The Hamiltonian of a free Majorana fermion field

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Abstract

In these notes, we express the quantum Hamiltonian of a free Majorana fermion field in 3+1 spacetime dimensions as a sum over Fourier modes. The full details of the calculation are provided. The computation is presented first using the four-component spinor formalism. The computation is then repeated using the two-component spinor formalism.

1 The Hamiltonian derived using four-component spinors

In four-component spinor notation, the Lagrangian density for a free Majorana field is given by

$$\mathscr{L} = \tfrac{1}{2} i \overline{\Psi}_M \gamma^\mu \partial_\mu \Psi_M - \tfrac{1}{2} m \overline{\Psi}_M \Psi_M \,, \tag{1}$$

where $\Psi_M = \Psi_M^C \equiv C\overline{\Psi}_M^{\mathsf{T}}$ and C is the charge conjugation matrix.[1] The Majorana field $\Psi_M(x)$ and its Dirac conjugate field $\overline{\Psi}_M(x)$ satisfy the free-field Dirac equation and its conjugate,

$$(i\gamma^\mu \partial_\mu - m)\Psi_M(x) = 0 \,, \qquad \overline{\Psi}_M(x)(i\gamma^\mu \overleftarrow{\partial_\mu} + m) = 0 \,. \tag{2}$$

To obtain the Hamiltonian, we first compute

$$\Pi_M = \frac{\partial \mathscr{L}}{\partial(\partial_0 \Psi_M)} = \tfrac{1}{2} i \overline{\Psi}_M \gamma^0 \,. \tag{3}$$

Then, the Hamiltonian density is given by

$$\begin{aligned} \mathscr{H} = \Pi_M \partial_0 \Psi_M - \mathscr{L} &= \tfrac{1}{2} i \overline{\Psi}_M \gamma^0 \partial_0 \Psi_M - \tfrac{1}{2} i \overline{\Psi}_M \gamma^\mu \partial_\mu \Psi_M + \tfrac{1}{2} m \overline{\Psi}_M \Psi_M \\ &= -\tfrac{1}{2} i \overline{\Psi}_M \vec{\boldsymbol{\gamma}} \boldsymbol{\cdot} \vec{\boldsymbol{\nabla}} \Psi_M + \tfrac{1}{2} m \overline{\Psi}_M \Psi_M \,. \end{aligned} \tag{4}$$

Finally, the Hamiltonian of the free Majorana field is given by

$$H = \frac{1}{2} \int d^3x \big[-i \overline{\Psi}_M(x) \vec{\boldsymbol{\gamma}} \boldsymbol{\cdot} \vec{\boldsymbol{\nabla}} \Psi_M(x) + m \overline{\Psi}_M(x) \Psi_M(x) \big] \,. \tag{5}$$

[1] For example, $C = i\gamma^0\gamma^2$ is the charge in the chiral representation of the gamma matrices (where γ_5 is diagonal). However, all computations in these notes are independent of the representation chosen for the gamma matrices and the spinor wave functions.

Next, we expand the Majorana field in Fourier modes which involve the creation and annihilation operators,

$$\Psi_M(x) = \sum_\lambda \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}} \left[u(\vec{\boldsymbol{p}},\lambda)a(\vec{\boldsymbol{p}},\lambda)e^{-ip\cdot x} + v(\vec{\boldsymbol{p}},\lambda)a^\dagger(\vec{\boldsymbol{p}},\lambda)e^{ip\cdot x}\right] , \tag{6}$$

where $p^\mu = (E_{\boldsymbol{p}}\,;\,\vec{\boldsymbol{p}})$ with $E_{\boldsymbol{p}} \equiv \left(|\vec{\boldsymbol{p}}|^2 + m^2\right)^{1/2}$ and the sum over helicities runs over $\lambda = -\frac{1}{2}, \frac{1}{2}$. Under the assumption that $\Psi_M(x)$ satisfies the free field equations [eq. (2)], the spinor wave functions $u(\vec{\boldsymbol{p}},\lambda)$ and $v(\vec{\boldsymbol{p}},\lambda)$ satisfy the momentum space Dirac equations,

$$(\not{p} - m)\, u(\vec{\boldsymbol{p}},s) = (\not{p} + m)\, v(\vec{\boldsymbol{p}},s) = 0\,, \tag{7}$$

$$\overline{u}(\vec{\boldsymbol{p}},s)\,(\not{p} - m) = \overline{v}(\vec{\boldsymbol{p}},s)\,(\not{p} + m) = 0\,, \tag{8}$$

Using the fact that a Majorana fermion field satisfies $\Psi_M = C\overline{\Psi}_M^{\mathsf{T}}$, eq. (6) implies that the u and v spinor wave functions are related as follows:[2]

$$v(\vec{\boldsymbol{p}},s) = C\overline{u}(\vec{\boldsymbol{p}},s)^{\mathsf{T}}\,, \qquad u(\vec{\boldsymbol{p}},s) = C\overline{v}(\vec{\boldsymbol{p}},s)^{\mathsf{T}}\,, \tag{9}$$

$$\overline{v}(\vec{\boldsymbol{p}},s) = -u(\vec{\boldsymbol{p}},s)^{\mathsf{T}}C^{-1}\,, \qquad \overline{u}(\vec{\boldsymbol{p}},s) = -v(\vec{\boldsymbol{p}},s)^{\mathsf{T}}C^{-1}\,. \tag{10}$$

The creation and annihilation operators $a^\dagger$ and a satisfy anticommutation relations:

$$\{a(\vec{\boldsymbol{p}},\lambda), a^\dagger(\vec{\boldsymbol{p}}\,',\lambda')\} = \delta^3(\vec{\boldsymbol{p}} - \vec{\boldsymbol{p}}')\delta_{\lambda\lambda'}\,, \tag{11}$$

with all other anticommutation relations vanishing. We have employed covariant normalization of the one-particle states given by

$$|\vec{\boldsymbol{p}},\lambda\rangle = (2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}a^\dagger(\boldsymbol{p},\lambda)|0\rangle\,. \tag{12}$$

Using eq. (6),

$$\overline{\Psi}_M(x) = \sum_\lambda \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}} \left[\overline{u}(\vec{\boldsymbol{p}},\lambda)a^\dagger(\vec{\boldsymbol{p}},\lambda)e^{ip\cdot x} + \overline{v}(\vec{\boldsymbol{p}},\lambda)a(\vec{\boldsymbol{p}},\lambda)e^{-ip\cdot x}\right] , \tag{13}$$

and

$$-i\vec{\boldsymbol{\nabla}}\Psi_M(x) = \sum_\lambda \int \frac{\vec{\boldsymbol{p}}\,d^3\vec{\boldsymbol{p}}}{(2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}} \left[u(\vec{\boldsymbol{p}},\lambda)a(\vec{\boldsymbol{p}},\lambda)e^{-ip\cdot x} - v(\vec{\boldsymbol{p}},\lambda)a^\dagger(\vec{\boldsymbol{p}},\lambda)e^{ip\cdot x}\right] . \tag{14}$$

[2]Since a theory of a Dirac fermion is equivalent to a theory of two mass-degenerate Majorana fermions, eqs. (9) and (10) also are valid for the u and v spinor wave functions employed in the Fourier mode decomposition of a Dirac fermion.

Inserting eqs. (6), (13), and (14) into eq. (5), we obtain

$$\int d^3x\big[-i\overline{\Psi}_M(x)\vec{\gamma}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\Psi_M(x)+m\overline{\Psi}_M(x)\Psi_M(x)\big]$$

$$\begin{aligned}
&=\int d^3x\int\frac{d^3\vec{\boldsymbol{p}}\,d^3\vec{\boldsymbol{p}}'}{(2\pi)^3(2E_{\boldsymbol{p}})^{1/2}(2E_{\boldsymbol{p}'})^{1/2}}\\
&\quad\times\sum_{\lambda,\lambda'}\Big\{\overline{u}(\vec{\boldsymbol{p}},\lambda)\big(\vec{\gamma}\boldsymbol{\cdot}\vec{\boldsymbol{p}}'+m\big)\,u(\vec{\boldsymbol{p}}',\lambda')a^\dagger(\vec{\boldsymbol{p}},\lambda)a(\vec{\boldsymbol{p}}',\lambda')e^{i(E_{\vec{\boldsymbol{p}}}-E_{\vec{\boldsymbol{p}}'})t}\,e^{-i(\vec{\boldsymbol{p}}-\vec{\boldsymbol{p}}')\boldsymbol{\cdot}\vec{\boldsymbol{x}}}\\
&\qquad+\overline{v}(\vec{\boldsymbol{p}},\lambda)\big(\vec{\gamma}\boldsymbol{\cdot}\vec{\boldsymbol{p}}'+m\big)\,u(\vec{\boldsymbol{p}}',\lambda')a(\vec{\boldsymbol{p}},\lambda)a(\vec{\boldsymbol{p}}',\lambda')e^{-i(E_{\vec{\boldsymbol{p}}}+E_{\vec{\boldsymbol{p}}'})t}\,e^{i(\vec{\boldsymbol{p}}+\vec{\boldsymbol{p}}')\boldsymbol{\cdot}\vec{\boldsymbol{x}}}\\
&\qquad-\overline{u}(\vec{\boldsymbol{p}},\lambda)\big(\vec{\gamma}\boldsymbol{\cdot}\vec{\boldsymbol{p}}'-m\big)\,v(\vec{\boldsymbol{p}}',\lambda')a^\dagger(\vec{\boldsymbol{p}},\lambda)a^\dagger(\vec{\boldsymbol{p}}',\lambda')e^{i(E_{\vec{\boldsymbol{p}}}+E_{\vec{\boldsymbol{p}}'})t}\,e^{-i(\vec{\boldsymbol{p}}+\vec{\boldsymbol{p}}')\boldsymbol{\cdot}\vec{\boldsymbol{x}}}\\
&\qquad-\overline{v}(\vec{\boldsymbol{p}},\lambda)\big(\vec{\gamma}\boldsymbol{\cdot}\vec{\boldsymbol{p}}'-m\big)\,v(\vec{\boldsymbol{p}}',\lambda')a(\vec{\boldsymbol{p}},\lambda)a^\dagger(\vec{\boldsymbol{p}}',\lambda')e^{-i(E_{\vec{\boldsymbol{p}}}-E_{\vec{\boldsymbol{p}}'})t}\,e^{i(\vec{\boldsymbol{p}}-\vec{\boldsymbol{p}}')\boldsymbol{\cdot}\vec{\boldsymbol{x}}}\Big\}.
\end{aligned}\tag{15}$$

We can now integrate over $\vec{\boldsymbol{x}}$ using

$$\frac{1}{(2\pi)^3}\int d^3\vec{\boldsymbol{x}}\,e^{i(\vec{\boldsymbol{p}}-\vec{\boldsymbol{p}}')\boldsymbol{\cdot}\vec{\boldsymbol{x}}}=\delta^3(\vec{\boldsymbol{p}}-\vec{\boldsymbol{p}}')\,,\tag{16}$$

and then use the delta function to integrate over $\vec{\boldsymbol{p}}'$. This procedure yields

$$\int d^3x\big[i\overline{\Psi}_M(x)\vec{\gamma}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\Psi_M(x)+m\overline{\Psi}_M(x)\Psi_M(x)\big]$$

$$\begin{aligned}
&=\int\frac{d^3\vec{\boldsymbol{p}}}{2E_{\boldsymbol{p}}}\sum_{\lambda,\lambda'}\Big\{\overline{u}(\vec{\boldsymbol{p}},\lambda)\big(\vec{\gamma}\boldsymbol{\cdot}\vec{\boldsymbol{p}}+m\big)\,u(\vec{\boldsymbol{p}},\lambda')a^\dagger(\vec{\boldsymbol{p}},\lambda)a(\vec{\boldsymbol{p}},\lambda')\\
&\qquad-\overline{v}(\vec{\boldsymbol{p}},\lambda)\big(\vec{\gamma}\boldsymbol{\cdot}\vec{\boldsymbol{p}}-m\big)\,u(-\vec{\boldsymbol{p}},\lambda')a(\vec{\boldsymbol{p}},\lambda)a(-\vec{\boldsymbol{p}},\lambda')e^{-2iE_{\vec{\boldsymbol{p}}}t}\\
&\qquad+\overline{u}(\vec{\boldsymbol{p}},\lambda)\big(\vec{\gamma}\boldsymbol{\cdot}\vec{\boldsymbol{p}}+m\big)\,v(-\vec{\boldsymbol{p}},\lambda')a^\dagger(\vec{\boldsymbol{p}},\lambda)a^\dagger(-\vec{\boldsymbol{p}},\lambda')e^{2iE_{\vec{\boldsymbol{p}}}t}\\
&\qquad-\overline{v}(\vec{\boldsymbol{p}},\lambda)\big(\vec{\gamma}\boldsymbol{\cdot}\vec{\boldsymbol{p}}-m\big)\,v(\vec{\boldsymbol{p}},\lambda')a(\vec{\boldsymbol{p}},\lambda)a^\dagger(\vec{\boldsymbol{p}},\lambda')\Big\}.
\end{aligned}\tag{17}$$

Next, we make use of the momentum space Dirac equations [eqs. (7) and (8)], which can be rewritten as

$$\big(\vec{\gamma}\boldsymbol{\cdot}\vec{\boldsymbol{p}}+m\big)\,u(\vec{\boldsymbol{p}},\lambda')=\gamma^0E_{\vec{\boldsymbol{p}}}\,u(\vec{\boldsymbol{p}},\lambda')\,,\qquad\big(\vec{\gamma}\boldsymbol{\cdot}\vec{\boldsymbol{p}}-m\big)\,v(\vec{\boldsymbol{p}},\lambda')=\gamma^0E_{\vec{\boldsymbol{p}}}\,v(\vec{\boldsymbol{p}},\lambda')\,.\tag{18}$$

$$\overline{u}(\vec{\boldsymbol{p}},\lambda)\big(\vec{\gamma}\boldsymbol{\cdot}\vec{\boldsymbol{p}}+m\big)=E_{\vec{\boldsymbol{p}}}\,\overline{u}(\vec{\boldsymbol{p}},\lambda)\gamma^0\,,\qquad\overline{v}(\vec{\boldsymbol{p}},\lambda)\big(\vec{\gamma}\boldsymbol{\cdot}\vec{\boldsymbol{p}}-m\big)=E_{\vec{\boldsymbol{p}}}\,\overline{v}(\vec{\boldsymbol{p}},\lambda)\gamma^0\,.\tag{19}$$

It then follows that

$$\begin{aligned}\overline{v}(\vec{\boldsymbol{p}},\lambda)\big(\vec{\boldsymbol{\gamma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}-m\big)\,u(-\vec{\boldsymbol{p}},\lambda')&=\tfrac{1}{2}\big[\overline{v}(\vec{\boldsymbol{p}},\lambda)\big(\vec{\boldsymbol{\gamma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}-m\big)\big]u(-\vec{\boldsymbol{p}},\lambda')+\tfrac{1}{2}\overline{v}(\vec{\boldsymbol{p}},\lambda)\big[\big(\vec{\boldsymbol{\gamma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}-m\big)u(-\vec{\boldsymbol{p}},\lambda')\big]\\&=\tfrac{1}{2}\overline{v}(\vec{\boldsymbol{p}},\lambda)\big[E_{\vec{\boldsymbol{p}}}\gamma^0-\gamma^0E_{\vec{\boldsymbol{p}}}\big]u(-\vec{\boldsymbol{p}},\lambda')=0\,.\end{aligned}\tag{20}$$

$$\begin{aligned}\overline{u}(\vec{\boldsymbol{p}},\lambda)\big(\vec{\boldsymbol{\gamma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}+m\big)\,v(-\vec{\boldsymbol{p}},\lambda')&=\tfrac{1}{2}\big[\overline{u}(\vec{\boldsymbol{p}},\lambda)\big(\vec{\boldsymbol{\gamma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}+m\big)\big]v(-\vec{\boldsymbol{p}},\lambda')+\tfrac{1}{2}\overline{u}(\vec{\boldsymbol{p}},\lambda)\big[\big(\vec{\boldsymbol{\gamma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}+m\big)\,v(-\vec{\boldsymbol{p}},\lambda')\big]\\&=\tfrac{1}{2}\overline{u}(\vec{\boldsymbol{p}},\lambda)\big[E_{\vec{\boldsymbol{p}}}\gamma^0-\gamma^0E_{\vec{\boldsymbol{p}}}\big]v(-\vec{\boldsymbol{p}},\lambda')=0\,.\end{aligned}\tag{21}$$

Inserting the results of eqs. (18)–(21) back into eq. (17) yields

$$\begin{aligned}&\int d^3x\big[i\overline{\Psi}_M(x)\vec{\boldsymbol{\gamma}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\Psi_M(x)+m\overline{\Psi}_M(x)\Psi_M(x)\big]\\&\quad=\frac{1}{2}\int d^3\vec{\boldsymbol{p}}\sum_{\lambda,\lambda'}\Big\{\overline{u}(\vec{\boldsymbol{p}},\lambda)\gamma^0u(\vec{\boldsymbol{p}},\lambda')a^\dagger(\vec{\boldsymbol{p}},\lambda)a(\vec{\boldsymbol{p}},\lambda')-\overline{v}(\vec{\boldsymbol{p}},\lambda)\gamma^0v(\vec{\boldsymbol{p}},\lambda')a(\vec{\boldsymbol{p}},\lambda)a^\dagger(\vec{\boldsymbol{p}},\lambda')\Big\}.\end{aligned}\tag{22}$$

To evaluate the remaining spinor products, we make use of the Bouchiat-Michel formulae given in eqs. (E.194) and (E.195) of Ref. 1 and convert the corresponding expressions into traces,

$$\overline{u}(\vec{\boldsymbol{p}},\lambda)\,\Gamma\,u(\vec{\boldsymbol{p}},\lambda')=\tfrac{1}{2}\,\mathrm{Tr}\left[\Gamma(\delta_{\lambda\lambda'}+\gamma_5\gamma_\mu\mathscr{S}^\mu_{\lambda\lambda'})(\not{p}+m)\right],\tag{23}$$

$$\overline{v}(\vec{\boldsymbol{p}},\lambda)\,\Gamma\,u(\vec{\boldsymbol{p}},\lambda')=\tfrac{1}{2}\,\mathrm{Tr}\left[\Gamma(\delta_{\lambda'\lambda}+\gamma_5\gamma_\mu\mathscr{S}^\mu_{\lambda'\lambda})(\not{p}-m)\right]\,,\tag{24}$$

where Γ is any product of gamma matrices and

$$\mathscr{S}^\mu_{\lambda\lambda'}=S^{a\mu}\tau^a_{\lambda\lambda'}\,,\tag{25}$$

with an implied sum over the repeated index $a\in\{1,2,3\}$. In eq. (25), the τ^a are the Pauli matrices and the spin vectors $S^{a\mu}$ satisfy $p\cdot S^a=0$ [for further details, see Section E.4 of Ref. 1].

We now can evaluate the following expressions:

$$\overline{u}(\vec{\boldsymbol{p}},\lambda)\gamma^0u(\vec{\boldsymbol{p}},\lambda')=\tfrac{1}{2}\,\mathrm{Tr}\left[\gamma^0(\delta_{\lambda\lambda'}+\gamma_5\gamma_\mu\mathscr{S}^\mu_{\lambda\lambda'})(\not{p}+m)\right]=2E_{\vec{\boldsymbol{p}}}\,\delta_{\lambda\lambda'}\,,\tag{26}$$

$$\overline{v}(\vec{\boldsymbol{p}},\lambda)\gamma^0v(\vec{\boldsymbol{p}},\lambda')=\tfrac{1}{2}\,\mathrm{Tr}\left[\gamma^0(\delta_{\lambda'\lambda}+\gamma_5\gamma_\mu\mathscr{S}^\mu_{\lambda'\lambda})(\not{p}-m)\right]=2E_{\vec{\boldsymbol{p}}}\,\delta_{\lambda'\lambda}\,.\tag{27}$$

Note that these results have been obtained without specifying any particular representations for the gamma functions and spinor wave functions. Inserting the above results into eq. (22) yields

$$\begin{aligned}H&=\frac{1}{2}\int d^3x\big[i\overline{\Psi}_M(x)\vec{\boldsymbol{\gamma}}\boldsymbol{\cdot}\vec{\boldsymbol{\nabla}}\Psi_M(x)+m\overline{\Psi}_M(x)\Psi_M(x)\big]\\&=\frac{1}{2}\sum_\lambda\int d^3\vec{\boldsymbol{p}}\;E_{\vec{\boldsymbol{p}}}\big[a^\dagger(\vec{\boldsymbol{p}},\lambda)a(\vec{\boldsymbol{p}},\lambda)-a(\vec{\boldsymbol{p}},\lambda)a^\dagger(\vec{\boldsymbol{p}},\lambda)\big]\,,\end{aligned}\tag{28}$$

after using the $\delta_{\lambda\lambda'}$ factors to perform the sum over λ'. Finally, we redefine the Hamiltonian to be the normal ordering of the above expression to eliminate the infinite zero point energy. In light of eq. (11),

$$:a(\vec{\boldsymbol{p}},\lambda)a^\dagger(\vec{\boldsymbol{p}},\lambda):=-a^\dagger(\vec{\boldsymbol{p}},\lambda)a(\vec{\boldsymbol{p}},\lambda)\,.\tag{29}$$

Hence, we end up with

$$H = \sum_\lambda \int d^3\vec{\boldsymbol{p}}\; E_{\vec{\boldsymbol{p}}}\, a^\dagger(\vec{\boldsymbol{p}},\lambda)a(\vec{\boldsymbol{p}},\lambda)\,, \tag{30}$$

as expected.

Added note:

Suppose we had not made use of the trick employed in eqs. (20) and (21) to prove that the corresponding spinor products vanished. Instead, we might have used the results of eq. (18) alone to obtain

$$\overline{v}(\vec{\boldsymbol{p}},\lambda)\big(\vec{\boldsymbol{\gamma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}} - m\big)\, u(-\vec{\boldsymbol{p}},\lambda') = -E_{\vec{\boldsymbol{p}}}\,\overline{v}(\vec{\boldsymbol{p}},\lambda)\gamma^0 u(-\vec{\boldsymbol{p}},\lambda')\,, \tag{31}$$

$$\overline{u}(\vec{\boldsymbol{p}},\lambda)\big(\vec{\boldsymbol{\gamma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}} + m\big)\, v(-\vec{\boldsymbol{p}},\lambda') = -E_{\vec{\boldsymbol{p}}}\,\overline{u}(\vec{\boldsymbol{p}},\lambda)\gamma^0 v(-\vec{\boldsymbol{p}},\lambda')\,. \tag{32}$$

Of course, we could have also have used the results of eq. (19) alone to obtain

$$\overline{v}(\vec{\boldsymbol{p}},\lambda)\big(\vec{\boldsymbol{\gamma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}} - m\big)\, u(-\vec{\boldsymbol{p}},\lambda') = E_{\vec{\boldsymbol{p}}}\,\overline{v}(\vec{\boldsymbol{p}},\lambda)\gamma^0 u(-\vec{\boldsymbol{p}},\lambda')\,, \tag{33}$$

$$\overline{u}(\vec{\boldsymbol{p}},\lambda)\big(\vec{\boldsymbol{\gamma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}} + m\big)\, v(-\vec{\boldsymbol{p}},\lambda') = E_{\vec{\boldsymbol{p}}}\,\overline{u}(\vec{\boldsymbol{p}},\lambda)\gamma^0 v(-\vec{\boldsymbol{p}},\lambda')\,. \tag{34}$$

Combining these two results then implies that

$$\overline{v}(\vec{\boldsymbol{p}},\lambda)\big(\vec{\boldsymbol{\gamma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}} - m\big)\, u(-\vec{\boldsymbol{p}},\lambda') = 0\,, \tag{35}$$

$$\overline{u}(\vec{\boldsymbol{p}},\lambda)\big(\vec{\boldsymbol{\gamma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}} + m\big)\, v(-\vec{\boldsymbol{p}},\lambda') = 0\,. \tag{36}$$

However, we can actually prove the stronger result,

$$\overline{v}(\vec{\boldsymbol{p}},\lambda)\gamma^0 u(-\vec{\boldsymbol{p}},\lambda') = \overline{u}(\vec{\boldsymbol{p}},\lambda)\gamma^0 v(-\vec{\boldsymbol{p}},\lambda') = 0\,, \tag{37}$$

as follows. Using eqs. (E.203) and (E.204) of Ref. 1,

$$\overline{u}(\vec{\boldsymbol{p}},\lambda)\,\Gamma\, v(-\vec{\boldsymbol{p}},\lambda') = -i\lambda'\,\mathrm{Tr}\left[\Gamma\gamma_5\gamma^0(\delta_{\lambda\lambda'} + \gamma_5\gamma_\mu\mathscr{S}^\mu_{\lambda\lambda'})(\not{p} + m)\right], \tag{38}$$

$$\overline{v}(\vec{\boldsymbol{p}},\lambda)\,\Gamma\, u(-\vec{\boldsymbol{p}},\lambda') = i\lambda'\,\mathrm{Tr}\left[\Gamma\gamma_5\gamma^0(\delta_{\lambda'\lambda} + \gamma_5\gamma_\mu\mathscr{S}^\mu_{\lambda'\lambda})(\not{p} - m)\right]. \tag{39}$$

We now evaluate the following expressions:

$$\begin{aligned}
\overline{v}(\vec{\boldsymbol{p}},\lambda)\gamma^0 u(-\vec{\boldsymbol{p}},\lambda') &= i\lambda'\,\mathrm{Tr}\left[\gamma^0\gamma_5\gamma^0(\delta_{\lambda'\lambda} + \gamma_5\gamma_\mu\mathscr{S}^\mu_{\lambda'\lambda})(\not{p} - m)\right] \\
&= -i\lambda'\,\mathrm{Tr}\left[\gamma_\mu\mathscr{S}^\mu_{\lambda'\lambda}(\not{p} - m)\right] = -4i\lambda' p_\mu\mathscr{S}^\mu_{\lambda'\lambda} = 0\,, \\
\overline{u}(\vec{\boldsymbol{p}},\lambda)\gamma^0 v(-\vec{\boldsymbol{p}},\lambda') &= -i\lambda'\,\mathrm{Tr}\left[\gamma^0\gamma_5\gamma^0(\delta_{\lambda\lambda'} + \gamma_5\gamma_\mu\mathscr{S}^\mu_{\lambda\lambda'})(\not{p} + m)\right] \\
&= i\lambda'\,\mathrm{Tr}\left[\gamma_\mu\mathscr{S}^\mu_{\lambda\lambda'}(\not{p} + m)\right] = 4i\lambda' p_\mu\mathscr{S}^\mu_{\lambda\lambda'} = 0\,,
\end{aligned} \tag{40}$$

after using $p_\mu\mathscr{S}^\mu_{\lambda\lambda'} = (p\cdot S^a)\tau^a_{\lambda\lambda'} = 0$, which confirms the result of eq. (37).

2 The Hamiltonian derived using two-component spinors

We can repeat the computation of Section 1 using two-component spinor formalism. Note that the four-component Majorana fermion field can be written in following form,

$$\Psi_M(x) \equiv \begin{pmatrix} \xi_\alpha(x) \\ \xi^{\dagger\dot{\alpha}}(x) \end{pmatrix} . \tag{41}$$

This form ensures that $\Psi_M = C\overline{\Psi}_M^{\mathsf{T}}$ in the chiral representation of the gamma matrices (cf. footnote 1). Moreover, the four-component spinor wave functions u and v can be expressed in terms of corresponding two-component spinor wave functions x and y as shown in Ref. 1,

$$u(\vec{\boldsymbol{p}}, s) = \begin{pmatrix} x_\alpha(\vec{\boldsymbol{p}}, s) \\ y^{\dagger\dot{\alpha}}(\vec{\boldsymbol{p}}, s) \end{pmatrix} , \qquad \overline{u}(\vec{\boldsymbol{p}}, s) = (y^\alpha(\vec{\boldsymbol{p}}, s),\ x^\dagger_{\dot{\alpha}}(\vec{\boldsymbol{p}}, s)) , \tag{42}$$

$$v(\vec{\boldsymbol{p}}, s) = \begin{pmatrix} y_\alpha(\vec{\boldsymbol{p}}, s) \\ x^{\dagger\dot{\alpha}}(\vec{\boldsymbol{p}}, s) \end{pmatrix} , \qquad \overline{v}(\vec{\boldsymbol{p}}, s) = (x^\alpha(\vec{\boldsymbol{p}}, s),\ y^\dagger_{\dot{\alpha}}(\vec{\boldsymbol{p}}, s)) . \tag{43}$$

However the calculations of this section will not make any reference to eqs. (41)–(43).

We begin with the Lagrangian density for a two-component fermion field $\xi_\alpha(x)$,

$$\mathscr{L} = i\xi^\dagger\,\overline{\sigma}^\mu\partial_\mu\xi - \tfrac{1}{2}m(\xi\xi + \xi^\dagger\,\xi^\dagger) , \tag{44}$$

where $\overline{\sigma}^\mu = (\mathbf{I}_{2\times 2}\,;\,-\vec{\boldsymbol{\sigma}})$ and $\xi^\dagger\overline{\sigma}^\mu\partial_\mu\xi \equiv \xi^\dagger_{\dot{\alpha}}\overline{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\xi_\alpha$. We also define

$$\xi\xi \equiv \xi^\alpha\xi_\alpha = \epsilon^{\alpha\beta}\xi_\beta\xi_\alpha , \qquad \xi^\dagger\xi^\dagger \equiv \xi^\dagger_{\dot{\alpha}}\xi^{\dagger\,\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\xi^{\dagger\,\dot{\beta}}\xi^{\dagger\,\dot{\alpha}} , \tag{45}$$

where the nonzero components of the epsilon symbols are $\epsilon^{12} = -\epsilon^{21} = \epsilon_{\dot{2}\dot{1}} = -\epsilon_{\dot{1}\dot{2}} = 1$. The two-component fermion fields $\xi(x)$ and $\xi^\dagger(x)$ satisfy the field equations,

$$i\overline{\sigma}^{\mu\dot{\alpha}\beta}\partial_\mu\xi_\beta(x) = m\xi^{\dagger\dot{\alpha}}(x) . \tag{46}$$

Then,

$$\Pi = \frac{\partial\mathscr{L}}{\partial(\partial_0\xi)} = i\xi^\dagger\overline{\sigma}^0 , \tag{47}$$

and the Hamiltonian density is given by

$$\begin{aligned} \mathscr{H} &= \Pi\,\partial_0\xi - \mathscr{L} = i\xi^\dagger\overline{\sigma}^0\partial_0\xi - i\xi^\dagger\,\overline{\sigma}^\mu\partial_\mu\xi + \tfrac{1}{2}m(\xi\xi + \xi^\dagger\,\xi^\dagger) \\ &= i\xi^\dagger\vec{\boldsymbol{\sigma}}\cdot\vec{\boldsymbol{\nabla}}\xi + \tfrac{1}{2}m(\xi\xi + \xi^\dagger\,\xi^\dagger) . \end{aligned} \tag{48}$$

The Hamiltonian is given by

$$H = \int d^3x \left\{ i\xi^\dagger(x)\vec{\boldsymbol{\sigma}}\cdot\vec{\boldsymbol{\nabla}}\xi(x) + \tfrac{1}{2}m[\xi(x)\xi(x) + \xi^\dagger(x)\,\xi^\dagger(x)] \right\} . \tag{49}$$

Next, we expand the two-component fermion field in Fourier modes which involve the creation and annihilation operators,

$$\xi_\alpha(x) = \sum_\lambda \int \frac{d^3\boldsymbol{p}}{(2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}} \left[x_\alpha(\vec{\boldsymbol{p}},\lambda)a(\vec{\boldsymbol{p}},\lambda)e^{-ip\cdot x} + y_\alpha(\vec{\boldsymbol{p}},\lambda)a^\dagger(\vec{\boldsymbol{p}},\lambda)e^{ip\cdot x}\right], \tag{50}$$

$$\xi^\dagger_{\dot\alpha}(x) = \sum_\lambda \int \frac{d^3\boldsymbol{p}}{(2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}} \left[x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)a^\dagger(\vec{\boldsymbol{p}},\lambda)e^{ip\cdot x} + y^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)a(\vec{\boldsymbol{p}},\lambda)e^{-ip\cdot x}\right], \tag{51}$$

where $p^\mu = (E_{\boldsymbol{p}}\,;\,\vec{\boldsymbol{p}})$ with $E_{\boldsymbol{p}} \equiv \left(|\vec{\boldsymbol{p}}|^2 + m^2\right)^{1/2}$, and the sum over helicities runs over $\lambda = -\frac{1}{2}, \frac{1}{2}$. Under the assumption that the fields ξ and $\xi^\dagger$ satisfy the field equations [eq. (46)], the x and y spinors satisfy the momentum space Dirac equations:

$$(p\cdot\overline{\sigma})^{\dot\alpha\beta}x_\beta = my^{\dagger\dot\alpha}\,, \qquad (p\cdot\sigma)_{\alpha\dot\beta}y^{\dagger\dot\beta} = mx_\alpha\,, \tag{52}$$

$$(p\cdot\sigma)_{\alpha\dot\beta}x^{\dagger\dot\beta} = -my_\alpha\,, \qquad (p\cdot\overline{\sigma})^{\dot\alpha\beta}y_\beta = -mx^{\dagger\dot\alpha}\,, \tag{53}$$

$$x^\alpha(p\cdot\sigma)_{\alpha\dot\beta} = -my^\dagger_{\dot\beta}\,, \qquad y^\dagger_{\dot\alpha}(p\cdot\overline{\sigma})^{\dot\alpha\beta} = -mx^\beta\,, \tag{54}$$

$$x^\dagger_{\dot\alpha}(p\cdot\overline{\sigma})^{\dot\alpha\beta} = my^\beta\,, \qquad y^\alpha(p\cdot\sigma)_{\alpha\dot\beta} = mx^\dagger_{\dot\beta}\,. \tag{55}$$

Taking the derivative of eq. (50) yields

$$i\vec{\boldsymbol{\nabla}}\xi_\alpha(x) = \sum_\lambda \int \frac{\vec{\boldsymbol{p}}\;d^3\boldsymbol{p}}{(2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}} \left[-x_\alpha(\vec{\boldsymbol{p}},\lambda)a(\vec{\boldsymbol{p}},\lambda)e^{-iE_{\vec{\boldsymbol{p}}}t+i\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{x}}} + y_\alpha(\vec{\boldsymbol{p}},\lambda)a^\dagger(\vec{\boldsymbol{p}},\lambda)e^{iE_{\vec{\boldsymbol{p}}}t-i\vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{x}}}\right]. \tag{56}$$

Inserting eqs. (50) and (56) into eq. (49), we obtain

$$\begin{aligned}
\int d^3x\left\{i\xi^\dagger(x)\vec{\boldsymbol{\sigma}}\cdot\vec{\boldsymbol{\nabla}}\xi(x) + \tfrac{1}{2}m[\xi(x)\xi(x) + \xi^\dagger(x)\,\xi^\dagger(x)]\right\} &= \int d^3x \int \frac{d^3\vec{\boldsymbol{p}}\;d^3\vec{\boldsymbol{p}}\boldsymbol{'}}{(2\pi)^3(2E_{\boldsymbol{p}})^{1/2}(2E_{\boldsymbol{p'}})^{1/2}} \\
\times \sum_{\lambda,\lambda'}\Big\{ &\left[\tfrac{1}{2}m\big(y(\vec{\boldsymbol{p}},\lambda)x(\vec{\boldsymbol{p}}\boldsymbol{'},\lambda') + x^\dagger(\vec{\boldsymbol{p}},\lambda)y^\dagger(\vec{\boldsymbol{p}}\boldsymbol{'},\lambda')\big) - x^\dagger(\vec{\boldsymbol{p}},\lambda)\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\boldsymbol{'}\,x(\vec{\boldsymbol{p}}\boldsymbol{'},\lambda')\right] \\
&\quad\times a^\dagger(\vec{\boldsymbol{p}},\lambda)a(\vec{\boldsymbol{p}}\boldsymbol{'},\lambda')e^{i(E_{\vec{\boldsymbol{p}}}-E_{\vec{\boldsymbol{p}}\boldsymbol{'}})t}\,e^{-i(\vec{\boldsymbol{p}}-\vec{\boldsymbol{p}}\boldsymbol{'})\cdot\vec{\boldsymbol{x}}} \\
+&\left[\tfrac{1}{2}m\big(x(\vec{\boldsymbol{p}},\lambda)x(\vec{\boldsymbol{p}}\boldsymbol{'},\lambda') + +y^\dagger(\vec{\boldsymbol{p}},\lambda)y^\dagger(\vec{\boldsymbol{p}}\boldsymbol{'},\lambda')\big) - y^\dagger(\vec{\boldsymbol{p}},\lambda)\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\boldsymbol{'}\,x(\vec{\boldsymbol{p}}\boldsymbol{'},\lambda')\right] \\
&\quad\times a(\vec{\boldsymbol{p}},\lambda)a(\vec{\boldsymbol{p}}\boldsymbol{'},\lambda')e^{-i(E_{\vec{\boldsymbol{p}}}+E_{\vec{\boldsymbol{p}}\boldsymbol{'}})t}\,e^{i(\vec{\boldsymbol{p}}+\vec{\boldsymbol{p}}\boldsymbol{'})\cdot\vec{\boldsymbol{x}}} \\
+&\left[\tfrac{1}{2}m\big(y(\vec{\boldsymbol{p}},\lambda)y(\vec{\boldsymbol{p}}\boldsymbol{'},\lambda') + x^\dagger(\vec{\boldsymbol{p}},\lambda)x^\dagger(\vec{\boldsymbol{p}}\boldsymbol{'},\lambda')\big) + x^\dagger(\vec{\boldsymbol{p}},\lambda)\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\boldsymbol{'}\,y(\vec{\boldsymbol{p}}\boldsymbol{'},\lambda')a^\dagger(\vec{\boldsymbol{p}},\lambda)\right] \\
&\quad\times a^\dagger(\vec{\boldsymbol{p}}\boldsymbol{'},\lambda')e^{i(E_{\vec{\boldsymbol{p}}}+E_{\vec{\boldsymbol{p}}\boldsymbol{'}})t}\,e^{-i(\vec{\boldsymbol{p}}+\vec{\boldsymbol{p}}\boldsymbol{'})\cdot\vec{\boldsymbol{x}}} \\
+&\left[\tfrac{1}{2}m\big(x(\vec{\boldsymbol{p}},\lambda)y(\vec{\boldsymbol{p}}\boldsymbol{'},\lambda') + y^\dagger(\vec{\boldsymbol{p}},\lambda)x^\dagger(\vec{\boldsymbol{p}}\boldsymbol{'},\lambda')\big) + y^\dagger(\vec{\boldsymbol{p}},\lambda)\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\boldsymbol{'}\,y(\vec{\boldsymbol{p}}\boldsymbol{'},\lambda')\right] \\
&\quad\times a(\vec{\boldsymbol{p}},\lambda)a^\dagger(\vec{\boldsymbol{p}}\boldsymbol{'},\lambda')e^{-i(E_{\vec{\boldsymbol{p}}}-E_{\vec{\boldsymbol{p}}\boldsymbol{'}})t}\,e^{i(\vec{\boldsymbol{p}}-\vec{\boldsymbol{p}}\boldsymbol{'})\cdot\vec{\boldsymbol{x}}}\Big\}.
\end{aligned} \tag{57}$$

Integrating over $\vec{\boldsymbol{x}}$ using eq. (16) and making use of the delta function to integrate over $\vec{\boldsymbol{p}}'$ yields

$$\int d^3x \left\{i\xi^\dagger(x)\vec{\boldsymbol{\sigma}}\cdot\vec{\boldsymbol{\nabla}}\xi(x) + \tfrac{1}{2}m[\xi(x)\xi(x) + \xi^\dagger(x)\,\xi^\dagger(x)]\right\} = \int \frac{d^3\vec{\boldsymbol{p}}}{2E_{\boldsymbol{p}}}\sum_{\lambda,\lambda'}$$

$$\times\Big\{\left[\tfrac{1}{2}m\big(y(\vec{\boldsymbol{p}},\lambda)x(\vec{\boldsymbol{p}},\lambda') + x^\dagger(\vec{\boldsymbol{p}},\lambda)y^\dagger(\vec{\boldsymbol{p}},\lambda')\big) - x^\dagger(\vec{\boldsymbol{p}},\lambda)\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\,x(\vec{\boldsymbol{p}},\lambda')\right]a^\dagger(\vec{\boldsymbol{p}},\lambda)a(\vec{\boldsymbol{p}},\lambda')$$

$$+\left[\tfrac{1}{2}m\big(x(\vec{\boldsymbol{p}},\lambda)x(-\vec{\boldsymbol{p}},\lambda') + y^\dagger(\vec{\boldsymbol{p}},\lambda)y^\dagger(-\vec{\boldsymbol{p}},\lambda')\big) + y^\dagger(\vec{\boldsymbol{p}},\lambda)\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\,x(-\vec{\boldsymbol{p}},\lambda')\right]a(\vec{\boldsymbol{p}},\lambda)a(-\vec{\boldsymbol{p}},\lambda')e^{-2iE_{\vec{\boldsymbol{p}}}t}$$

$$+\left[\tfrac{1}{2}m\big(y(\vec{\boldsymbol{p}},\lambda)y(-\vec{\boldsymbol{p}},\lambda') + x^\dagger(\vec{\boldsymbol{p}},\lambda)x^\dagger(-\vec{\boldsymbol{p}},\lambda')\big) - x^\dagger(\vec{\boldsymbol{p}},\lambda)\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\,y(-\vec{\boldsymbol{p}},\lambda')\right]a^\dagger(\vec{\boldsymbol{p}},\lambda)a^\dagger(-\vec{\boldsymbol{p}},\lambda')e^{2iE_{\vec{\boldsymbol{p}}}t}$$

$$+\left[\tfrac{1}{2}m\big(x(\vec{\boldsymbol{p}},\lambda)y(\vec{\boldsymbol{p}},\lambda') + y^\dagger(\vec{\boldsymbol{p}},\lambda)x^\dagger(\vec{\boldsymbol{p}},\lambda')\big) + y^\dagger(\vec{\boldsymbol{p}},\lambda)\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\,y(\vec{\boldsymbol{p}},\lambda')\right]a(\vec{\boldsymbol{p}},\lambda)a^\dagger(\vec{\boldsymbol{p}},\lambda')\Big\}. \tag{58}$$

We now use the momentum space Dirac equations [eqs. (52)–(55)], which can be rewritten as

$$(\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}})^{\dot{\alpha}\beta}x_\beta = my^{\dagger\dot{\alpha}} - E_{\vec{\boldsymbol{p}}}(\overline{\sigma}^0 x)^{\dot{\alpha}}\ , \qquad (\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}})_{\alpha\dot{\beta}}y^{\dagger\dot{\beta}} = E_{\vec{\boldsymbol{p}}}(\sigma^0 y^\dagger)_\alpha - mx_\alpha\ , \tag{59}$$

$$(\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}})_{\alpha\dot{\beta}}x^{\dagger\dot{\beta}} = E_{\vec{\boldsymbol{p}}}(\sigma^0 x^\dagger)_\alpha + my_\alpha\ , \qquad (\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}})^{\dot{\alpha}\beta}y_\beta = -E_{\vec{\boldsymbol{p}}}(\overline{\sigma}^0 y)^{\dot{\alpha}} - mx^{\dagger\dot{\alpha}}\ , \tag{60}$$

$$x^\alpha(\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}})_{\alpha\dot{\beta}} = E_{\vec{\boldsymbol{p}}}(x\sigma^0)_{\dot{\beta}} + my^\dagger_{\dot{\beta}}\ , \qquad y^\dagger_{\dot{\alpha}}(\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}})^{\dot{\alpha}\beta} = -E_{\vec{\boldsymbol{p}}}(y^\dagger\overline{\sigma}^0)^\beta - mx^\beta\ , \tag{61}$$

$$x^\dagger_{\dot{\alpha}}(\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}})^{\dot{\alpha}\beta} = -E_{\vec{\boldsymbol{p}}}(x^\dagger\overline{\sigma}^0)^\beta + my^\beta\ , \qquad y^\alpha(\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}})_{\alpha\dot{\beta}} = E_{\vec{\boldsymbol{p}}}(y\sigma^0)_{\dot{\beta}} - mx^\dagger_{\dot{\beta}}\ . \tag{62}$$

We can make use of eqs. (59)–(62) to simplify the expressions in eq. (58) as follows:

$$\begin{aligned}
&\tfrac{1}{2}m\big(y(\vec{\boldsymbol{p}},\lambda)x(\vec{\boldsymbol{p}},\lambda') + x^\dagger(\vec{\boldsymbol{p}},\lambda)y^\dagger(\vec{\boldsymbol{p}}',\lambda')\big) - x^\dagger(\vec{\boldsymbol{p}},\lambda)\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\,x(\vec{\boldsymbol{p}},\lambda')\\
&\qquad = \tfrac{1}{2}\Big\{m\big[y(\vec{\boldsymbol{p}},\lambda) - x^\dagger(\vec{\boldsymbol{p}},\lambda)\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\big]x(\vec{\boldsymbol{p}},\lambda') + x^\dagger(\vec{\boldsymbol{p}},\lambda)\big[my^\dagger(\vec{\boldsymbol{p}}',\lambda') - \vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\,x(\vec{\boldsymbol{p}},\lambda')\big]\Big\}\\
&\qquad = E_{\vec{\boldsymbol{p}}}\,x^\dagger(\vec{\boldsymbol{p}},\lambda)\overline{\sigma}^0 x(\vec{\boldsymbol{p}},\lambda')\,,
\end{aligned} \tag{63}$$

$$\begin{aligned}
&\tfrac{1}{2}m\big(x(\vec{\boldsymbol{p}},\lambda)y(\vec{\boldsymbol{p}},\lambda') + y^\dagger(\vec{\boldsymbol{p}},\lambda)x^\dagger(\vec{\boldsymbol{p}},\lambda')\big) + y^\dagger(\vec{\boldsymbol{p}},\lambda)\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\,y(\vec{\boldsymbol{p}},\lambda')]\\
&\qquad = \tfrac{1}{2}\Big\{m\big[x(\vec{\boldsymbol{p}},\lambda) + y^\dagger(\vec{\boldsymbol{p}},\lambda)\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\big]y(\vec{\boldsymbol{p}},\lambda') + y^\dagger(\vec{\boldsymbol{p}},\lambda)\big[mx^\dagger(\vec{\boldsymbol{p}},\lambda') + \vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\,y(\vec{\boldsymbol{p}},\lambda')\big]\Big\}\\
&\qquad = -E_{\vec{\boldsymbol{p}}}\,y^\dagger(\vec{\boldsymbol{p}},\lambda)\overline{\sigma}^0 y(\vec{\boldsymbol{p}},\lambda')\,,
\end{aligned} \tag{64}$$

$$\begin{aligned}
&\tfrac{1}{2}m\big(x(\vec{\boldsymbol{p}},\lambda)x(-\vec{\boldsymbol{p}},\lambda') + y^\dagger(\vec{\boldsymbol{p}},\lambda)y^\dagger(-\vec{\boldsymbol{p}},\lambda')\big) + y^\dagger(\vec{\boldsymbol{p}},\lambda)\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\,x(-\vec{\boldsymbol{p}},\lambda')]\\
&\qquad = \tfrac{1}{2}\Big\{m\big[x(\vec{\boldsymbol{p}},\lambda) + y^\dagger(\vec{\boldsymbol{p}},\lambda)\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\big]x(-\vec{\boldsymbol{p}},\lambda') + y^\dagger(\vec{\boldsymbol{p}},\lambda)\big[my^\dagger(-\vec{\boldsymbol{p}},\lambda') + \vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\,x(-\vec{\boldsymbol{p}},\lambda')\big]\Big\} = 0,
\end{aligned} \tag{65}$$

$$\begin{aligned}
&\tfrac{1}{2}m\big(y(\vec{\boldsymbol{p}},\lambda)y(-\vec{\boldsymbol{p}},\lambda') + x^\dagger(\vec{\boldsymbol{p}},\lambda)x^\dagger(-\vec{\boldsymbol{p}},\lambda')\big) - x^\dagger(\vec{\boldsymbol{p}},\lambda)\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\,y(-\vec{\boldsymbol{p}},\lambda')\\
&\qquad = \tfrac{1}{2}\Big\{m\big[y(\vec{\boldsymbol{p}},\lambda) - x^\dagger(\vec{\boldsymbol{p}},\lambda)\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\big]y(-\vec{\boldsymbol{p}},\lambda') + x^\dagger(\vec{\boldsymbol{p}},\lambda)\big[mx^\dagger(-\vec{\boldsymbol{p}},\lambda') - \vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\vec{\boldsymbol{p}}\,y(-\vec{\boldsymbol{p}},\lambda')\big]\Big\} = 0.
\end{aligned} \tag{66}$$

Hence, eq. (58) reduces to

$$\int d^3x\,\left\{i\xi^\dagger(x)\vec{\boldsymbol{\sigma}}\cdot\vec{\boldsymbol{\nabla}}\xi(x)+\tfrac{1}{2}m[\xi(x)\xi(x)+\xi^\dagger(x)\,\xi^\dagger(x)]\right\}$$

$$=\frac{1}{2}\int d^3p\sum_{\lambda,\lambda'}\bigl[x^\dagger(\vec{\boldsymbol{p}},\lambda)\overline{\sigma}^0x(\vec{\boldsymbol{p}},\lambda')a^\dagger(\vec{\boldsymbol{p}},\lambda)a(\vec{\boldsymbol{p}},\lambda')-y^\dagger(\vec{\boldsymbol{p}},\lambda)\overline{\sigma}^0y(\vec{\boldsymbol{p}},\lambda')a(\vec{\boldsymbol{p}},\lambda)a^\dagger(\vec{\boldsymbol{p}},\lambda')\bigr]\,. \quad (67)$$

To evaluate the remaining spinor products, we make use of the Bouchiat-Michel formulae given in eqs. (E.133) and (E.138) of Ref. 1 and convert the corresponding expressions into traces,

$$x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)\Gamma^{\dot\alpha\beta}x_\beta(\vec{\boldsymbol{p}},\lambda')=\tfrac{1}{2}\mathrm{Tr}\bigl[\Gamma\bigl(p\cdot\sigma\delta_{\lambda\lambda'}-m\sigma_\mu\mathscr{S}^\mu_{\lambda\lambda'}\bigr)\bigr]\,, \quad (68)$$

$$y^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)\Gamma^{\dot\alpha\beta}y_\beta(\vec{\boldsymbol{p}},\lambda')=\tfrac{1}{2}\mathrm{Tr}\bigl[\Gamma\bigl(p\cdot\sigma\delta_{\lambda\lambda'}+m\sigma_\mu\mathscr{S}^\mu_{\lambda\lambda'}\bigr)\bigr]\,, \quad (69)$$

where Γ is any product of sigma matrices and $\mathscr{S}^\mu_{\lambda\lambda'}$ is defined in eq. (25). Using $\Gamma=\overline{\sigma}^0$ in the above formulae, it follows that

$$x^\dagger(\vec{\boldsymbol{p}},\lambda)\overline{\sigma}^0x(\vec{\boldsymbol{p}},\lambda')=\mathrm{Tr}\bigl[\overline{\sigma}^0\bigl(p\cdot\sigma\delta_{\lambda\lambda'}-m\sigma_\mu\mathscr{S}^\mu_{\lambda\lambda'}\bigr)\bigr]\,,$$

$$y^\dagger(\vec{\boldsymbol{p}},\lambda)\overline{\sigma}^0y(\vec{\boldsymbol{p}},\lambda')=\mathrm{Tr}\bigl[\overline{\sigma}^0\bigl(p\cdot\sigma\delta_{\lambda\lambda'}+m\sigma_\mu\mathscr{S}^\mu_{\lambda\lambda'}\bigr)\bigr]\,.$$

Employing $\mathrm{Tr}(\overline{\sigma}^\mu\sigma^\nu)=2g^{\mu\nu}$ and $p_\mu\mathscr{S}^\mu_{\lambda\lambda'}=(p\cdot S^a)\tau^a_{\lambda\lambda'}=0$, it follows that

$$x^\dagger(\vec{\boldsymbol{p}},\lambda)\overline{\sigma}^0x(\vec{\boldsymbol{p}},\lambda')=y^\dagger(\vec{\boldsymbol{p}},\lambda)\overline{\sigma}^0y(\vec{\boldsymbol{p}},\lambda')=2E_{\vec{\boldsymbol{p}}}\,\delta_{\lambda\lambda'}\,. \quad (70)$$

Inserting this result back into eq. (67), we finally obtain

$$H=\int d^3x\,\left\{i\xi^\dagger(x)\vec{\boldsymbol{\sigma}}\cdot\vec{\boldsymbol{\nabla}}\xi(x)+\tfrac{1}{2}m[\xi(x)\xi(x)+\xi^\dagger(x)\,\xi^\dagger(x)]\right\}$$

$$=\frac{1}{2}\sum_\lambda\int d^3\vec{\boldsymbol{p}}\;E_{\vec{\boldsymbol{p}}}\bigl[a^\dagger(\vec{\boldsymbol{p}},\lambda)a(\vec{\boldsymbol{p}},\lambda)-a(\vec{\boldsymbol{p}},\lambda)a^\dagger(\vec{\boldsymbol{p}},\lambda)\bigr]\,, \quad (71)$$

after using the $\delta_{\lambda\lambda'}$ factors to perform the sum over λ'.

Finally, we redefine the Hamiltonian to be the normal ordering of the above expression to eliminate the infinite zero point energy. Using eq. (29), we end up with

$$H=\sum_\lambda\int d^3\vec{\boldsymbol{p}}\;E_{\vec{\boldsymbol{p}}}\,a^\dagger(\vec{\boldsymbol{p}},\lambda)a(\vec{\boldsymbol{p}},\lambda)\,, \quad (72)$$

thereby reproducing the result obtained in eq. (30) using the four-component spinor formalism.

References

1. Herbi K. Dreiner, Howard E. Haber, and Stephen P. Martin, *From Spinors to Supersymmetry* (Cambridge University Press, Cambridge, UK, 2023). See also Ref. 2.

2. Herbi K. Dreiner, Howard E. Haber, and Stephen P. Martin, *Two-component spinor techniques and Feynman rules for quantum field theory and supersymmetry*, Physics Reports **494** (2010) 1–196 [arXiv:0812.1594 [hep-ph]].