\frac{\ln(1-z)}{z}+\frac{2-z}{2(1-z)}\right] = \frac{(2-z)^2}{6(1-z)}\,{}_2F_1(\tfrac{1}{2},1;\tfrac{5}{2};w)\,. \tag{69} $$

It then follows that

$$ {}_2F_1(\tfrac{1}{2},1;\tfrac{5}{2};z^2/(2-z)^2) = \frac{3(2-z)}{z^2}\left[\frac{(1-z)\ln(1-z)}{z}+\frac{2-z}{2}\right]. \tag{70} $$

Using eq. (65), we end up with

$$ {}_2F_1(1,2;4;z) = \frac{6(1-z)\ln(1-z)}{z^3}+\frac{3(2-z)}{z^2}. \tag{71} $$

Setting $z=-1$ yields,

$$ {}_2F_1(1,2;4;-1) = -12\ln 2+9\,. \tag{72} $$

Hence, by using eq. (27) we end up with

$$ \mathcal{I}_1 = \ln 2 - \tfrac{3}{4}\,. \tag{73} $$

Compare this result with eqs. (40) and (41). In particular, after employing eq. (68), we obtain

$$ \begin{aligned} r_1 &= \frac{2}{27}\frac{d}{dw}\left\{\frac{1}{\sqrt{w}}\ln\left(\frac{1+\sqrt{w}}{1-\sqrt{w}}\right)-\frac{\ln 2}{\sqrt{w}}\right\}\bigg|_{w=1/9} \\ &= \frac{2}{27}\left\{\frac{1}{w(1-w)}-\frac{1}{2w^{3/2}}\ln\left(\frac{1+\sqrt{w}}{1-\sqrt{w}}\right)+\frac{\ln 2}{2w^{3/2}}\right\}\bigg|_{w=1/9} \\ &= \frac{2}{27}\cdot\frac{81}{8}=\frac{3}{4}\,, \end{aligned} \tag{74} $$

in agreement with eq. (73).

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