Adventures in Nima-Land

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Part 1: The mathematics behind transcendental tuning

A closer look at Nima's examples—rational approximations of π and $\ln 2$.

Reference: H.E. Haber, "*Rational approximations of* $\ln 2$," http://scipp.ucsc.edu/~haber/

On April 6, 2021, Nima Arkani-Hamed gave a very stimulating talk at the BSM Pandemic Seminar via zoom[1] entitled "*Some new thoughts on the hierarchy problem*."

He proposed a mechanism to explain the cancellation of tree effects ("rational") against loop effects ("transcendental") which otherwise would look fine-tuned.

A simple example: the decay width of ortho-positronium,

$$\Gamma(\text{o–Ps} \to \gamma\gamma\gamma) = \frac{2\left(\pi^2 - 9\right) m_e \alpha^6}{9\pi} (1 + \mathcal{O}(\alpha)).$$

Note that $\pi^2 - 9 \simeq 0.8696$ which is an order of magnitude smaller than π^2 and 9.

[1] Check it out here: `https://indico.cern.ch/event/1025750/`.

Nima provided two mathematical examples of this mechanism:

$$\mathcal{J}_n \equiv 4(-1)^n \int_0^1 \frac{dx}{1+x^2} \left(\frac{x^2(1-x)^2}{2}\right)^{2n} = \pi - p_n \,,$$

$$\mathcal{I}_n \equiv (-1)^n \int_0^1 \frac{dx}{1+x} \left(\frac{x(1-x)}{2}\right)^{n} = \ln 2 - r_n \,.$$

Observe that $0 < (-1)^n \mathcal{J}_n < 2^{-10n}$ and $0 < (-1)^n \mathcal{I}_n < 2^{-3n}$.

Hence,

$$\lim_{n\to\infty} p_n = \pi \,, \qquad \lim_{n\to\infty} r_n = \ln 2 \,.$$

Similar results can be obtained for other transcendental numbers that are *periods*,[2] which play a starring role in modern scattering amplitude methods.

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[2] "Periods" is a generic term used to designate the numbers arising as integrals of algebraic functions over domains described by algebraic equations or inequalities with rational coefficients. See, e.g., M. Kontsevich and D. Zagier, "Periods," in *Mathematics Unlimited—2001 and Beyond*, edited by B. Engquist and W. Schmid (Springer-Verlag, Berlin, 2001) pp. 771–808.

A closed form expression for p_n was given in the literature[3] (but I could not find a similar expression for r_n),

$$p_n = \sum_{k=0}^{n-1} (-1)^k \frac{2^{4-2k}(4k)!(4k+3)!}{(8k+7)!} \left(820k^3 + 1533k^2 + 902k + 165\right)$$

This formula was derived using a magical identity (no proof of this identity was given):

$$\frac{x^{4n}(1-x)^{4n}}{1+x^2} = \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4\right) \sum_{k=0}^{n-1} (-4)^{n-1-k} x^{4k}(1-x)^{4k} + \frac{(-4)^n}{1+x^2} \, .$$

The result for p_n yields an infinite series for π,

$$\pi = \frac{22}{7} - \frac{19}{15015} + \frac{543}{594914320} - \frac{77}{104187267600} + \cdots$$

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[3] S.K. Lukas, "Approximations to π derived from integrals with nonnegative integrands," The American Mathematical Monthly, 162, 166 (2009).

Challenge: prove that

$$\frac{x^{4n}(1-x)^{4n}}{1+x^2} = \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4\right)\sum_{k=0}^{n-1}(-4)^{n-1-k}x^{4k}(1-x)^{4k} + \frac{(-4)^n}{1+x^2}\,.$$

Start with the identity,

$$\frac{x^4(1-x)^4}{1+x^2} = P(x) + \frac{R}{1+x^2}\,,$$

for some polynomial $P(x)$ and constant R to be determined. Considering the residue at the poles, $x = \pm i$, yields $R = -4$. Then $P(x)$ is determined:

$$\begin{aligned} P(x) &= \frac{x^4(1-x)^4 + 4}{1+x^2} \\ &= x^6 - 4x^5 + 5x^4 - 4x^2 + 4\,. \end{aligned}$$

Now, you can use the identity, $x^4(1-x)^4 = (1+x^2)P(x) - 4$, repeatedly, to derive,

$$\begin{aligned}
\frac{x^8(1-x)^8}{1+x^2} &= x^4(1-x)^4\left[P(x) - \frac{4}{1+x^2}\right] \\
&= x^4(1-x)^4P(x) - \frac{4}{1+x^2}\left[(1+x^2)P(x) - 4\right] \\
&= \left[x^4(1-x)^4 - 4\right]P(x) + \frac{(-4)^2}{1+x^2},
\end{aligned}$$

$$\frac{x^{12}(1-x)^{12}}{1+x^2} = \left[x^8(1-x)^8 - 4x^4(1-x)^4 + (-4)^2\right]P(x) + \frac{(-4)^3}{1+x^2},$$

and so on. After n steps we are done!

$$\frac{x^{4n}(1-x)^{4n}}{1+x^2} = P(x)\sum_{k=0}^{n-1}(-4)^{n-1-k}x^{4k}(1-x)^{4k} + \frac{(-4)^n}{1+x^2}.$$

A similar (and much simpler calculation) yields a closed form expression for r_n. The corresponding magical identity is,

$$\frac{x^n(1-x)^n}{1+x} = (2-x)\sum_{k=0}^{n-1}(-2)^{n-1-k}x^k(1-x)^k + \frac{(-2)^n}{1+x}\,.$$

Then,

$$\mathcal{I}_n = \ln 2 + \sum_{k=0}^{n-1}(-2)^{-1-k}\int_0^1 x^k(1-x)^k(2-x)\,dx\,.$$

The integrals can be expressed in terms of Beta functions, $B(r,s) \equiv \Gamma(r)\Gamma(s)/\Gamma(r+s)$,

$$\mathcal{I}_n = \ln 2 + \sum_{k=0}^{n-1}(-2)^{-1-k}\left[2B(k+1,k+1) - B(k+2,k+1)\right].$$

n	a_{n-1}	r_n	numerical value
1	$\frac{3}{4}$	$\frac{3}{4}$	0.75
2	$-\frac{1}{16}$	$\frac{11}{16}$	0.6875
3	$\frac{1}{160}$	$\frac{111}{160}$	0.69375
4	$-\frac{3}{4480}$	$\frac{621}{896}$	0.69308035714
5	$\frac{1}{13440}$	$\frac{2329}{3360}$	0.69315476190
6	$-\frac{1}{118272}$	$\frac{19519}{28160}$	0.69314630682
7	$\frac{1}{1025024}$	$\frac{3552463}{5125120}$	0.69314728241
8	$-\frac{1}{8785920}$	$\frac{42629549}{61501440}$	0.69314716859
9	$\frac{1}{74680320}$	$\frac{241567449}{348508160}$	0.69314718198
10	$-\frac{3}{1891901440}$	$\frac{834505731}{1203937280}$	0.69314718039

Hence,

$$r_n = \sum_{k=0}^{n-1} a_k.$$

Explicitly,

$$\boxed{r_n = \frac{3}{4} \sum_{k=0}^{n-1} \frac{(-1)^k [k!]^2}{2^k (2k+1)!}}$$

With ten digit accuracy,

$$\ln 2 \simeq 0.6931471806$$

Previously, I had never encountered the expansion,

$$\ln 2 = \frac{3}{4}\sum_{k=0}^{\infty}\frac{(-1)^k[k!]^2}{2^k(2k+1)!}\,.$$

Here is an independent proof. Start with

$${}_2F_1(\tfrac{1}{2},1,\tfrac{3}{2};z^2) = \frac{1}{2z}\ln\left(\frac{1+z}{1-z}\right)\,,$$

where ${}_2F_1$ is the Gauss hypergeometric function. Using the identity

$${}_2F_1(a,b;c;z^2) = (1-z^2)^{-b}\,{}_2F_1\big(c-a,b;c,z^2/(z^2-1)\big)\,,$$

and setting $z=\frac{1}{3}$ yields,

$$\ln 2 = \tfrac{3}{4}\,{}_2F_1\big(1,1;\tfrac{3}{2},-\tfrac{1}{8})\,.$$

Employing the series representation of ${}_2F_1$ and making use of the duplication formula for the gamma function yields the series representation for $\ln 2$ given above.

Part 2: Exceptional regions of the 2HDM parameter space

Based on the following two papers,

1. H.E. Haber and J.P. Silva, "Exceptional regions of the 2HDM parameter space," arXiv:2102.07136 [hep-ph], Phys. Rev. D (2021), in press.

2. P. Draper, A. Ekstedt and H.E. Haber, "A natural mechanism for approximate Higgs alignment in the 2HDM," arXiv:2011.13159 [hep-ph], JHEP (2021), in press.

The Nima challenge

At the KITP in December, 2012, I asked Nima whether he would still advocate for a fine-tuned electroweak scale due to selection effects[4] if additional scalar states of an extended Higgs sector were discovered at LHC. Nima said he would abandon this proposal because he would not be able to explain the additional fine-tunings required to accommodate light scalars of the extended Higgs sector. Taking up Nima's challenge, in 2016 Patrick Draper, Josh Ruderman and I provided a proof of principle that extended Higgs sectors with only one fine-tuning were viable.[5]

Our model employed an exceptional region of the 2HDM parameter space. Subsequently, I realized that this idea could be repurposed to provide a natural explanation for approximate Higgs alignment without decoupling.

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[4] *Selection effects* is a more palatable term that avoids mentioning by name the anthropic principle.

[5] P. Draper, H.E. Haber and J.T. Ruderman, "Partially Natural Two Higgs Doublet Models," JHEP **06**, 124 (2016) [arXiv:1605.03237 [hep-ph]].

Outline

- The complex 2HDM (C2HDM) framework
- Exceptional Regions of the 2HDM Parameter Space (ERPS and ERPS4)
- Reviewing the Higgs basis and the Higgs alignment limit
- Reviewing Family and Generalized CP symmetries of the 2HDM
- Exceptional features of the ERPS4
  - CP-conserving scalar potential when $\mathrm{Im}\big(\lambda_5^*[m_{12}^2]^2\big) \neq 0$
  - Exact Higgs alignment due to a (softly-broken) symmetry
  - Custodial symmetric scalar potential with exact Higgs alignment
- What about the Yukawa couplings?

Why the C2HDM?

Let us focus on the two-Higgs doublet model (2HDM) as a prototype for an extended Higgs sector. Consider the 2HDM scalar potential (in the Φ-basis),

$$\begin{aligned}\mathcal{V} = {} & m_{11}^2\Phi_1^\dagger\Phi_1 + m_{22}^2\Phi_2^\dagger\Phi_2 - [m_{12}^2\Phi_1^\dagger\Phi_2 + \text{h.c.}] + \tfrac{1}{2}\lambda_1(\Phi_1^\dagger\Phi_1)^2 \\ & + \tfrac{1}{2}\lambda_2(\Phi_2^\dagger\Phi_2)^2 + \lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1) \\ & + \left\{\tfrac{1}{2}\lambda_5(\Phi_1^\dagger\Phi_2)^2 + \left[\lambda_6(\Phi_1^\dagger\Phi_1) + \lambda_7(\Phi_2^\dagger\Phi_2)\right]\Phi_1^\dagger\Phi_2 + \text{h.c.}\right\}\,.\end{aligned}$$

The Φ_i are hypercharge $Y = 1$ doublets. After minimizing the scalar potential, $\langle\Phi_1^0\rangle = vc_\beta/\sqrt{2}$ and $\langle\Phi_2^0\rangle = vs_\beta e^{i\xi}/\sqrt{2}$, where $v \equiv 2m_W/g = 246$ GeV, $s_\beta \equiv \sin\beta$ and $c_\beta \equiv \cos\beta$, with $0 \leq \beta \leq \frac{1}{2}\pi$.

The complex 2HDM (C2HDM) is defined as the 2HDM such that $\lambda_6 = \lambda_7 = 0$ and $\text{Im}\big(\lambda_5^*[m_{12}^2]^2\big) \neq 0$.

Comments

- We have imposed a (softly-broken) $\mathbb{Z}_2$ symmetry so that $\lambda_6 = \lambda_7 = 0$ (but allow for $m_{12}^2 \neq 0$). This provides a framework for avoiding tree-level Higgs mediated FCNCs.

- We have allowed λ_5 and m_{12}^2 to be complex. This may be forced on you due to the presence of a CP-violating phase in the Higgs-quark Yukawa couplings.

  See D. Fontes, M. Löschner, J. C. Romão and J. P. Silva, arXiv:2103.05002.

- The number of independent parameters (dofs) that govern the 2HDM scalar potential has been reduced from 11 to 9.

“Well-known” facts about the C2HDM

- Expressions for the m_{ij}^2 and the λ_i look quite different under a unitary change of scalar field basis, $\Phi_a \to V_{ab}\Phi_b$.

- $\text{Im}\big(\lambda_5^*[m_{12}^2]^2\big) \neq 0$ implies a CP-violating scalar potential.

- Exact Higgs alignment, corresponding to the existence of a neutral scalar whose tree-level properties are those of the SM Higgs boson (without decoupling of heavy scalar states), requires a fine-tuning of scalar potential parameters.

- Imposing a custodial symmetric scalar potential requires a CP-odd scalar that is degenerate in mass with $H^\pm$.

An Exceptional Region of the 2HDM Parameter Space

ERPS: $m_{11}^2 = m_{22}^2$, $m_{12}^2 = 0$, $\lambda_1 = \lambda_2$, and $\lambda_7 = -\lambda_6$ (5 dofs)

ERPS4: $\lambda_1 = \lambda_2$ and $\lambda_7 = -\lambda_6$ (8 dofs)

What happened to the condition that $\lambda_6 = \lambda_7 = 0$?

Theorem: If $\lambda_1 = \lambda_2$ and $\lambda_7 = -\lambda_6$, then a basis of scalar fields exists (which is not unique) such that $\lambda_6 = \lambda_7 = 0$ and $\lambda_5 \in \mathbb{R}$.

The ERPS corresponds to a regime in which the scalar potential respects a generalized CP-symmetry called GCP2. The ERPS4 corresponds to a softly-broken GCP2-symmetric scalar potential.

Exceptional features of the ERPS4

- If $\lambda_1 = \lambda_2$ and $\lambda_7 = -\lambda_6$ is satisfied in one scalar field basis, then it is satisfied in any choice of scalar field basis.

- $\mathrm{Im}\big(\lambda_5^*[m_{12}^2]^2\big) \neq 0$ does not necessarily imply a CP-violating scalar potential.[6]

- Exact Higgs alignment holds $naturally$ in the ERPS and can also be achieved in a significant region of the ERPS4.

- The custodial symmetric subregion of the ERPS4 can allow for a CP-even scalar that is degenerate in mass with $H^\pm$.

[6] In the ERPS where $m_{12}^2 = 0$, CP is automatically conserved by the scalar potential and vacuum.

The Higgs basis and the Higgs alignment limit

Define the scalar doublet fields of the Higgs basis,

$$\mathcal{H}_1 = \begin{pmatrix} \mathcal{H}_1^+ \\ \mathcal{H}_1^0 \end{pmatrix} \equiv c_\beta \Phi_1 + s_\beta e^{-i\xi} \Phi_2 \,, \qquad \mathcal{H}_2 = \begin{pmatrix} \mathcal{H}_2^+ \\ \mathcal{H}_2^0 \end{pmatrix} \equiv e^{i\eta}\big(-s_\beta e^{i\xi} \Phi_1 + c_\beta \Phi_2\big),$$

such that $\langle \mathcal{H}_1^0 \rangle = v/\sqrt{2}$ and $\langle \mathcal{H}_2^0 \rangle = 0$. The Higgs basis is uniquely defined up to an overall rephasing that is parameterized by the phase angle η. [See R. Boto, T. V. Fernandes, H.E. Haber, J.C. Romão and J.P. Silva, Phys. Rev. D **101**, 055023 (2020)]

The neutral scalar $\mathcal{H}_1^0$ is *aligned* in field space with the vacuum expectation value v. If $\sqrt{2}\,\mathrm{Re}\mathcal{H}_1^0 - v$ were a mass eigenstate, then its tree-level properties would coincide with those of the SM Higgs boson.

In the Higgs basis, the scalar potential is given by:

$$\mathcal{V} = Y_1\mathcal{H}_1^\dagger\mathcal{H}_1 + Y_2\mathcal{H}_2^\dagger\mathcal{H}_2 + [Y_3e^{-i\eta}\mathcal{H}_1^\dagger\mathcal{H}_2 + \text{h.c.}] + \tfrac{1}{2}Z_1(\mathcal{H}_1^\dagger\mathcal{H}_1)^2$$
$$+\tfrac{1}{2}Z_2(\mathcal{H}_2^\dagger\mathcal{H}_2)^2 + Z_3(\mathcal{H}_1^\dagger\mathcal{H}_1)(\mathcal{H}_2^\dagger\mathcal{H}_2) + Z_4(\mathcal{H}_1^\dagger\mathcal{H}_2)(\mathcal{H}_2^\dagger\mathcal{H}_1)$$
$$+\left\{\tfrac{1}{2}Z_5e^{-2i\eta}(\mathcal{H}_1^\dagger\mathcal{H}_2)^2 + \left[Z_6e^{-i\eta}(\mathcal{H}_1^\dagger\mathcal{H}_1) + Z_7e^{-i\eta}(\mathcal{H}_2^\dagger\mathcal{H}_2)\right]\mathcal{H}_1^\dagger\mathcal{H}_2 + \text{h.c.}\right\}.$$

Minimize the scalar potential: $Y_1 = -\frac{1}{2}Z_1v^2$ and $Y_3 = -\frac{1}{2}Z_6v^2$.

Remark:

Exact Higgs alignment $\Longleftrightarrow$ $Z_6 = 0$ (and $Y_3 = 0$ via the scalar potential minimum conditions), which implies no $\mathcal{H}_1^0$–$\mathcal{H}_2^0$ mixing.

Only the terms highlighted in red can yield an $\mathcal{H}_1^\dagger\mathcal{H}_2 + \text{h.c.}$ contribution to the quadratic terms of the scalar potential after imposing $\langle\mathcal{H}_1^0\rangle = v/\sqrt{2}$ and $\langle\mathcal{H}_2^0\rangle = 0$.

Approximate Higgs alignment in the CP-conserving 2HDM

With respect to Higgs basis states, $\{\sqrt{2}\,\mathrm{Re}\,H_1^0 - v\,,\sqrt{2}\,\mathrm{Re}\,H_2^0\}$,

$$\mathcal{M}_H^2 = \begin{pmatrix} Z_1 v^2 & Z_6 v^2 \\ Z_6 v^2 & m_A^2 + Z_5 v^2 \end{pmatrix}, \qquad \text{where } Z_5,\ Z_6 \in \mathbb{R}\,.$$

The CP-even Higgs bosons are h and H with $m_h \leq m_H$. Approximate Higgs alignment arises in two limiting cases:

1. $m_A^2 \gg (Z_1 - Z_5)v^2$. This is the *decoupling limit*, where h is SM-like and $m_A^2 \sim m_H^2 \sim m_{H^\pm}^2 \gg m_h^2 \simeq Z_1 v^2$.

2. $|Z_6| \ll 1$. Then, h is SM-like if $m_A^2 + (Z_5 - Z_1)v^2 > 0$; otherwise, H is SM-like. $\Longrightarrow$ *Alignment without decoupling*.

Achieving exact Higgs alignment in the 2HDM

The inert doublet model (IDM): There is a $\mathbb{Z}_2$ symmetry in the Higgs basis such that $\mathcal{H}_2 \to -\mathcal{H}_2$ is the only $\mathbb{Z}_2$-odd field. Then $Z_6 = 0$, and tree-level alignment is exact. Deviations from SM behavior can appear at loop level due to the virtual exchange of the scalar states that reside in $\mathcal{H}_2$.

Approximate Higgs alignment without decoupling : If present,

- is this a result of an accidental choice of model parameters?
- is this a consequence of an approximate (softly-broken) symmetry? Not possible in the IDM; possible in the ERPS4.

Family and Generalized CP symmetries of the 2HDM

Higgs family symmetries

$\mathbb{Z}_2$: $\Phi_1 \to \Phi_1$, $\Phi_2 \to -\Phi_2$

Π_2 : $\Phi_1 \longleftrightarrow \Phi_2$

U(1)$_{\rm PQ}$ [Peccei-Quinn]: $\Phi_1 \to e^{-i\theta}\Phi_1$, $\Phi_2 \to e^{i\theta}\Phi_2$

SO(3): $\Phi_a \to U_{ab}\Phi_b$, $U \in \mathrm{U}(2)/\mathrm{U}(1)_{\rm Y}$

Generalized CP (GCP) transformations

$$\text{GCP1}: \quad \Phi_1 \to \Phi_1^*, \qquad \Phi_2 \to \Phi_2^*$$

$$\text{GCP2}: \quad \Phi_1 \to \Phi_2^*, \qquad \Phi_2 \to -\Phi_1^*$$

$$\text{GCP3}: \quad \Phi_1 \to \Phi_1^* c_\theta + \Phi_2^* s_\theta, \qquad \Phi_2 \to -\Phi_1^* s_\theta + \Phi_2^* c_\theta, \quad \text{for any } 0 < \theta < \tfrac{1}{2}\pi$$

where $c_\theta \equiv \cos\theta$ and $s_\theta \equiv \sin\theta$.

Possible symmetries of the 2HDM scalar potential

A complete classification of possible Higgs family and generalized CP symmetries of the scalar potential (in the Φ-basis) has been obtained.[7]

symmetry	m_{22}^2	m_{12}^2	λ_2	λ_4	$\mathrm{Re}\lambda_5$	$\mathrm{Im}\lambda_5$	λ_6	λ_7
$\mathbb{Z}_2$		0					0	0
Π_2	m_{11}^2	real	λ_1			0		λ_6^*
$\mathbb{Z}_2 \otimes \Pi_2$	m_{11}^2	0	λ_1			0	0	0
U(1)		0			0	0	0	0
U(1)$\otimes\Pi_2$	m_{11}^2	0	λ_1		0	0	0	0
SO(3)	m_{11}^2	0	λ_1	$\lambda_1 - \lambda_3$	0	0	0	0
GCP1		real				0	real	real
GCP2	m_{11}^2	0	λ_1					$-\lambda_6$
GCP3	m_{11}^2	0	λ_1		$\lambda_1 - \lambda_3 - \lambda_4$	0	0	0

Note that $\Pi_2 \Longleftrightarrow \mathbb{Z}_2$ symmetry in a different Φ'-basis; $\mathbb{Z}_2 \otimes \Pi_2 \Longleftrightarrow$ GCP2 in a different basis; U(1)$\otimes\Pi_2 \Longleftrightarrow$ GCP3 in a different basis, where $\Phi' = V\Phi$ for a suitably chosen V.

[7] I.P. Ivanov, Phys. Rev. D **77**, 015017 (2008) [arXiv:0710.3490]; P.M. Ferreira, H.E. Haber and J.P. Silva, Phys. Rev. D **79**, 116004 (2009) [arXiv:0902.1537].

More family and GCP symmetries of the 2HDM

Higgs family symmetries

Π_2' : $\quad \Phi_1 \to \Phi_2, \qquad \Phi_2 \to -\Phi_1$

U(1)′: $\quad \Phi_1 \to \Phi_1 c_\theta + \Phi_2 s_\theta \qquad \Phi_2 \to -\Phi_1 s_\theta + \Phi_2 c_\theta$

U(1)″: $\quad \Phi_1 \to \Phi_1 c_\theta + i\Phi_2 s_\theta \qquad \Phi_2 \to i\Phi_1 s_\theta + \Phi_2 c_\theta$

GCP transformations

$\mathrm{GCP1}'$: $\quad \Phi_1 \to \Phi_2^*, \qquad \Phi_2 \to \Phi_1^*$

GCP3′ : $\quad \Phi_1 \to \Phi_1^* c_\theta - i\Phi_2^* s_\theta \qquad \Phi_2 \to i\Phi_1^* s_\theta - \Phi_2^* c_\theta\,,$ for any $0 < \theta < \frac{1}{2}\pi$

symmetry	m_{22}^2	m_{12}^2	λ_2	$\mathrm{Re}\lambda_5$	$\mathrm{Im}\lambda_5$	λ_6	λ_7
Π_2'	m_{11}^2	pure imaginary	λ_1		0		$-\lambda_6^*$
$\Pi_2 \otimes \Pi_2'$	m_{11}^2	0	λ_1		0	0	0
U(1)$'$	m_{11}^2	pure imaginary	λ_1	$\lambda_1 - \lambda_3 - \lambda_4$	0	0	0
U(1)$''$	m_{11}^2	real	λ_1	$\lambda_3 + \lambda_4 - \lambda_1$	0	0	0
U(1)$' \otimes \mathbb{Z}_2$	m_{11}^2	0	λ_1	$\lambda_1 - \lambda_3 - \lambda_4$	0	0	0
U(1)$'' \otimes \mathbb{Z}_2$	m_{11}^2	0	λ_1	$\lambda_3 + \lambda_4 - \lambda_1$	0	0	0
GCP1$'$	m_{11}^2		λ_1				λ_6
GCP3$'$	m_{11}^2	0	λ_1	$\lambda_3 + \lambda_4 - \lambda_1$	0	0	0

Note that $\Pi_2' \Longleftrightarrow \mathbb{Z}_2$ symmetry in a different basis; GCP1$'$ $\Longleftrightarrow$ GCP1 in a different basis; GCP3$'$ $\Longleftrightarrow$ GCP3 in a different basis. Moreover, the constraints on the scalar potential parameters due to the $\mathbb{Z}_2 \otimes \Pi_2$, GCP3 and GCP3$'$ symmetries coincide with those of the $\Pi_2 \otimes \Pi_2'$, U(1)$' \otimes \mathbb{Z}_2$ and U(1)$'' \otimes \mathbb{Z}_2$ symmetries, respectively.

The ERPS4 with $\mathrm{Im}(\lambda_5^*[m_{12}^2]^2) \neq 0$

Outside of the ERPS4, $\mathrm{Im}(\lambda_5^*[m_{12}^2]^2) \neq 0$ necessarily implies a CP-violating scalar potential. However, consider the following two cases:

- softly-broken $Z_2 \otimes \Pi_2$ with complex m_{12}^2 and $\beta = \frac{1}{4}\pi$.

  Softly-broken $Z_2 \otimes \Pi_2$ implies that λ_5 is real and nonzero.[8] Hence, $\mathrm{Im}(\lambda_5^*[m_{12}^2]^2) \neq 0$. Nevertheless the scalar potential and vacuum are CP-conserving. This model possesses an unbroken GCP1′ symmetry,

$$\mathrm{GCP1}': \qquad \Phi_1 \to \Phi_2^*, \qquad\qquad \Phi_2 \to \Phi_1^*$$

  which imposes the conditions $m_{11}^2 = m_{22}^2$, $\lambda_1 = \lambda_2$ and $\lambda_6 = \lambda_7$, consistent with the $Z_2 \otimes \Pi_2$ symmetry constraints (since $\beta = \frac{1}{4}\pi$ requires $m_{11}^2 = m_{22}^2$). However, no reality condition is imposed on m_{12}^2 or λ_5. Moreover, since $\langle \Phi_1^0 \rangle = \langle \Phi_2^0 \rangle$, the vacuum also respects the $\mathrm{GCP1}'$ symmetry!

[8] If $\lambda_5 = 0$ then the $\mathbb{Z}_2 \otimes \Pi_2$ symmetry is promoted to a U(1)$\otimes\Pi_2$ symmetry.

- softly-broken GCP3 with complex m_{12}^2, for arbitrary $\tan\beta$.

  Softly broken GCP3 implies that $\lambda_5 = \lambda_1 - \lambda_3 - \lambda_4$ is real and nonzero.[9] Hence, $\mathrm{Im}\big(\lambda_5^*[m_{12}^2]^2\big) \neq 0$. Nevertheless the scalar potential and vacuum are CP-conserving. One can construct the relevant GCP transformation that is preserved (its ugly!).

  However, it is easier to transform to the scalar field basis where the U(1)$\otimes\Pi_2$ symmetry is manifestly realized. In this basis, m_{12}^2 is still complex but $\lambda_5 = 0$. That is, in this basis $\mathrm{Im}\big(\lambda_5^*[m_{12}^2]^2\big) = 0$ and there is no possibility of spontaneous CP violation.

Remark:

In the ERPS4, an explicit CP-violating scalar potential arises if $s_{2\beta} \neq 0$, $\sin 2\xi \neq 0$, $m_{11}^2 \neq m_{22}^2$, and $\mathrm{Im}[m_{12}^2]^2 \neq 0$. If latter condition is replaced by $\mathrm{Im}[m_{12}^2]^2 = 0$, then spontaneous CP violation arises if $0 < |m_{12}^2| < \frac{1}{2}\lambda_5 v^2 s_{2\beta}$.

[9] If $\lambda_5 = 0$, the GCP3 symmetry is promoted to an SO(3) symmetry.

Symmetry origin for exact Higgs alignment

In the Φ-basis, $\langle\Phi_1^0\rangle = vc_\beta/\sqrt{2}$ and $\langle\Phi_2^0\rangle = vs_\beta e^{i\xi}/\sqrt{2}$. The scalar potential parameters in this basis are related to the corresponding Higgs basis parameters; e.g.,

$$Y_3 = \left[\tfrac{1}{2}(m_{22}^2 - m_{11}^2)s_{2\beta} - \mathrm{Re}(m_{12}^2 e^{i\xi})c_{2\beta} - i\mathrm{Im}(m_{12}^2 e^{i\xi})\right]e^{-i\xi}\,.$$

If $m_{11}^2 = m_{22}^2$ and $m_{12}^2 = 0$, then $Y_3 = 0$. The scalar potential minimum condition ($Y_3 = -\frac{1}{2}Z_6 v^2$) then yields $Z_6 = 0$, i.e. exact Higgs alignment.

That is, any 2HDM scalar potential that satisfies $m_{11}^2 = m_{22}^2$ and $m_{12}^2 = 0$ due to a symmetry will yield exact Higgs alignment naturally!

Exact Higgs alignment arises when the following symmetries of the 2HDM scalar potential are unbroken.

symmetry	m_{22}^2	m_{12}^2	λ_2	λ_4	λ_5	λ_6	λ_7
$\mathbb{Z}_2 \otimes \Pi_2$	m_{11}^2	0	λ_1		real	0	0
GCP2	m_{11}^2	0	λ_1				$-\lambda_6$
U(1)$\otimes\Pi_2$	m_{11}^2	0	λ_1		0	0	0
GCP3	m_{11}^2	0	λ_1		$\lambda_1 - \lambda_3 - \lambda_4$ (real)	0	0
SO(3)	m_{11}^2	0	λ_1	$\lambda_1 - \lambda_3$	0	0	0

As previously noted, $\mathbb{Z}_2 \otimes \Pi_2$ and U(1)$\otimes\Pi_2$ are not independent symmetries, since a change of basis can be performed in each case to a new basis in which the GCP2 and GCP3 symmetries, respectively, are manifestly realized.

However, it is remarkable that in many cases, exact alignment is preserved even if the above symmetries are softly broken (corresponding to the ERPS4). In all such cases, exact Higgs alignment is achieved in the *inert limit* where $Y_3 = Z_6 = Z_7 = 0$. A complete classification of 2HDM scalar potentials with Higgs alignment due to a symmetry has been obtained.

Symmetry	soft-breaking	parameter constraints	residual unbroken symmetry of scalar potential	vacuum
$\mathbb{Z}_2$	none	$s_{2\beta}=0$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
U(1)	none	$s_{2\beta}=0$	U(1)	U(1)
$\mathbb{Z}_2\otimes\Pi_2$	$m_{11}^2\neq m_{22}^2$	$s_{2\beta}=0$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
$\mathbb{Z}_2\otimes\Pi_2$	$\mathrm{Re}m_{12}^2\neq 0$	$c_{2\beta}=\sin\xi=0$	Π_2	Π_2
$\mathbb{Z}_2\otimes\Pi_2$	$\mathrm{Im}m_{12}^2\neq 0$	$c_{2\beta}=\cos\xi=0$	Π_2'	Π_2'
$\mathbb{Z}_2\otimes\Pi_2$	none	$s_{2\beta}=0$	$\mathbb{Z}_2\otimes\Pi_2$	$\mathbb{Z}_2$
$\mathbb{Z}_2\otimes\Pi_2$	none	$c_{2\beta}=\sin 2\xi=0$	$\mathbb{Z}_2\otimes\Pi_2$	Π_2
U(1)$\otimes\Pi_2$	$m_{11}^2\neq m_{22}^2$	$s_{2\beta}=0$	U(1)	U(1)
U(1)$\otimes\Pi_2$	$\mathrm{Re}(m_{12}^2e^{i\xi})\neq 0$	$c_{2\beta}=0$	$\Pi_2^{(\xi)}$	$\Pi_2^{(\xi)}$
U(1)$\otimes\Pi_2$	none	$s_{2\beta}=0$	U(1)$\otimes\Pi_2$	U(1)
U(1)$\otimes\Pi_2$	none	$c_{2\beta}=0$	U(1)$\otimes\Pi_2$	Π_2

Part I of the classification of symmetries of the 2HDM scalar potential that yield exact Higgs alignment. Note that $m_{11}^2=m_{22}^2$ and $\mathrm{Re}(m_{12}^2e^{i\xi})=\mathrm{Im}(m_{12}^2e^{i\xi})=0$ unless otherwise indicated. All such basis choices are consistent with the ERPS4 with $\lambda_6=\lambda_7=0$ and real λ_5. In cases where the vacuum preserves a U(1) symmetry, $m_H=m_A\neq 0$. Since GCP3 is equivalent to $\mathrm{U}(1)\otimes\Pi_2$ when expressed in a different scalar field basis, there is a one-to-one mapping between the corresponding entries in this Table and the one that follows.

Symmetry	soft-breaking	parameter constraints	residual unbroken symmetry of scalar potential	residual unbroken symmetry of vacuum
GCP3	$m'^2_{11} \neq m'^2_{22}$, $\text{Re}m'^2_{12} \neq 0$	$s_{2\beta'}c_{2\beta'} \neq 0$, $\sin\xi' = 0$	$\overline{\Pi}_2^{(\alpha)}$	$\overline{\Pi}_2^{(\alpha)}$
GCP3	$m'^2_{11} \neq m'^2_{22}$	$s_{2\beta'} = 0$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
GCP3	$\text{Re}m'^2_{12} \neq 0$	$c_{2\beta'} = 0$, $\sin\xi' = 0$	Π_2	Π_2
GCP3	$\text{Im}m'^2_{12} \neq 0$	$c_{2\beta'} = 0$, $\cos\xi' = 0$	U(1)$'$	U(1)$'$
GCP3	none	$s_{2\beta'} = 0$	U(1)$' \otimes \mathbb{Z}_2$	$\mathbb{Z}_2$
GCP3	none	$s_{2\beta'} \neq 0$, $\sin\xi' = 0$	U(1)$' \otimes \mathbb{Z}_2$	$\overline{\Pi}_2^{(\alpha)}$
GCP3	none	$c_{2\beta'} = 0$, $\cos\xi' = 0$	U(1)$' \otimes \mathbb{Z}_2$	U(1)$'$
SO(3)	$m'^2_{11} \neq m'^2_{22}$, $\text{Re}(m'^2_{12}e^{i\xi'}) \neq 0$	$s_{2\beta'}c_{2\beta'} \neq 0$	U(1)$_{\rm H}$	U(1)$_{\rm H}$
SO(3)	$\text{Re}(m'^2_{12}e^{i\xi'}) \neq 0$	$c_{2\beta'} = 0$	U(1)$_{\rm H}$	U(1)$_{\rm H}$
SO(3)	$m'^2_{11} \neq m'^2_{22}$	$s_{2\beta'} = 0$	U(1)	U(1)
SO(3)	none	none	SO(3)	U(1)$_{\rm H}$

Part II of the classification of symmetries of the 2HDM scalar potential that yield exact Higgs alignment. Note that $m'^2_{11} = m'^2_{22}$ and $\text{Re}(m'^2_{12}e^{i\xi'}) = \text{Im}(m'^2_{12}e^{i\xi'}) = 0$ unless otherwise indicated, where the primed parameters correspond to the GCP3 scalar field basis. All such basis choices are consistent with the ERPS4 with $\lambda'_6 = \lambda'_7 = 0$ and real λ'_5. The symmetry group U(1)$_{\rm H}$ refers to a Peccei-Quinn U(1) symmetry that is manifestly realized in the Higgs basis. In cases where the vacuum preserves a U(1) symmetry, $m_H = m_A \neq 0$ (with the exception of the unbroken SO(3)-symmetric scalar potential where both H and A are massless).

What is natural Higgs alignment?

In P.S. Bhupal Dev and A. Pilaftsis, JHEP **1412**, 024 (2014), "natural" alignment is defined as Higgs alignment due to a symmetry that is independent of $\tan\beta$. That is, the solution to $Y_3 = Z_6 = 0$ should not depend on $\tan\beta$,

$$Z_6 = \Big\{ -\tfrac{1}{2}s_{2\beta}\left[\lambda_1 c_\beta^2 - \lambda_2 s_\beta^2 - \lambda_{345}c_{2\beta} - i\mathrm{Im}(\lambda_5 e^{2i\xi})\right] + c_\beta c_{3\beta}\mathrm{Re}(\lambda_6 e^{i\xi}) + s_\beta s_{3\beta}\mathrm{Re}(\lambda_7 e^{i\xi}) + ic_\beta^2\mathrm{Im}(\lambda_6 e^{i\xi}) + is_\beta^2\mathrm{Im}(\lambda_7 e^{i\xi})\Big\} e^{-i\xi}\,,$$

where $\lambda_{345} \equiv \lambda_3 + \lambda_4 + \mathrm{Re}(\lambda_5 e^{2i\xi})$.

Application: GCP3-conserving scalar potential with $m_{11}^2 = m_{22}^2$, $m_{12}^2 = 0$, $\lambda_1 = \lambda_2 = \lambda_3 + \lambda_4 + \mathrm{Re}\lambda_5$, and $\mathrm{Im}\lambda_5 = \lambda_6 = \lambda_7 = 0$, produces

$$Z_6 = i\lambda_5 s_{2\beta}\sin\xi e^{-i\xi}\big(\cos\xi + ic_{2\beta}\sin\xi\big)\,.$$

The potential minimum conditions yield $s_{2\beta}\sin 2\xi = c_{2\beta}\sin^2\xi = 0$, which implies that $\sin\xi = 0$ for arbitrary β or $\cos\xi = 0$ for $\beta = \frac{1}{4}\pi$. Bhupal Dev and Pilaftsis assumed that $\sin\xi = 0$, implying $Z_6 = 0$ independently of $\tan\beta$.

Transforming to the U(1)$\otimes\Pi_2$ basis, the corresponding minimum conditions of the scalar potential yield $s_{2\beta}c_{2\beta} = 0 \iff \beta = 0$, $\frac{1}{4}\pi$ or $\frac{1}{2}\pi$. For these values, $Y_3 = Z_6 = 0$. Is this still an example of "natural" alignment?

Likewise, $Z_6 = 0$ after applying the minimum conditions for the GCP2 and $\mathbb{Z}_2 \otimes \Pi_2$-conserving scalar potentials, which are not viewed by Bhupal Dev and Pilaftsis as examples of "natural" alignment.

I believe that what Bhupal Dev and Pilaftsis really meant by "natural" alignment is that $Y_3 = Z_6 = 0$ *independently of the scalar potential minimum conditions*. With this definition, neither the GCP2 nor GCP3-conserving scalar potentials exhibit natural alignment. Only the SO(3)-conserving scalar potential (i.e., GCP3 with $\lambda_5 = 0$) would qualify.

I prefer the term "natural alignment" to imply Higgs alignment as a consequence of a symmetry (which may be softly broken). That is, naturalness in the sense of 't Hooft, where a symmetry is enlarged when a parameter is set to zero.

Deviations from exact Higgs alignment in the ERPS4

Scalar potentials with a softly-broken $\mathbb{Z}_2 \otimes \Pi_2$ symmetry

β	$\sin 2\xi$	$m_{11}^2,\, m_{22}^2$	m_{12}^2	CP-violation?	comment
$s_{2\beta} \neq 0$	$\neq 0$	$m_{11}^2 \neq m_{22}^2$	complex	explicit	$\mathrm{Im}\left[m_{12}^2\right]^2 \neq 0$
$s_{2\beta} \neq 0$	$\neq 0$	$m_{11}^2 \neq m_{22}^2$	$\mathrm{Im}\left[m_{12}^2\right]^2 = 0$	spontaneous	$0 < \lvert m_{12}^2 \rvert < \frac{1}{2}\lambda_5 v^2 s_{2\beta}$
$s_{2\beta} \neq 0$	$\neq 0$	$m_{11}^2 \neq m_{22}^2$	$\mathrm{Im}\left[m_{12}^2\right]^2 = 0$	no	$\lvert m_{12}^2 \rvert > \frac{1}{2}\lambda_5 v^2 s_{2\beta}$
$c_{2\beta} = 0$	$\neq 0$	$m_{11}^2 = m_{22}^2$	complex	no	$m_{12}^2 \neq 0$
$s_{2\beta} c_{2\beta} \neq 0$	0	$m_{11}^2 \neq m_{22}^2$	$\mathrm{Im}\left[m_{12}^2\right]^2 = 0$	no	

Scalar potentials with a softly-broken U(1)$\otimes\Pi_2$ symmetry (e.g., tree-level MSSM Higgs sector)

β	$m_{11}^2,\, m_{22}^2$	$m_{12}^2 e^{i\xi}$	R	comment
$s_{2\beta} c_{2\beta} \neq 0$	$m_{11}^2 \neq m_{22}^2$	> 0	$R \neq 1$	
$s_{2\beta} c_{2\beta} \neq 0$	$m_{11}^2 \neq m_{22}^2$	0	$\lvert R \rvert < 1$	$m_A^2 = 0$

$R \equiv (\lambda_3 + \lambda_4)/\lambda_1$ and $\lambda_5 = 0$.

Scalar potentials with a softly-broken GCP3 symmetry

β'	ξ'	$m_{11}^{\prime 2},\, m_{22}^{\prime 2}$	$m_{12}^{\prime 2} e^{i\xi'}$
$s_{2\beta'} c_{2\beta'} \neq 0$	$\sin 2\xi' \neq 0$	$m_{11}^{\prime 2} \neq m_{22}^{\prime 2}$	complex
$s_{2\beta'} c_{2\beta'} \neq 0$	$\cos\xi' = 0$	$m_{11}^{\prime 2} \neq m_{22}^{\prime 2}$	real
$c_{2\beta'} = 0$	$\sin 2\xi' \neq 0$	$m_{11}^{\prime 2} = m_{22}^{\prime 2}$	real ($\neq 0$)

Primed parameters refer to the GCP3 basis.

Custodial-symmetric scalar potential with exact Higgs alignment

references: H.E. Haber and D. O'Neil, Phys. Rev. D **83**, 055017 (2011); A. Pilaftsis, Phys. Lett. B **706**, 465 (2012); M. Aiko and S. Kanemura, JHEP **02**, 046 (2021).

The scalar potential respects the custodial symmetry if the Higgs basis parameters satisfy,

$$Z_4 = Z_5 e^{-2i\eta} \in \mathbb{R}\,, \qquad Y_3 e^{-i\eta} = -\tfrac{1}{2} Z_6 e^{-i\eta} v^2 \in \mathbb{R}\,, \qquad Z_7 e^{-i\eta} \in \mathbb{R}\,.$$

Hence, one can choose η such that the parameters of the scalar potential in the Higgs basis are all real. In particular, in a real Higgs basis, either

$$Z_4 = Z_5 \qquad (\text{if } \eta = 0)\,,$$

or

$$Z_4 = \pm |Z_5|\,, \quad \text{and} \quad Y_3 = Z_6 = Z_7 = 0 \qquad (\text{if } \eta = \tfrac{1}{2}\pi)\,.$$

The custodial-symmetric potential is CP-conserving. Outside of the inert limit, $Z_4 = Z_5$ implies that $m_{H^\pm} = m_A$, where A is a CP-odd neutral scalar.

Exact Higgs alignment via a symmetry implies the inert limit where the sign of Z_5 is unphysical.[10] In the custodial limit, $H^\pm$ is degenerate in mass with either H or A. Although H and A are *relatively* CP-odd, the individual CP quantum numbers of H and A are not fixed by the bosonic sector.

The CP quantum numbers of H and A *may* be resolved by the Yukawa couplings,[11] in which case,

$$m_{H^\pm} = \begin{cases} m_A & \text{if } Z_4 = Z_5 \quad \text{and} \quad Z_6 = Z_7 = 0\,, \\ m_H & \text{if } Z_4 = -Z_5 \quad \text{and} \quad Z_6 = Z_7 = 0\,, \end{cases}$$

in a real Higgs basis. Indeed, the transformation $\mathcal{H}_2 \to i\mathcal{H}_2$ changes the sign of Z_5 while also changing the scalar Yukawa coupling into a pseudoscalar Yukawa coupling and vice versa.

[10] Note that in the inert limit, $\mathcal{H}_2 \to i\mathcal{H}_2$ maintains the reality of the Higgs basis.

[11] In the IDM, the individual CP quantum numbers of H and A remain undefined.

A complete classification of custodial-symmetric 2HDM scalar potentials with Higgs alignment due to a symmetry has been obtained.

Higgs basis conditions (all cases satisfy $Y_3 = Z_6 = Z_7 = 0$)	custodial symmetry conditions	additional real Φ-basis constraints	scalar Lagrangian symmetry
	$Z_4 = \pm Z_5 \neq 0$	$s_{2\beta} = 0$	$\mathbb{Z}_2$
	$Z_4 = Z_5 = 0$	$s_{2\beta} = 0$	U(1)
$Z_1 = Z_2 \neq Z_{345}$	$Z_4 = \pm Z_5 \neq 0$	$c_{2\beta} \sin 2\xi = 0$, $\lambda = \lambda_3$ or $\lambda_4 = \pm\lambda_5$	$\mathbb{Z}_2 \otimes \Pi_2$
$Z_1 = Z_2 \neq Z_{345}$	$Z_4 = \pm Z_5 \neq 0$	$s_{2\beta} = 0$, $\lambda_4 = \pm\lambda_5$	$\mathbb{Z}_2 \otimes \Pi_2$
$Z_1 = Z_2 = Z_3 + 2Z_4$	$Z_4 = \pm Z_5 \neq 0$	$c_{2\beta} = 0$, $\lambda = \lambda_3$, $\lambda_4 \neq 0$	$\mathrm{U}(1) \otimes \Pi_2$
$Z_1 = Z_2 = Z_3$	$Z_4 = \pm Z_5 \neq 0$	$c_{2\beta} = 0$, $\lambda \neq \lambda_3$, $\lambda_4 = 0$	$\mathrm{U}(1) \otimes \Pi_2$
$Z_1 = Z_2 \neq Z_3$	$Z_4 = Z_5 = 0$	$s_{2\beta} = 0$, $\lambda \neq \lambda_3$, $\lambda_4 = 0$	$\mathrm{U}(1) \otimes \Pi_2$
$Z_1 = Z_2 = Z_3 + 2Z_4$	$Z_4 = Z_5 \neq 0$	$s_{2\beta'} \sin \xi' = 0$, $\lambda'_4 = \lambda'_5 \neq 0$	GCP3
$Z_1 = Z_2 = Z_3$	$Z_4 = -Z_5 \neq 0$	$s_{2\beta'} \sin \xi' = 0$, $\lambda'_4 = -\lambda'_5 \neq 0$	GCP3
$Z_1 = Z_2 \neq Z_3$	$Z_4 = Z_5 = 0$	$c_{2\beta'} = \cos \xi' = 0$, $\lambda'_4 = -\lambda'_5 \neq 0$	GCP3
$Z_1 = Z_2 = Z_3$	$Z_4 = Z_5 = 0$	$\lambda_4 = \lambda_5 = 0$	$\mathrm{SO}(3)$

Classification of 2HDM scalar potentials that possess an unbroken custodial symmetry and satisfy the inert conditions, $Y_3 = Z_6 = Z_7 = 0$, thereby exhibiting exact Higgs alignment. The Higgs basis field $\mathcal{H}_2$ has been rephased such that Z_5 is real. In the symmetry limit, the scalar Lagrangian symmetry that is manifestly realized in the Φ-basis is shown. Excluding the first two lines of the table, all entries correspond to the ERPS4 regime. The corresponding ERPS symmetry may be softly-broken if $m_{11}^2 \neq m_{22}^2$ and/or $m_{12}^2 \neq 0$. The primed parameters correspond to the GCP3 basis. Since GCP3 is equivalent to $\mathrm{U}(1) \otimes \Pi_2$ when expressed in a different scalar field basis, there is a one-to-one mapping between their corresponding entries.

What about the Yukawa couplings?

Unfortunately, none of the Higgs family and GCP symmetries of the ERPS can be extended to the Yukawa interactions without generating a massless quark or some other phenomenologically untenable feature.[12] That is, the Yukawa couplings constitute a hard breaking of the ERPS symmetries.

There are two two options:

1. Treat the symmetry conditions as being implemented at a very high energy scale (e.g. the Planck scale) by some unknown UV physics. Use RG-evolution to determine the deviation of the parameters at the electroweak scale from their ERPS symmetry values. Check if the violations of the alignment limit and the custodial limit are consistent with experimental constraints.

P.S. Bhupal Dev and A. Pilaftsis, JHEP **12**, 024 (2014); N. Darvishi and A. Pilaftsis, Phys. Rev. D **99**, 115014 (2019); M. Aiko and S. Kanemura, JHEP **02**, 046 (2021).

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[12] P.M. Ferreira and J.P. Silva, Eur. Phys. J. C **69**, 45 (2010).

2. Extend the Yukawa Lagrangian to include vectorlike quark and lepton partners. In this case, one can construct a Yukawa Lagrangian that is consistent with the ERPS4 regime.

Example: Consider a U(1)$\otimes\Pi_2$-symmetric scalar potential, where

$$m_{11}^2 = m_{22}^2\,, \quad \lambda \equiv \lambda_1 = \lambda_2\,, \quad m_{12}^2 = \lambda_5 = \lambda_6 = \lambda_7 = 0\,.$$

To extend this symmetry to the Yukawa sector, we introduce vector-like fermions U and $\overline{U}$. SM two-component fermions are denoted by lower case letters (e.g. doublet fields $q = (u, d)$ with $Y = 1/3$ and singlet fields $\bar{u}$ with $Y = -4/3$); vector-like singlet two-component fermions by upper case letters. Note that $Y_{\bar{u}} = Y_{\bar{U}} = -Y_U$. Under the U(1)$\otimes\Pi_2$ symmetry,[13]

symmetry	Φ_1	Φ_2	q	$\bar{u}$	$\overline{U}$	U
Π_2	Φ_2	Φ_1	q	$\overline{U}$	$\bar{u}$	U
U(1)	$e^{-i\theta}\Phi_1$	$e^{i\theta}\Phi_2$	q	$e^{-i\theta}\bar{u}$	$e^{-i\theta}\overline{U}$	$e^{\pm i\theta}U$

[13]Down-type fermions and leptons can also be included by introducing the appropriate vector-like fermions.

The Yukawa couplings consistent with the $\mathrm{U}(1) \otimes \Pi_2$ symmetry and the SU(2)×U(1)$_\mathrm{Y}$ gauge symmetry are

$$\mathscr{L}_{\mathrm{Yuk}} \supset y_t \left(q\Phi_2 \bar{u} + q\Phi_1 \overline{U} \right) + \mathrm{h.c.}$$

The model is not phenomenologically viable due to

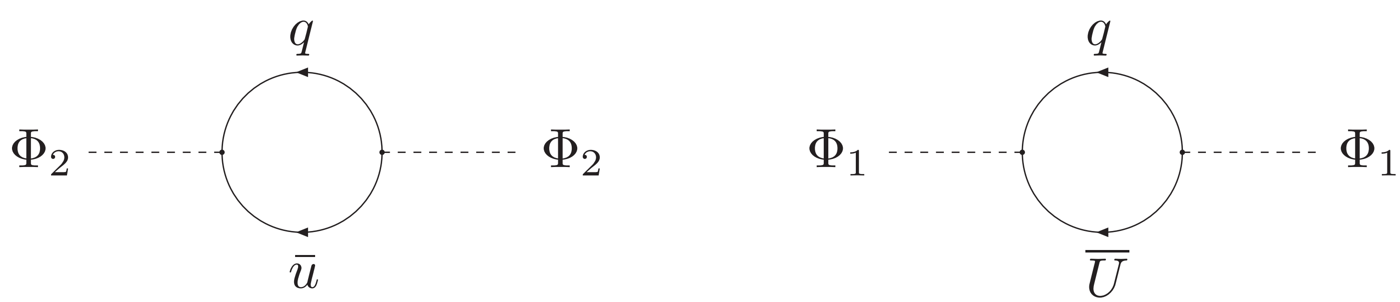
- experimental limits on vector-like fermion masses
- existence of a massless scalar if the global U(1) is spontaneously broken

Thus, we introduce SU(2)×U(1)$_\mathrm{Y}$ preserving mass terms,

$$\mathscr{L}_{\mathrm{mass}} \supset M_U \overline{U} U + M_u \bar{u} U + \mathrm{h.c.}$$

The U(1) symmetry is explicitly broken if $M_U M_u \neq 0$. The Π_2 discrete symmetry is also explicitly broken if $M_U \neq M_u$. The symmetry breaking is soft, so that corrections to the scalar potential squared-mass parameters are protected from quadratic sensitivity to the cutoff scale Λ of the theory.

Effects of the softly-broken symmetries



$$\Delta m^2 \equiv m_{22}^2 - m_{11}^2 \sim \kappa(M_U^2 - M_u^2) - \frac{3y_t^2(M_U^2 - M_u^2)}{4\pi^2} \ln(\Lambda/M) \,,$$

where $M \equiv (M_U^2 + M_u^2)^{1/2}$. The above result includes a finite threshold corrections proportional to κ. Note that when $M_U = M_u$, the Π_2 symmetry is unbroken and hence the relation $m_{11}^2 = m_{22}^2$ is protected. Likewise,

$$m_{12}^2 \sim \kappa_{12} M_U M_u + \frac{3y_t^2 M_U M_u}{4\pi^2} \ln(\Lambda/M) \,,$$

which includes a finite threshold corrections proportional to κ_{12}. In our numerical scans we chose $\ln(\Lambda/M) = 3$ and examined two benchmark points, $\gamma = 0.1$ and $\gamma = 0.3$, where $\tan\gamma \equiv M_u/M_U$.

Regions of approximate alignment without decoupling

In addition to a SM-like Higgs boson (consistent with LHC data), we have also imposed:

- Non-SM Higgs bosons in the parameter regime of Higgs alignment without decoupling should have so far evaded LHC detection.

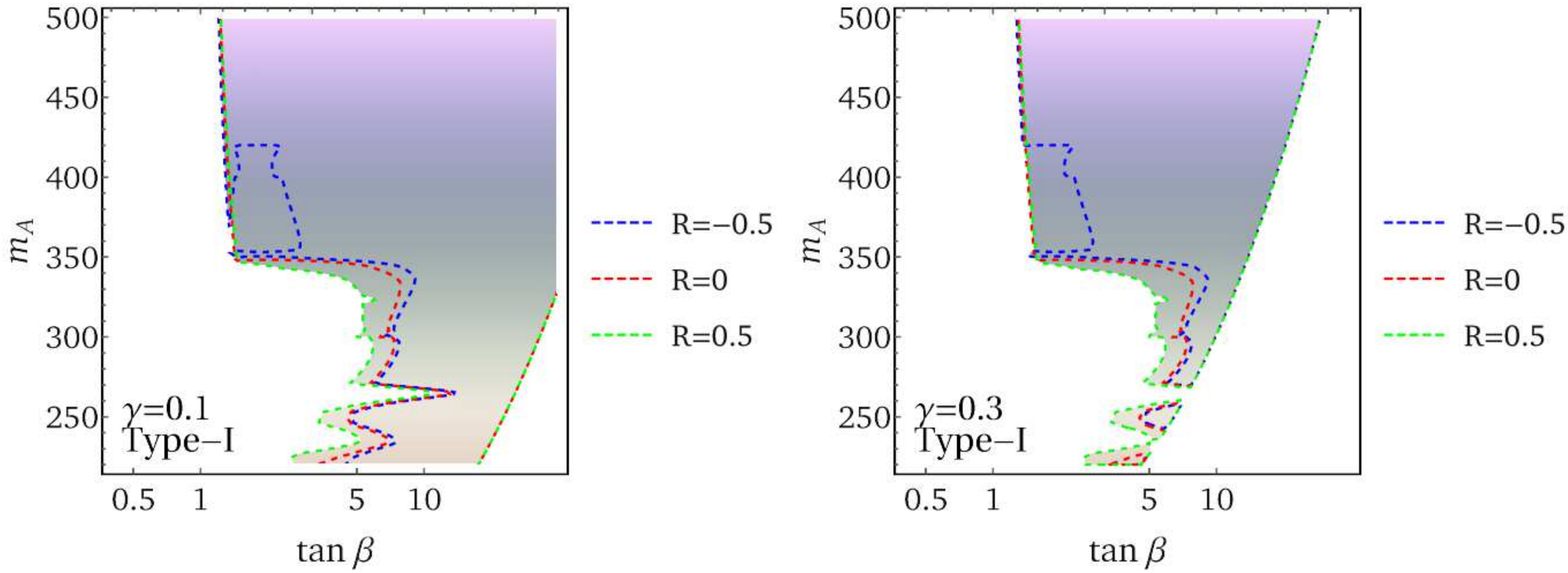
- Constraints on the charged Higgs mass from flavor constraints in the Type-I 2HDM.

- Vectorlike top quark mass bounds [we chose $M_T \gtrsim 1.5$ TeV].

- Constraints on mixing between the top quark and its vectorlike fermion partner (the mixing is governed by the parameters γ, β, m_t and M_T).[14]

- Avoid excessive fine-tuning while keeping small the size of the effects due to the soft breaking of the $\mathrm{U}(1)\otimes\Pi_2$ symmetry.

[14]See, e.g., A. Arhrib et al., Phys. Rev. D 97, 095015 (2018).



Regions allowed by experimental bounds and tuning constraints for different values of $R \equiv (\lambda_3 + \lambda_4)/\lambda$, with an m_{12}^2 and Δm^2 tuning of at most 5% [assuming that $\ln(\Lambda/M) = 3$]. Each panel shows three different R curves; the white regions of the parameter space are ruled out. The ruled out areas expand somewhat as R decreases, with the borders of the allowed shaded regions indicated by the corresponding contours. For $R = -0.5$, the area enclosed by the closed dashed blue contour in panel (a) is also ruled out. Type-I Yukawa couplings are employed and, two choices for γ are shown. Taken from P. Draper, A. Ekstedt and H.E. Haber, arXiv:2011.13159 [hep-ph].

Note: The shrinking of the allowed parameter space as γ increases is due primarily to the behavior of the measure of fine-tuning of the parameter m_{12}^2.

Summary of Part 2

- The Exceptional Region of the 2HDM Parameter Space
  - ERPS: $m_{11}^2 = m_{22}^2$, $m_{12}^2 = 0$, $\lambda_1 = \lambda_2$, $\lambda_7 = -\lambda_6$ (with $\lambda_{5,6,7}$ generically complex), corresponding to a GCP2-symmetric scalar potential.
  - ERPS4: $\lambda_1 = \lambda_2$ and $\lambda_7 = -\lambda_6$, corresponding to a softly-broken GCP2-symmetric scalar potential.

- Exceptional features of the ERPS4
  - constrained C2HDM; however CP violation is not necessarily guaranteed by $\mathrm{Im}\big(\lambda_5^*[m_{12}^2]^2\big) \neq 0$.
  - (softly-broken) symmetry governing the Higgs alignment limit can provide a viable framework for the observed SM-like Higgs boson.
  - implications of imposing an additional custodial symmetry constraint.

- Challenges of the Yukawa sector

Backup slides

Equivalence of $\mathbb{Z}_2 \otimes \Pi_2$ and GCP2 symmetries of the 2HDM

The 2HDM scalar potential in the Φ-basis can be written as,

$$\mathcal{V}(\Phi) = Y_{ab}(\Phi_a^\dagger \Phi_b) + \tfrac{1}{2} Z_{ac,bd}(\Phi_a^\dagger \Phi_b)(\Phi_c^\dagger \Phi_d).$$

Define a three-vector whose components P_B (for $B = 1, 2, 3$) are given by

$$P_B = \tfrac{1}{4}(Z_{ab,cd}+\overline{Z}_{ab,cd})\delta_{ca}\sigma^B_{db} = \Big(\mathrm{Re}(\lambda_6+\lambda_7) \quad -\mathrm{Im}(\lambda_6+\lambda_7) \quad \tfrac{1}{2}(\lambda_1-\lambda_2)\Big)\,,$$

and a 3×3 real symmetric matrix whose elements D_{AB} are given by

$$D_{AB} = \tfrac{1}{4}(Z_{ab,cd}+\overline{Z}_{ab,cd})\sigma^A_{ca}\sigma^B_{db} - \tfrac{1}{12}(Z_{ab,ab}+\overline{Z}_{ab,ab})\delta^{AB}$$

$$= \begin{pmatrix} -\frac{1}{3}\Delta + \mathrm{Re}\,\lambda_5 & -\mathrm{Im}\,\lambda_5 & \mathrm{Re}\,(\lambda_6-\lambda_7) \\ -\mathrm{Im}\,\lambda_5 & -\frac{1}{3}\Delta - \mathrm{Re}\,\lambda_5 & -\mathrm{Im}\,(\lambda_6-\lambda_7) \\ \mathrm{Re}\,(\lambda_6-\lambda_7) & -\mathrm{Im}\,(\lambda_6-\lambda_7) & \frac{2}{3}\Delta \end{pmatrix},$$

where $\Delta \equiv \frac{1}{2}(\lambda_1+\lambda_2) - \lambda_3 - \lambda_4$ and $\overline{Z}_{ab,cd} \equiv Z_{ba,cd} = Z_{ab,dc}$.

Under a change of scalar field basis, $\Phi \to \Phi' = V\Phi$ (where V is unitary),

$$P_B \to P'_B = \mathcal{R}_{BD} P_D \,, \qquad D_{AB} \to D'_{AB} = \mathcal{R}_{AC}\mathcal{R}_{BD} D_{CD} = (\mathcal{R} D \mathcal{R}^{\mathsf{T}})_{AB} \,,$$

after employing the identity $V^\dagger \sigma^A V = \mathcal{R}_{AB}\sigma^B$, where $\mathcal{R}$ is a real orthogonal matrix that is explicitly given by $\mathcal{R}_{AB} = \frac{1}{2}\mathrm{Tr}(V^\dagger \sigma^A V \sigma^B)$.

Theorem: If $\lambda_1 = \lambda_2$ and $\lambda_7 = -\lambda_6$ in the Φ-basis, then there exists a Φ'-basis, defined by $\Phi' = U\Phi$, in which $\lambda'_1 = \lambda'_2$ and $\mathrm{Im}\lambda'_5 = \lambda'_6 = \lambda'_7 = 0$.

If $\lambda_1 = \lambda_2$ and $\lambda_7 = -\lambda_6$ in the Φ-basis [GCP2 symmetry] then it follows that $P = 0$. Moreover, D is a real traceless symmetric matrix, which can always be transformed into a real diagonal matrix via an orthogonal similarity transformation. Thus, there exists a real orthogonal matrix $\mathcal{R}$ such that

$$P' = \mathcal{R}P = 0 \quad \text{and} \quad D' = \mathcal{R} D \mathcal{R}^{\mathsf{T}} \text{ is diagonal.}$$

Noting the explicit forms of P and D previously given, it follows that $\lambda'_1 = \lambda'_2$ and $\mathrm{Im}\lambda'_5 = \lambda'_6 = \lambda'_7 = 0$ in the Φ'-basis [$\mathbb{Z}_2 \otimes \Pi_2$ symmetry].

Translation between the U(1)$\otimes\Pi_2$ basis and the GCP3 basis

Consider the following unitary basis transformation, $\Phi \to \Phi' = V\Phi$, where

$$V = \frac{e^{i\phi}}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\,, \qquad \text{where } e^{i\phi} = \frac{c_\beta + is_\beta e^{-i\xi}}{(1+s_{2\beta}\sin\xi)^{1/2}}.$$

Starting from the U(1)$\otimes\Pi_2$-basis,

$$\lambda' = \lambda_1' = \lambda_2' = \tfrac{1}{2}\lambda(1+R)\,,$$

$$\lambda_3' = \lambda_3 + \tfrac{1}{2}\lambda(1-R)\,,$$

$$\lambda_4' = \lambda_4 + \tfrac{1}{2}\lambda(1-R)\,,$$

$$\lambda_5' = -\tfrac{1}{2}\lambda(1-R)\,,$$

$$\lambda_6' = -\lambda_7' = 0\,,$$

where $R \equiv (\lambda_3+\lambda_4)/\lambda$. In particular, $\lambda_5' = \lambda' - \lambda_3' - \lambda_4'$ is real and $\lambda_6' = \lambda_7' = 0$, corresponding to the GCP3 basis.

The corresponding soft-breaking squared mass parameters are,

$$m'^{\,2}_{11} = \tfrac{1}{2}(m^2_{11} + m^2_{22}) + \mathrm{Im}m^2_{12}\,,$$

$$m'^{\,2}_{22} = \tfrac{1}{2}(m^2_{11} + m^2_{22}) - \mathrm{Im}m^2_{12}\,,$$

$$m'^{\,2}_{12} = \mathrm{Re}m^2_{12} + \tfrac{1}{2}i(m^2_{22} - m^2_{11})\,.$$

The vevs, $v'_1 \equiv vc_{\beta'}$ and $v'_2 \equiv vs_{\beta'}$ are real and positive,

$$c_{\beta'} = \frac{1}{\sqrt{2}}\left(1 + s_{2\beta}\sin\xi\right)^{1/2}, \qquad\qquad s_{\beta'} = \frac{1}{\sqrt{2}}\left(1 - s_{2\beta}\sin\xi\right)^{1/2},$$

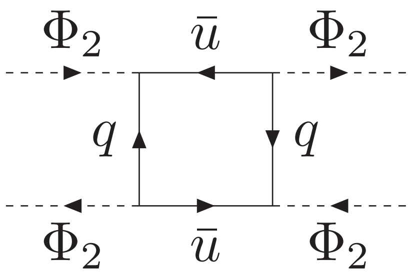
which yields, $s^2_{2\beta'} = 1 - s^2_{2\beta}\sin^2\xi$. Likewise, the relative phase angle, ξ' is given by

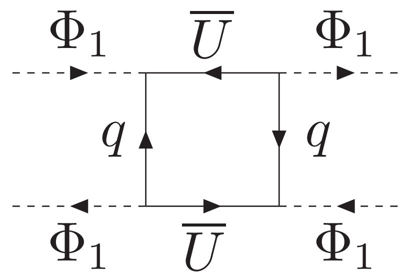
$$\sin\xi' = \frac{-c_{2\beta}}{(1 - s^2_{2\beta}\sin^2\xi)^{1/2}}\,, \qquad\qquad \cos\xi' = \frac{s_{2\beta}\cos\xi}{(1 - s^2_{2\beta}\sin^2\xi)^{1/2}}\,.$$

Finally, if $\beta = \frac{1}{4}\pi$ and $\sin\xi = \pm 1$, then one of the vevs vanishes. It then follows that $s_{2\beta'} = 0$, in which case ξ' is indeterminate if $s_{\beta'} = 0$ and $\xi' = 0$ if $c_{\beta'} = 0$.

Small corrections to the ERPS4 conditions

Integrating out the vector-like fermions below the scale M, one generates a small splitting between λ_1 and λ_2 and nonzero values of $\lambda_{5,6,7}$. For example, above the scale M, the diagrams





contribute equally to $\lambda_2(\Phi_2^\dagger\Phi_2)^2$ and $\lambda_1(\Phi_1^\dagger\Phi_1)^2$, respectively. Below the scale M, the diagrams with internal U lines decouple, which then yields

$$\Delta\lambda \equiv |\lambda_1 - \lambda_2| \sim \frac{3y_t^4}{4\pi^2}\left(\frac{M_U^2 - M_u^2}{M_U^2 + M_u^2}\right)\log(M/m_t) \sim \mathcal{O}(0.1)\,,$$

for $M \sim \mathcal{O}(1~\text{TeV})$. This is a small correction, which in first approximation can be neglected in our analysis. Likewise, explicit breaking of the U(1) symmetry will generate small nonzero values of λ_5, λ_6 and λ_7.

Top quark–vectorlike quark mixing

After electroweak symmetry breaking, the fermion mass eigenstates are obtain by Takagi-diagonalization of the following 4×4 mass matrix.

$$-\mathscr{L}_{\rm mass} = \tfrac{1}{2}(u \quad U \quad \bar{u} \quad \overline{U}) \begin{pmatrix} 0 & 0 & Y s_\beta & Y c_\beta \\ 0 & 0 & M_u & M_U \\ Y s_\beta & M_u & 0 & 0 \\ Y c_\beta & M_U & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ U \\ \bar{u} \\ \overline{U} \end{pmatrix} + {\rm h.c.}\,,$$

where $Y \equiv y_t v/\sqrt{2}$. States with the same electric charge, i.e. $\{u, U\}$ and $\{\bar{u}, \overline{U}\}$, can separately mix (with mixing angles θ_L and θ_R, respectively). This yields two Dirac fermions—the top quark t and its vector-like top partner T, with corresponding masses and mixing angles (assuming $m_t \ll M_T$),

$$m_t \simeq Y|s_{\beta-\gamma}|\left(1 - \frac{Y}{M}c_{\beta-\gamma}\right)\,, \qquad M_T \simeq M\left[1 + \frac{m_t^2}{2M^2}\cot^2(\beta-\gamma)\right]\,,$$

$$\theta_L \simeq \frac{m_t}{M_T}|\cot(\beta-\gamma)|\,, \qquad \theta_R \simeq \gamma + \frac{m_t^2}{M_T^2}\cot(\beta-\gamma)\,.$$

The Higgs sector of the softly-broken U(1)⊗Π_2-symmetric 2HDM

The important parameters of the scalar potential are:

$$m^2 \equiv \tfrac{1}{2}(m_{11}^2 + m_{22}^2)\,, \qquad \Delta m^2 \equiv m_{22}^2 - m_{11}^2\,, \qquad R \equiv \frac{\lambda_3+\lambda_4}{\lambda}\,, \qquad m_{12}^2\,,$$

with $\lambda \equiv \lambda_1 = \lambda_2$ and $\lambda_5 = \lambda_6 = \lambda_7 = 0$. We impose $\lambda > 0$ and $R > -1$ to ensure that the vacuum is bounded from below. Solving for the potential minimum yields,

$$2m^2 = \bar{m}^2 - \tfrac{1}{2}\lambda v^2(1+R)\,, \qquad \Delta m^2 = \epsilon\left(\bar{m}^2 + \tfrac{1}{2}\lambda v^2(1-R)\right)\,,$$

where $\bar{m}^2 \equiv 2m_{12}^2/\sin 2\beta$ and

$$\tan\beta \equiv \frac{v_2}{v_1} = \sqrt{\frac{1-\epsilon}{1+\epsilon}}\,, \qquad \text{where} \quad \epsilon \equiv \cos 2\beta\,.$$

The positivity of v_1^2 and v_2^2 requires $|\epsilon| < 1$.

Approximate alignment without decoupling

The relevant Higgs basis parameters are given by,

$$Z_1 = \tfrac{1}{2}\lambda\big[1 + R + \epsilon^2(1-R)\big]\,,$$

$$m_A^2 + Z_5 v^2 = 2m^2 + \lambda v^2\big[1 - \tfrac{1}{2}\epsilon^2(1-R)\big]\,,$$

$$Z_6 = \tfrac{1}{2}\lambda(R-1)\epsilon\sqrt{1-\epsilon^2}\,,$$

Approximate alignment without decoupling requires that $|Z_6| \ll 1$ and $m^2 \sim \mathcal{O}(v^2)$. To avoid $\tan\beta$ very large or very small, we consider two limiting cases: $|\epsilon| \ll 1$ and $|R-1| \ll 1$.

In the limit of $|\epsilon| \ll 1$,

$$m_h^2 = \tfrac{1}{2}\lambda v^2(1+R)\,, \qquad m_H^2 = 2m^2 + \lambda v^2\,, \qquad c_{\beta-\alpha} = \frac{\lambda v^2(1-R)\epsilon}{4m^2 + \lambda v^2(1-R)}\,.$$

In the limit of $|R-1| \ll 1$,

$$m_h^2 = \lambda v^2\,, \qquad m_H^2 = 2m^2 + \lambda v^2\,, \qquad c_{\beta-\alpha} = \frac{\lambda v^2(1-R)\epsilon\sqrt{1-\epsilon^2}}{4m^2}\,.$$