Electron wave function and mass renormalization in QED

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Abstract

In these notes, we compute the renormalized 1PI two-point Green function for electrons in QED at one loop order. Renormalization is carried out in the modified minimal subtraction scheme ($\overline{\text{MS}}$) and the on-shell (OS) schemes, in a general covariant gauge using dimensional regularization. The wave function renormalization and mass renormalization constants, Z_2 and Z_m are explicitly evaluated. Special attention is given to the dependence on the gauge parameter. In the OS scheme, Z_2 exhibits an infrared divergence for (almost) all possible values of the gauge parameter, with one exception (corresponding to the Yennie gauge).

I. Introduction

The bare QED Lagrangian is given by

$$\mathscr{L}_{\rm QED} = -\tfrac{1}{4}F_B^{\mu\nu}F_{B\mu\nu} + \overline{\psi}_B(i\partial\!\!\!/ + e_B A\!\!\!/)\psi_B - m\overline{\psi}_B\psi_B - \frac{1}{2\xi_B}(\partial_\mu A_B^\mu)^2\,, \tag{1}$$

where the subscript B refers to bare parameters and fields. Introducing the renormalization constants to relate bare quantities to renormalized quantities (the latter with the B subscript removed), the renormalized parameters are defined via

$$e = Z_e^{-1}\mu^{-\epsilon}e_B\,, \qquad m = Z_m^{-1}m_B\,, \qquad \xi = Z_\xi^{-1}\xi_B\,, \tag{2}$$

where $\epsilon \equiv 2 - \frac{1}{2}n$ appears so that the renormalized coupling e is dimensionless when one-loop integrals are evaluated in n dimensions using dimensional regularization. Likewise, the renormalized fields are defined via

$$\psi = Z_2^{-1/2}\psi_B\,, \qquad A^\mu = Z_3^{-1/2}A_B^\mu\,. \tag{3}$$

It is traditional to introduce the vertex renormalization constant via

$$e\,\overline{\psi}A\!\!\!/\psi = \mu^{-\epsilon}Z_1^{-1}e_B\overline{\psi}_B A\!\!\!/_B\psi_B\,,$$

in which case we identify $Z_e = Z_1 Z_2^{-1}Z_3^{-1/2}$. One can also prove that

$$\frac{1}{2\xi}(\partial_\mu A^\mu)^2 = \frac{1}{2\xi_B}(\partial_\mu A_B^\mu)^2\,, \tag{4}$$

as a consequence of the Ward identities, which implies that $Z_\xi = Z_3$.

Inserting eqs. (2) and (3) into eq. (1) yields

$$\mathscr{L}_{\rm QED} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \overline{\psi}(i\partial\!\!\!/ + \mu^\epsilon e A\!\!\!/)\psi - m\overline{\psi}\psi - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \mathcal{L}_{\rm CT}\,,$$

where the counterterm Lagrangian is given by

$$\mathcal{L}_{\rm CT} = -(Z_3-1)\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + (Z_2-1)i\overline{\psi}\partial\!\!\!/\psi - (Z_m Z_2 - 1)m\overline{\psi}\psi + (Z_1-1)\mu^\epsilon e\overline{\psi}A\!\!\!/\psi\,.$$

Note that the counterterm Lagrangian does not contain a term proportional to the gauge parameter ξ in light of eq. (4). The counterterm Lagrangian is treated as a perturbation, which introduces additional Feynman rules for QED Green functions.

Working to one-loop order, it is convenient to define

$$\delta Z_i \equiv Z_i - 1\,, \quad (\text{for } i=1,2,3)\,, \qquad \delta Z_m \equiv Z_m - 1\,.$$

At one loop, $\delta Z_i\,,\,\delta Z_m \sim \mathcal{O}(\alpha)$, where $\alpha \equiv e^2/(4\pi)$. Hence we can rewrite the counterterm Lagrangian at one-loop as

$$\mathcal{L}_{\rm CT} = -\tfrac{1}{4}\delta Z_3 F^{\mu\nu}F_{\mu\nu} + i\delta Z_2\overline{\psi}\partial\!\!\!/\psi - (\delta Z_m + \delta Z_2)m\overline{\psi}\psi + \delta Z_1\mu^\epsilon e\overline{\psi}A\!\!\!/\psi\,.$$

II. The 1PI electron two-point function

We now turn to the 1PI electron two-point function in momentum space,

$$i\Gamma^{(2)}(p) = i(p\!\!\!/ - m) - i\Sigma(p)\,, \tag{5}$$

where p is the four-momentum of the electron. Here, we have denoted the sum of the loop contributions to $i\Gamma^{(2)}(p)$ by $-i\Sigma(p)$. At one-loop the two contributing Feynman graphs are

where the cross indicates the contribution of the terms $i\delta Z_2\overline{\psi}\partial\!\!\!/\psi - (\delta Z_m+\delta Z_2)m\overline{\psi}\psi$ of the counterterm Lagrangian. Thus, at one loop,

$$-i\Sigma(p) = (i\mu^\epsilon e)^2\int\frac{d^nq}{(2\pi)^n}\frac{\gamma^\nu(q\!\!\!/ + p\!\!\!/ + m)\gamma^\mu}{q^2\big[(q+p)^2 - m^2\big]}\left(g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\right) + i\delta Z_2 p\!\!\!/ - im(\delta Z_m + \delta Z_2)\,.$$

Using Dirac algebra in $n = 4-2\epsilon$ dimensions,

$$\gamma^\mu(q\!\!\!/ + p\!\!\!/ + m)\gamma_\mu = 2(\epsilon-1)(q\!\!\!/ + p\!\!\!/) + (4-2\epsilon)m\,,$$

$$q\!\!\!/ p\!\!\!/ q\!\!\!/ = (2q\cdot p - p\!\!\!/ q\!\!\!/)q\!\!\!/ = 2q\cdot p\, q\!\!\!/ - q^2 p\!\!\!/\,,$$

it follow that

$$\int \frac{d^nq}{(2\pi)^n} \frac{\gamma^\nu(\not{q}+\not{p}+m)\gamma^\mu}{q^2\big[(q+p)^2-m^2\big]} \left(g_{\mu\nu}-(1-\xi)\frac{q_\mu q_\nu}{q^2}\right)$$

$$= \int \frac{d^nq}{(2\pi)^n} \frac{2(\epsilon-1)(\not{q}+\not{p})+(4-2\epsilon)m}{q^2\big[(q+p)^2-m^2\big]} - (1-\xi)\int \frac{d^nq}{(2\pi)^n} \frac{q^2(\not{q}-\not{p}+m)+2q\cdot p\,\not{q}}{q^4\big[(q+p)^2-m^2\big]} .$$

$$= \int \frac{d^nq}{(2\pi)^n} \frac{2(\epsilon-1)(\not{q}+\not{p})+(4-2\epsilon)m-(1-\xi)(\not{q}-\not{p}+m)}{q^2\big[(q+p)^2-m^2\big]}$$

$$-2(1-\xi)p_\mu\gamma_\nu \int \frac{d^nq}{(2\pi)^n} \frac{q^\mu q^\nu}{q^4\big[(q+p)^2-m^2\big]} . \tag{6}$$

Introducing Feynman parameters,

$$\int \frac{d^nq}{(2\pi)^n} \frac{1}{q^2\big[(q+p)^2-m^2\big]} = \int_0^1 dx \int \frac{d^nq}{(2\pi)^n} \frac{1}{\big[q^2+2q\cdot p\,x+x(p^2-m^2)\big]^2}$$

$$= i(4\pi)^{\epsilon-2}\Gamma(\epsilon)\int_0^1 dx\, x^{-\epsilon}\big[m^2-p^2(1-x)\big]^{-\epsilon}, \tag{7}$$

$$\int \frac{d^nq}{(2\pi)^n} \frac{q^\mu}{q^2\big[(q+p)^2-m^2\big]} = \int_0^1 dx \int \frac{d^nq}{(2\pi)^n} \frac{q^\mu}{\big[q^2+2q\cdot p\,x+x(p^2-m^2)\big]^2}$$

$$= -i(4\pi)^{\epsilon-2}\Gamma(\epsilon)p^\mu \int_0^1 dx\, x^{1-\epsilon}\big[m^2-p^2(1-x)\big]^{-\epsilon},$$

$$\int \frac{d^nq}{(2\pi)^n} \frac{q^\mu q^\nu}{q^4\big[(q+p)^2-m^2\big]} = 2\int_0^1 (1-x)\,dx \int \frac{d^nq}{(2\pi)^n} \frac{q^\mu q^\nu}{\big[q^2+2q\cdot p\,x+x(p^2-m^2)\big]^3}$$

$$= -i(4\pi)^{\epsilon-2}\Gamma(\epsilon)\int_0^1 dx\,(1-x)x^{-1-\epsilon}\big[m^2-p^2(1-x)\big]^{-1-\epsilon}\Big\{\epsilon x^2p^\mu p^\nu - \tfrac{1}{2}g^{\mu\nu}x\big[m^2-p^2(1-x)\big]\Big\}.$$

It then follows that

$$\int \frac{d^nq}{(2\pi)^n} \frac{\gamma^\nu(\not{q}+\not{p}+m)\gamma^\mu}{q^2\big[(q+p)^2-m^2\big]} \left(g_{\mu\nu}-(1-\xi)\frac{q_\mu q_\nu}{q^2}\right)$$

$$= i(4\pi)^{\epsilon-2}\Gamma(\epsilon)\Bigg\{\Big[(3+\xi-2\epsilon)m-(1+\xi-2\epsilon)\not{p}\Big]\int_0^1 dx\, x^{-\epsilon}\big[m^2-p^2(1-x)\big]^{-\epsilon}$$

$$+(3-\xi-2\epsilon)\not{p}\int_0^1 dx\, x^{1-\epsilon}\big[m^2-p^2(1-x)\big]^{-\epsilon}$$

$$+2(1-\xi)\not{p}\int_0^1 dx\,(1-x)x^{-1-\epsilon}\big[m^2-p^2(1-x)\big]^{-1-\epsilon}\Big(\epsilon x^2p^2-\tfrac{1}{2}x\big[m^2-p^2(1-x)\big]\Big)\Bigg\}.$$

Hence, we end up with

$$\Sigma(p) = -\not{p}A(p^2) + mB(p^2)\,, \tag{8}$$

where

$$A(p^2) = \delta Z_2 + \frac{\alpha}{2\pi}(4\pi)^\epsilon\,\Gamma(\epsilon)\int_0^1 dx\,x^{-\epsilon}\left[\frac{m^2-p^2(1-x)}{\mu^2}\right]^{-\epsilon}$$
$$\times\left\{(1-x)(1-\epsilon) - x(1-\xi) - \frac{\epsilon(1-\xi)x(1-x)p^2}{m^2-p^2(1-x)}\right\} \tag{9}$$

$$B(p^2) = \delta Z_m + \delta Z_2 + \frac{\alpha}{2\pi}(4\pi)^\epsilon\,\Gamma(\epsilon)\left[\tfrac{1}{2}(3+\xi) - \epsilon\right]\int_0^1 dx\,x^{-\epsilon}\left[\frac{m^2-p^2(1-x)}{\mu^2}\right]^{-\epsilon}\,, \tag{10}$$

after putting $e^2 = 4\pi\alpha$.

We recognize the ultraviolet divergence due to the presence of $\Gamma(\epsilon)$. But, if we attempt to take the on-shell limit, $p^2 = 0$, we obtain an indeterminant quantity in the $\epsilon \to 0$ limit.

III. The limit of a zero mass electron

The limit of $m = 0$ is a subtle one. In this case, $\Sigma(p) = -\not{p}\big[\delta Z_2 + A_0(p^2)\big]$, where

$$A_0(p^2) = \frac{\alpha}{2\pi}(4\pi)^\epsilon\,(1-\epsilon)\,\Gamma(\epsilon)\left(-\frac{p^2}{\mu^2}\right)^{-\epsilon}\int_0^1 x^{-\epsilon}(1-x)^{-\epsilon}(1-2x+\xi x)\,dx\,, \tag{11}$$

after employing the result of eq. (9). Noting the symmetry of the integrand under $x \to 1-x$, it follows that

$$\begin{aligned} A_0(p^2) &= \frac{\alpha\,\xi}{2\pi}(4\pi)^\epsilon\,(1-\epsilon)\,\Gamma(\epsilon)\left(-\frac{p^2}{\mu^2}\right)^{-\epsilon}\int_0^1 x^{1-\epsilon}(1-x)^{-\epsilon}\,dx \\ &= \frac{\alpha\,\xi}{2\pi}(4\pi)^\epsilon\,\frac{\Gamma(\epsilon)\,\Gamma^2(2-\epsilon)}{\Gamma(3-2\epsilon)}\left(-\frac{p^2}{\mu^2}\right)^{-\epsilon}\,. \end{aligned} \tag{12}$$

Recalling that $\epsilon = 2 - \frac{1}{2}n$, where n is the number of spacetime dimensions, we recognize the ultraviolet divergence due to the presence of $\Gamma(\epsilon)$. In particular, one must assume that $n < 4$ or equivalently $\epsilon > 0$, prior to the analytic continuation to $\epsilon = 0$. But, if we attempt to take the on-shell limit, $p^2 = 0$, we obtain an indeterminate quantity in the $\epsilon \to 0$ limit. One suggested strategy for dealing with this ambiguity is outlined on pp. 118–119 of Ref. [1].

However, a more direct approach can be adopted by setting $p^2 = m^2 = 0$ in eq. (6). It then follows that,

$$\begin{aligned} &\int \frac{d^nq}{(2\pi)^n}\frac{\gamma^\nu(\not{q}+\not{p})\gamma^\mu}{q^2(q+p)^2}\left(g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\right) \\ &\quad = \int \frac{d^nq}{(2\pi)^n}\frac{2(\epsilon-1)(\not{q}+\not{p}) - (1-\xi)(\not{q}-\not{p})}{q^2(q+p)^2} - 2(1-\xi)p_\mu\gamma_\nu\int\frac{d^nq}{(2\pi)^n}\frac{q^\mu q^\nu}{q^4(q+p)^2}\,. \end{aligned} \tag{13}$$

If we write $(q+p)^2 = q^2 + 2q\cdot p$, then

$$\frac{1}{q^2(q^2-2q\cdot p)} = \int_0^1 \frac{dx}{\left[q^2+2xq\cdot p\right]^2}\,, \qquad\qquad \frac{1}{q^4(q^2-2q\cdot p)} = 2\int_0^1 \frac{(1-x)\,dx}{\left[q^2+2xq\cdot p\right]^3}\,. \tag{14}$$

Defining a new integration variable, $Q = q + xp$, it follows that $Q^2 = q^2 + 2xq\cdot p$ and

$$\begin{aligned}\int \frac{d^nq}{(2\pi)^n}\frac{2(\epsilon-1)(\not{q}+\not{p})-(1-\xi)(\not{q}-\not{p})}{q^2(q+p)^2} &= \not{p}\int_0^1 \left[2(\epsilon-1)(1-x)+(1-\xi)(1+x)\right]dx \int \frac{d^nQ}{(2\pi)^n}\frac{1}{Q^4} \\ &= \not{p}\left[\tfrac{1}{2}+\epsilon-\tfrac{3}{2}\xi\right]\int \frac{d^nQ}{(2\pi)^n}\frac{1}{Q^4}\,. \end{aligned}\tag{15}$$

and

$$p_\mu\gamma_\nu \int \frac{d^nq}{(2\pi)^n}\frac{q^\mu q^\nu}{q^4(q+p)^2} = 2p_\mu\gamma_\nu\int_0^1 (1-x)\,dx\int \frac{d^nQ}{(2\pi)^n}\frac{Q^\mu Q^\nu}{Q^6} = \frac{1}{n}\not{p}\int\frac{d^nQ}{(2\pi)^n}\frac{1}{Q^4}\,. \tag{16}$$

Hence, after putting $n = 4 - 2\epsilon$,

$$\int \frac{d^nq}{(2\pi)^n}\frac{\gamma^\nu(\not{q}+\not{p})\gamma^\mu}{q^2(q+p)^2}\left(g_{\mu\nu}-(1-\xi)\frac{q_\mu q_\nu}{q^2}\right) = \not{p}\left[\frac{1}{2}+\epsilon-\frac{3\xi}{2}-\frac{1-\xi}{2-\epsilon}\right]\int\frac{d^nQ}{(2\pi)^n}\frac{1}{Q^4}\,. \tag{17}$$

Strictly speaking, the integral

$$\int \frac{d^nQ}{(2\pi)^n}\frac{1}{Q^4}\,, \tag{18}$$

is undefined for any value of n. In particular, it exhibits an infrared divergence as $Q \to 0$ if $n \leq 4$ and an ultraviolet divergence as $Q \to \infty$ if $n \geq 4$. Following the conventions of dimensional regularization (see, e.g., Ref. [2]), one defines integrals with no explicit scale to be zero,

$$\int \frac{d^nQ}{(2\pi)^n}\frac{1}{Q^p} = 0\,, \tag{19}$$

for any power p. This result implies that $A_0(p^2=0)=0$ due to a cancellation of the would be infrared and ultraviolet divergences.

In order to see this cancellation explicitly, we shall rewrite eq. (18) following eq. (C.22) of Ref. [3],

$$\int \frac{d^nQ}{(2\pi)^n}\frac{1}{Q^4} = \int \frac{d^nQ}{(2\pi)^n}\frac{1}{Q^2(Q^2-m^2)} - \int \frac{d^nQ}{(2\pi)^n}\frac{m^2}{Q^4(Q^2-m^2)}\,. \tag{20}$$

This result clearly exhibits the infrared and ultraviolet divergences, but relegates them to separate integrals. It then follows from eq. (7) that

$$\begin{aligned}\int \frac{d^nQ}{(2\pi)^n}\frac{1}{Q^2(Q^2-m^2)} &= i(4\pi)^{\epsilon-2}\Gamma(\epsilon)(m^2)^{-\epsilon}\int_0^1 x^{-\epsilon}\,dx \\ &= \frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon}-\gamma+\ln(4\pi)+1-\ln m^2\right)+\mathcal{O}(\epsilon)\,, \end{aligned}\tag{21}$$

where $n = 4 - 2\epsilon$, which exhibits an ultraviolet divergence when the limit of $\epsilon \to 0$ is taken.

The second integral on the right-hand side of eq. (20) is infrared divergent. To distinguish this divergence from the ultraviolet divergence in eq. (21), we will write $n = 4 - 2\epsilon'$ in the following computation,

$$\begin{aligned}
\int \frac{d^nQ}{(2\pi)^n} \frac{m^2}{Q^4(Q^2-m^2)} &= 2m^2 \int_0^1 x\,dx \int \frac{d^nq}{(2\pi)^n} \frac{1}{\big[q^2-(1-x)m^2)\big]^3} \\
&= -i(4\pi)^{\epsilon'-2}(m^2)^{-\epsilon'}\,\Gamma(1+\epsilon') \int_0^1 x(1-x)^{-1-\epsilon'}\,dx \\
&= -i(4\pi)^{\epsilon'-2}(m^2)^{-\epsilon'}\,\frac{\Gamma(1+\epsilon')\Gamma(-\epsilon')}{\Gamma(2-\epsilon')} = \frac{i}{(4\pi)^2}\left(\frac{4\pi}{m^2}\right)^{\epsilon'} \frac{\Gamma(1+\epsilon')}{\epsilon'(1-\epsilon')} \\
&= \frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon'} - \gamma + \ln(4\pi) + 1 - \ln m^2\right) + \mathcal{O}(\epsilon')\,. \qquad (22)
\end{aligned}$$

Hence, eq. (20) yields,

$$\int \frac{d^nQ}{(2\pi)^n} \frac{1}{Q^4} = \frac{i}{16\pi^2}\left(\frac{1}{\epsilon} - \frac{1}{\epsilon'}\right) = 0\,, \qquad (23)$$

after using $n = 4 - 2\epsilon = 4 - \epsilon'$, which demonstrates the exact cancellation of the infrared and ultraviolet divergences as asserted below eq. (19).

Of course, the statement that $A_0(p^2 = 0)$ does not imply that the divergence is absent. Indeed, δZ_2 can be unambiguously determined the $\overline{\rm MS}$ scheme where no infrared divergences are present as we will show in Sections IV. In the computation of physical observables at one-loop, the implication of $A(p^2 = 0) = 0$ in massless QED is simply that one can neglect Feynman diagrams that contain self-energy corrections on the outgoing electrons and positron lines. The counterterms on the external legs of the diagram are still present, and will end up being reinterpreted as contributing to the infrared divergence, which will ultimately cancel infrared divergences arising from other Feynman graphs, since physical observables are necessarily infrared safe.

IV. The renormalized 1PI electron two-point function in the $\overline{\rm MS}$ scheme

If we use $\overline{\rm MS}$ subtraction to fix the counterterms, then

$$\begin{aligned}
\delta Z_2^{\overline{\rm MS}} &= -\frac{\alpha}{2\pi}(4\pi)^\epsilon\,\Gamma(\epsilon) \int_0^1 dx\,(1-2x+x\xi) = -\frac{\alpha\,\xi}{4\pi}(4\pi)^\epsilon\,\Gamma(\epsilon)\,, \\
\delta Z_m^{\overline{\rm MS}} + \delta Z_2^{\overline{\rm MS}} &= -\frac{\alpha(3+\xi)}{4\pi}(4\pi)^\epsilon\,\Gamma(\epsilon)\,.
\end{aligned}$$

Hence,

$$\delta Z_2^{\overline{\rm MS}} = -\frac{\alpha\,\xi}{4\pi}(4\pi)^\epsilon\,\Gamma(\epsilon)\,, \qquad \delta Z_m^{\overline{\rm MS}} = -\frac{3\alpha}{4\pi}(4\pi)^\epsilon\,\Gamma(\epsilon)\,, \qquad (24)$$

where

$$(4\pi)^\epsilon\,\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + \ln 4\pi + \mathcal{O}(\epsilon)\,.$$

Note that δZ_2 is gauge dependent, whereas δZ_m is gauge independent.

Inserting eq. (24) back into eqs. (9) and (10) and taking the $\epsilon \to 0$ limit, we obtain

$$A(p^2)_{\overline{\rm MS}} = -\frac{\alpha}{2\pi}\left\{\int_0^1 \left[1 - x + \frac{x(1-x)(1-\xi)p^2}{m^2 - p^2(1-x)}\right] dx \right.$$

$$\left. + \int_0^1 (1 - 2x + x\xi)\left[\ln x + \ln\left(\frac{m^2 - p^2(1-x)}{\mu^2}\right)\right] dx\right\},$$

$$B(p^2)_{\overline{\rm MS}} = -\frac{\alpha}{2\pi}\left\{1 + \tfrac{1}{2}(3+\xi)\int_0^1 \left[\ln x + \ln\left(\frac{m^2 - p^2(1-x)}{\mu^2}\right)\right] dx\right\},$$

The relevant integrals are

$$\int_0^1 \frac{x(1-x)dx}{m^2 - p^2(1-x)} = \frac{2m^2 - p^2}{2p^4} - \frac{m^2}{p^4}\left(1 - \frac{m^2}{p^2}\right)\ln\left(1 - \frac{p^2}{m^2}\right),$$

$$\int_0^1 x^n \ln x\, dx = -\frac{1}{(n+1)^2}, \quad \text{for } n = 0, 1, 2, \ldots,$$

$$\int_0^1 \ln\left(\frac{m^2 - p^2(1-x)}{\mu^2}\right) dx = \ln\left(\frac{m^2 - p^2}{\mu^2}\right) - \frac{m^2}{p^2}\ln\left(1 - \frac{p^2}{m^2}\right) - 1,$$

$$\int_0^1 (1-x)\ln\left(\frac{m^2 - p^2(1-x)}{\mu^2}\right) dx = \frac{1}{2}\ln\left(\frac{m^2 - p^2}{\mu^2}\right) - \frac{m^4}{2p^4}\ln\left(1 - \frac{p^2}{m^2}\right) - \frac{m^2}{2p^2} - \frac{1}{4}.$$

It follows that

$$A(p^2)_{\overline{\rm MS}} = \frac{\alpha\,\xi}{4\pi}\left[1 + \frac{m^2}{p^2} - \ln\left(\frac{m^2 - p^2}{\mu^2}\right) + \frac{m^4}{p^4}\ln\left(1 - \frac{p^2}{m^2}\right)\right],$$

$$B(p^2)_{\overline{\rm MS}} = \frac{\alpha}{2\pi}\left\{2 + \xi - \tfrac{1}{2}(3+\xi)\left[\ln\left(\frac{m^2 - p^2}{\mu^2}\right) - \frac{m^2}{p^2}\ln\left(1 - \frac{p^2}{m^2}\right)\right]\right\}.$$

Note that A and B are finite for $p^2 = m^2$,

$$A(m^2)_{\overline{\rm MS}} = \frac{\alpha\,\xi}{4\pi}\left[2 - \ln\left(\frac{m^2}{\mu^2}\right)\right], \qquad B(m^2)_{\overline{\rm MS}} = \frac{\alpha}{2\pi}\left[2 + \xi - \tfrac{1}{2}(3+\xi)\ln\left(\frac{m^2}{\mu^2}\right)\right]. \quad (25)$$

In light of eqs. (5) and (8), the one-loop correction to the inverse propagator is

$$\Gamma^{(2)}(p)_{\overline{\rm MS}} = \not{p} - m - \Sigma(p)_{\overline{\rm MS}} = \not{p}\big[1 + A(p^2)_{\overline{\rm MS}}\big] - m\big[1 + B(p^2)_{\overline{\rm MS}}\big]$$

$$= \not{p}\left\{1 + \frac{\alpha\,\xi}{4\pi}\left[1 + \frac{m^2}{p^2} - \ln\left(\frac{m^2 - p^2}{\mu^2}\right) + \frac{m^4}{p^4}\ln\left(1 - \frac{p^2}{m^2}\right)\right]\right\}$$

$$-m\left\{1 + \frac{\alpha}{2\pi}\left(2 + \xi - \tfrac{1}{2}(3+\xi)\left[\ln\left(\frac{m^2 - p^2}{\mu^2}\right) - \frac{m^2}{p^2}\ln\left(1 - \frac{p^2}{m^2}\right)\right]\right)\right\}. \quad (26)$$

In eq. (26), $m \equiv m(\mu)$ is the renormalized mass, which differs from the physical pole mass. The definition of the $\overline{\rm MS}$ mass is obtained by setting $\mu = m$. That is, the $\overline{\rm MS}$ mass is defined as $m_R \equiv m(m)$. Thus, we set $\mu = m$ in eq. (26) and obtain,

$$\Gamma^{(2)}(p)_{\overline{\rm MS}} = \not{p}\left\{1 + \frac{\alpha\,\xi}{4\pi}\left[1 + \frac{m_R^2}{p^2} - \left(1 - \frac{m_R^4}{p^4}\right)\ln\left(1 - \frac{p^2}{m_R^2}\right)\right]\right\}$$
$$-m_R\left\{1 + \frac{\alpha}{2\pi}\left[2 + \xi - \tfrac{1}{2}(3+\xi)\left(1 - \frac{m_R^2}{p^2}\right)\ln\left(1 - \frac{p^2}{m_R^2}\right)\right]\right\}. \qquad (27)$$

The physical pole mass, denoted by m_e, corresponds to a zero of the inverse propagator. That is, m_e is defined by the condition

$$\Gamma^{(2)}(p)\Big|_{\not{p}=m_e} = 0\,. \qquad (28)$$

The simplest way to obtain an expression for m_e at one loop accuracy is to rewrite eq. (26) as

$$\Gamma^{(2)}(p)_{\overline{\rm MS}} = \left[1 + A(p^2)_{\overline{\rm MS}}\right]\left[\not{p} - m\left(\frac{1 + B(p^2)_{\overline{\rm MS}}}{1 + A(p^2)_{\overline{\rm MS}}}\right)\right].$$

Since $A(p^2)_{\overline{\rm MS}}$ and $B(p^2)_{\overline{\rm MS}}$ are quantities of $\mathcal{O}(\alpha)$, then to one-loop accuracy,

$$\Gamma^{(2)}(p)_{\overline{\rm MS}} \simeq \left[1 + A(p^2)_{\overline{\rm MS}}\right]\left[\not{p} - m\big(1 + B(p^2)_{\overline{\rm MS}} - A(p^2)_{\overline{\rm MS}}\big)\right].$$

We can then immediately identify

$$m_e = m\left[1 + B(m_e^2)_{\overline{\rm MS}} - A(m_e^2)_{\overline{\rm MS}}\right].$$

At one-loop accuracy, $A(m_e^2) = A(m^2)\big|_{\mu=m}$ and $B(m_e^2) = B(m^2)\big|_{\mu=m}$. Hence, using eq. (25), we end up with

$$m_e = m_R\left(1 + \frac{\alpha}{\pi}\right).$$

Although the quantity $B(p^2)_{\overline{\rm MS}} - A(p^2)_{\overline{\rm MS}}$ is gauge-invariant on-shell, it depends on the gauge parameter ξ off-shell. In particular,

$$B(p^2)_{\overline{\rm MS}} - A(p^2)_{\overline{\rm MS}} = \frac{\alpha}{4\pi}\left\{4 + \xi\left(1 - \frac{m^2}{p^2}\right) - 3\ln\left(\frac{m^2 - p^2}{\mu^2}\right) + \frac{m^2}{p^2}\left[3 + \xi\left(1 - \frac{m^2}{p^2}\right)\right]\ln\left(1 - \frac{p^2}{m^2}\right)\right\}.$$

One can easily check that $B(m^2) - A(m^2) = \alpha/\pi$ as required.

Finally, we note that the $\overline{\rm MS}$ scheme is a mass-independent scheme. In particular, no infrared divergences appear in the evaluation of the 1PI electron two-point function in this scheme.

V. The renormalized 1PI electron two-point function in the on-shell (OS) scheme

Consider the on-shell (OS) renormalization scheme, where we identify the parameter m as the pole mass. In this case, we expand,

$$\Sigma(p) = \Sigma(m) + (\not{p} - m)\Sigma'(m) + \mathcal{O}\big((\not{p} - m)^2\big).$$

The renormalization conditions that ensure that the residue of the propagator is unity and the pole of the propagator is the pole mass m are:

$$\Sigma(m)_{\rm OS} = 0\,, \qquad\qquad \Sigma'(m)_{\rm OS} = 0\,. \tag{29}$$

It then follows that the inverse propagator can be written as

$$\begin{aligned}\Gamma^{(2)}(p)_{\rm OS} = \not p - m - \Sigma(p)_{\rm OS} &= \big[1+\Sigma'(m)_{\rm OS}\big](\not p - m) - \Sigma(m)_{\rm OS} + \mathcal{O}\big((\not p - m)^2\big) \\ &= \not p - m + \mathcal{O}\big((\not p - m)^2\big)\,.\end{aligned}$$

Employing eq. (8), we can rewrite the boundary conditions specified in eq. (29) as

$$A(m^2)_{\rm OS} = B(m^2)_{\rm OS}\,, \qquad\qquad A(m^2)_{\rm OS} = 2m^2\left[\left(\frac{\partial B_{\rm OS}}{\partial p^2}\right) - \left(\frac{\partial A_{\rm OS}}{\partial p^2}\right)\right]_{p^2=m^2}\,, \tag{30}$$

where we have used $\not p \not p = p^2$ and

$$\frac{\partial}{\partial \not p} = 2\not p \frac{\partial}{\partial p^2}\,.$$

Using eqs. (9) and (10),

$$A(m^2)_{\rm OS} = \delta Z_2 + \frac{\alpha}{2\pi}(4\pi)^\epsilon\Gamma(\epsilon)\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\int_0^1 dx\, x^{-2\epsilon}\big[(1-x)\big(1-(2-\xi)\epsilon\big) - x(1-\xi)\big]$$

$$B(m^2)_{\rm OS} = \delta Z_m + \delta Z_2 + \frac{\alpha}{2\pi}\left(\tfrac{1}{2}(3+\xi) - \epsilon\right)(4\pi)^\epsilon\Gamma(\epsilon)\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\int_0^1 dx\, x^{-2\epsilon}\,.$$

The integrals above are elementary; the end result is

$$A(m^2)_{\rm OS} = \delta Z_2^{\rm OS} + \frac{\alpha\,\xi}{4\pi}(4\pi)^\epsilon\Gamma(\epsilon)\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\frac{1}{1-2\epsilon}\,, \tag{31}$$

$$B(m^2)_{\rm OS} = \delta Z_m^{\rm OS} + \delta Z_2^{\rm OS} + \frac{\alpha}{2\pi}\left(\tfrac{1}{2}(3+\xi) - \epsilon\right)(4\pi)^\epsilon\Gamma(\epsilon)\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\frac{1}{1-2\epsilon}\,. \tag{32}$$

Using eq. (30), we conclude that

$$\delta Z_m^{\rm OS} = -\frac{\alpha}{4\pi}(4\pi)^\epsilon\Gamma(\epsilon)\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\left(\frac{3-2\epsilon}{1-2\epsilon}\right) = -\frac{3\alpha}{4\pi}\left[(4\pi)^\epsilon\Gamma(\epsilon) + \tfrac{4}{3} - \ln\left(\frac{m^2}{\mu^2}\right)\right]\,,$$

after dropping terms of $\mathcal{O}(\epsilon)$.

Next, we compute derivatives of eqs. (9) and (10) with respect to p^2,

$$\begin{aligned}\frac{\partial A_{\rm OS}}{\partial p^2} = \frac{\alpha}{2\pi\mu^2}(4\pi)^\epsilon\Gamma(1+\epsilon)\int_0^1 dx\, x^{-\epsilon}(1-x)\left[\frac{m^2 - p^2(1-x)}{\mu^2}\right]^{-1-\epsilon} \\ \times\left\{(1-x)(1-\epsilon) - 2x(1-\xi) - \frac{(1-\xi)(1+\epsilon)x(1-x)p^2}{m^2-p^2(1-x)}\right\}\,,\end{aligned} \tag{33}$$

$$\frac{\partial B_{\rm OS}}{\partial p^2} = \frac{\alpha}{2\pi\mu^2}(4\pi)^\epsilon\Gamma(1+\epsilon)\big[\tfrac{1}{2}(3+\xi) - \epsilon\big]\int_0^1 dx\, x^{-\epsilon}(1-x)\left[\frac{m^2 - p^2(1-x)}{\mu^2}\right]^{-1-\epsilon}\,. \tag{34}$$

Note that these quantities are ultraviolet finite, after using $\epsilon\Gamma(\epsilon) = \Gamma(1+\epsilon)$. If we now set $p^2 = m^2$, we see that the integrands in eqs. (33) and (34) behave as $x^{-1+2\epsilon}$ as $x \to 0$. Thus integrating over x generates infrared divergences, which are regulated when $\epsilon \neq 0$. The resulting integrals are elementary, and it follows that

$$\left(\frac{\partial A_{\rm OS}}{\partial p^2}\right)_{p^2=m^2} = -\frac{\alpha\,\xi}{4\pi m^2}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\frac{(4\pi)^\epsilon\Gamma(1+\epsilon)}{\epsilon(1-2\epsilon)}\,, \tag{35}$$

$$\left(\frac{\partial B_{\rm OS}}{\partial p^2}\right)_{p^2=m^2} = -\frac{\alpha}{4\pi m^2}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\frac{(4\pi)^\epsilon\Gamma(1+\epsilon)}{\epsilon(1-2\epsilon)}\left[\tfrac{1}{2}(3+\xi)-\epsilon\right]. \tag{36}$$

In light of eqs. (30) and (31),

$$\delta Z_2^{\rm OS} = -\frac{\alpha\,\xi}{4\pi}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\frac{(4\pi)^\epsilon\Gamma(\epsilon)}{1-2\epsilon} + \frac{\alpha}{4\pi}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\frac{(4\pi)^\epsilon\Gamma(1+\epsilon)}{\epsilon(1-2\epsilon)}\left[\xi-3+2\epsilon\right]. \tag{37}$$

The term on the right hand side of eq. (37) proportional to $\Gamma(\epsilon)$ represents the ultraviolet divergence [cf. eq. (24)]. The last term on the right hand side of eq. (37) which contains a pole at $\epsilon = 0$ corresponds to the infrared divergence. Note that the infrared divergence at one loop is absent in the Yennie gauge, which corresponds to $\xi = 3$.

We can add the two terms on the right hand side of eq. (37), if we are not concerned about the mixing of the infrared and the ultraviolet divergences. The end result is

$$\delta Z_2^{\rm OS} = -\frac{\alpha}{4\pi}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\frac{(3-2\epsilon)(4\pi)^\epsilon\Gamma(1+\epsilon)}{\epsilon(1-2\epsilon)}\,. \tag{38}$$

The ultraviolet divergence cancels part of the infrared divergence. Remarkably, the end result is independent of the gauge parameter ξ.

Using eqs. (9) and (10), we can determine $A(p^2)$ and $B(p^2)$ in the on-shell scheme by writing

$$A(p^2)_{\rm OS} = A(p^2)_{\overline{\rm MS}} + \delta Z_2^{\rm OS} - \delta Z_2^{\overline{\rm MS}}\,,$$

$$B(p^2)_{\rm OS} = B(p^2)_{\overline{\rm MS}} + \delta Z_m^{\rm OS} + \delta Z_2^{\rm OS} - \delta Z_m^{\overline{\rm MS}} - \delta Z_2^{\overline{\rm MS}}\,.$$

Eqs. (24), (33) and (37) yield,

$$\delta Z_2^{\rm OS} - \delta Z_2^{\overline{\rm MS}} = -\frac{\alpha\,\xi}{2\pi}\left[1-\tfrac{1}{2}\ln\left(\frac{m^2}{\mu^2}\right)\right] + \frac{\alpha}{4\pi}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}\frac{(4\pi)^\epsilon\Gamma(1+\epsilon)}{\epsilon(1-2\epsilon)}\left[\xi-3+2\epsilon\right], \tag{39}$$

$$\delta Z_m^{\rm OS} - \delta Z_m^{\overline{\rm MS}} = -\frac{\alpha}{\pi}\left[1-\tfrac{3}{4}\ln\left(\frac{m^2}{\mu^2}\right)\right]. \tag{40}$$

The infrared divergence is explicitly exhibited in eq. (39). Expanding about $\epsilon = 0$ yields

$$\delta Z_2^{\rm OS} - \delta Z_2^{\overline{\rm MS}} = \frac{\alpha(\xi-3)}{4\pi}(4\pi)^\epsilon\Gamma(\epsilon) - \frac{\alpha}{\pi}\left[1-\tfrac{3}{4}\ln\left(\frac{m^2}{\mu^2}\right)\right]. \tag{41}$$

Thus, both $A(p^2)_{\rm OS}$ and $B(p^2)_{\rm OS}$ are infrared divergent if $\xi \neq 3$. However the difference $B(p^2)_{\rm OS} - A(p^2)_{\rm OS}$ is infrared finite, although it depends on the gauge parameter ξ for $p^2 \neq m^2$.

VI. The functions $A(p^2)$ and $B(p^2)$ in terms of hypergeometric functions

It is sometimes convenient to evaluate the functions $A(p^2)$ and $B(p^2)$ prior to taking the $\epsilon \to 0$ limit. In this case, $A(p^2)$ and $B(p^2)$ can be expressed in terms of hypergeometric functions. First, we define the following family of integrals:

$$I_{n,\ell}(p^2) \equiv \int_0^1 dx\, x^{n-\epsilon} \left[\frac{m^2 - p^2(1-x)}{m^2}\right]^{-\epsilon-\ell}, \tag{42}$$

Using the integral representation of the Gauss hypergeometric function,

$$F(a\,,\,b\,;\,c\,;\,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 x^{b-1}(1-x)^{c-b-1}(1-xz)^{-a}\,dx\,,$$

for $\mathrm{Re}\,c > \mathrm{Re}\,b > 0$ and $|\arg(1-z)| < \pi$, and the functional relation,

$$F(a\,,\,b\,;\,c\,;\,z) = (1-z)^{-a} F\left(a\,,\,c-b\,;\,c\,;\,\frac{z}{z-1}\right)\,,$$

it follows that

$$I_{n,\ell}(p^2) = \frac{1}{n+1-\epsilon}\, F\left(\ell+\epsilon\,,\,1\,;\,n+2-\epsilon\,;\,\frac{p^2}{m^2}\right)\,. \tag{43}$$

It is convenient to rewrite $I_{n,0}(\epsilon)$ in another form using the recursion relation,

$$(c-a-b)F(a\,,\,b\,;\,c\,;\,z) + a(1-z)F(a+1\,,\,b\,;\,c\,;\,z) - (c-b)F(a\,,\,b-1\,;\,c\,;\,z) = 0\,.$$

It follows that

$$I_{n,0}(p^2) = \frac{1}{n+1-2\epsilon}\left[1 - \frac{\epsilon}{n+1-\epsilon}\left(1-\frac{p^2}{m^2}\right) F\left(1+\epsilon\,,\,1\,;\,n+2-\epsilon\,;\,\frac{p^2}{m^2}\right)\right]. \tag{44}$$

Using eqs. (9) and (10),

$$A(p^2) = \delta Z_2 + \frac{\alpha}{2\pi}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon} (4\pi)^\epsilon\,\Gamma(\epsilon)\Big\{(1-\epsilon)\left[I_{0,0}(p^2) - I_{1,0}(p^2)\right] - (1-\xi)I_{1,0}(\epsilon)$$
$$-\epsilon(1-\xi)\frac{p^2}{m^2}\left[I_{1,1}(p^2) - I_{2,1}(p^2)\right]\Big\} \tag{45}$$

$$B(p^2) = \delta Z_m + \delta Z_2 + \frac{\alpha}{2\pi}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon} (4\pi)^\epsilon\,\Gamma(\epsilon)\left[\tfrac{1}{2}(3+\xi) - \epsilon\right] I_{0,0}(p^2)\,. \tag{46}$$

From eqs. (45) and (46), one can easily perform the expansion in ϵ. In particular,

$$I_{n,0}(p^2) = \frac{1}{n+1}\left[1 + \frac{2\epsilon}{n+1} - \frac{\epsilon}{n+1}\left(1-\frac{p^2}{m^2}\right) F\left(1\,,\,1\,;\,n+2\,;\,\frac{p^2}{m^2}\right)\right] + \mathcal{O}(\epsilon^2)\,,$$

$$I_{n,1}(p^2) = \frac{1}{n+1} F\left(1\,,\,1\,;\,n+2\,;\,\frac{p^2}{m^2}\right) + \mathcal{O}(\epsilon)\,,$$

where

$$F\,(1\,,\,1\,;\,n+2\,;\,z)=-\frac{(1+n)(1-z)^n}{n!}\frac{d^n}{dz^n}\left[\frac{\ln(1-z)}{z}\right]\,.$$

In particular,

$$F\,(1\,,\,1\,;\,2\,;\,z)=-\frac{\ln(1-z)}{z}\,, \tag{47}$$

$$F\,(1\,,\,1\,;\,3\,;\,z)=2\left[\frac{1}{z}+\frac{(1-z)\ln(1-z)}{z^2}\right]\,, \tag{48}$$

$$F\,(1\,,\,1\,;\,4\,;\,z)=-\frac{3}{2}\left[\frac{2-3z}{z^2}+\frac{2(1-z)^2\ln(1-z)}{z^3}\right]\,. \tag{49}$$

It follows that

$$\begin{aligned}A(p^2)=\delta Z_2+\frac{\alpha\,\xi}{4\pi}(4\pi)^\epsilon\Gamma(\epsilon)+\frac{\alpha}{2\pi}\Bigg\{&\frac{3}{2}-\left(1-\frac{p^2}{m^2}\right)F\left(1\,,\,1\,;\,2\,;\,\frac{p^2}{m^2}\right)\\&-(1-\tfrac{1}{2}\xi)\left[1-\frac{1}{2}\left(1-\frac{p^2}{m^2}\right)F\left(1\,,\,1\,;\,3\,;\,\frac{p^2}{m^2}\right)\right]\\&-(1-\xi)\frac{p^2}{m^2}\left[\tfrac{1}{2}F\left(1\,,\,1\,;\,3\,;\,\frac{p^2}{m^2}\right)-\tfrac{1}{3}F\left(1\,,\,1\,;\,4\,;\,\frac{p^2}{m^2}\right)\right]-\tfrac{1}{2}\xi\ln\left(\frac{m^2}{\mu^2}\right)\Bigg\}+\mathcal{O}(\epsilon)\,.\end{aligned}$$

$$\begin{aligned}B(p^2)=\delta Z_m+\delta Z_2+\frac{\alpha}{4\pi}(3+\xi)(4\pi)^\epsilon\,\Gamma(\epsilon)&\\+\frac{\alpha}{2\pi}\Bigg\{2+\xi-\tfrac{1}{2}(3+\xi)\left[\left(1-\frac{p^2}{m^2}\right)F\left(1\,,\,1\,;\,2\,;\,\frac{p^2}{m^2}\right)+\ln\left(\frac{m^2}{\mu^2}\right)\right]\Bigg\}+\mathcal{O}(\epsilon)\,.&\end{aligned}$$

Inserting the results of eqs. (47)–(49) yields

$$A(p^2)=\delta Z_2+\frac{\alpha\,\xi}{4\pi}(4\pi)^\epsilon\Gamma(\epsilon)+\frac{\alpha\,\xi}{4\pi}\left\{\left(1+\frac{m^2}{p^2}\right)\left[1-\left(1-\frac{m^2}{p^2}\right)\ln\left(1-\frac{p^2}{m^2}\right)\right]-\ln\left(\frac{m^2}{\mu^2}\right)\right\}+\mathcal{O}(\epsilon)\,,$$

$$\begin{aligned}B(p^2)=\delta Z_m+\delta Z_2+\frac{\alpha}{4\pi}(3+\xi)(4\pi)^\epsilon\,\Gamma(\epsilon)&\\+\frac{\alpha}{2\pi}\left\{2+\xi-\tfrac{1}{2}(3+\xi)\left[\left(1-\frac{m^2}{p^2}\right)\ln\left(1-\frac{p^2}{m^2}\right)+\ln\left(\frac{m^2}{\mu^2}\right)\right]\right\}+\mathcal{O}(\epsilon)\,,&\end{aligned}$$

which are equivalent to the results previously obtained.

In the OS scheme, we also need to compute the derivatives of $A(p^2)$ and $B(p^2)$ with respect to p^2. It is straightforward to obtain,

$$\frac{\partial I_{n,0}}{\partial p^2}=\frac{\epsilon}{m^2(n+1-\epsilon)(n+2-\epsilon)}\,F\left(1+\epsilon\,,\,2\,;\,n+3-\epsilon\,;\,\frac{p^2}{m^2}\right)\,,$$

$$\frac{\partial(p^2I_{n,1})}{\partial p^2}=\frac{1}{n+1-\epsilon}\,F\left(1+\epsilon\,,\,2\,;\,n+2-\epsilon\,;\,\frac{p^2}{m^2}\right)\,.$$

Taking the derivative of eq. (45), it follows that

$$\frac{\partial A}{\partial p^2}=\frac{\alpha\,\xi}{2\pi m^2}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}(4\pi)^\epsilon\,\frac{\Gamma(1+\epsilon)}{2-\epsilon}\left\{F\left(1+\epsilon\,,\,2\,;\,3-\epsilon\,;\,\frac{p^2}{m^2}\right)-\left(\frac{1-\epsilon}{3-\epsilon}\right)F\left(1+\epsilon\,,\,2\,;\,4-\epsilon\,;\,\frac{p^2}{m^2}\right)\right\}.$$

Likewise, we can compute $\partial B/\partial p^2$ by taking the derivative of eq. (46). We can simplify the expression for $\partial A/\partial p^2$ by using the recursion relation,

$$(c-b-1)F(a\,,\,b\,;\,c\,;\,z)+bF(a\,,\,b+1\,;\,c\,;\,z)-(c-1)F(a\,,\,b\,;\,c-1\,;\,z)=0\,.$$

The end result is

$$\frac{\partial A}{\partial p^2}=\frac{\alpha\,\xi}{\pi m^2}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}(4\pi)^\epsilon\,\frac{\Gamma(1+\epsilon)}{(2-\epsilon)(3-\epsilon)}\,F\left(1+\epsilon\,,\,3\,;\,4-\epsilon\,;\,\frac{p^2}{m^2}\right)\,, \qquad (50)$$

$$\frac{\partial B}{\partial p^2}=\frac{\alpha}{2\pi m^2}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}(4\pi)^\epsilon\,\frac{\Gamma(1+\epsilon)}{(1-\epsilon)(2-\epsilon)}\left[\tfrac{1}{2}(3+\xi)-\epsilon\right]F\left(1+\epsilon\,,\,2\,;\,3-\epsilon\,;\,\frac{p^2}{m^2}\right)\,. \qquad (51)$$

The infrared divergence emerges when $p^2=m^2$. In this limit, we can employ the identity,

$$F(a\,,\,b\,;\,c\,;\,1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,,$$

in eqs. (50) and (51) to recover the results of eqs. (35) and (36).

Finally, to demonstrate the fact that the infrared divergence in δZ_2 cancels in the Yennie gauge, we make use of another recursion relation,

$$cF(a\,,\,b\,;\,c\,;\,z)-bzF(a\,,\,b+1\,;\,c+1\,;\,z)-cF(a-1\,,\,b\,;\,c\,;\,z)=0\,,$$

to write

$$(3-\epsilon)\left[F\left(1+\epsilon\,,\,2\,;\,3-\epsilon\,;\,\frac{p^2}{m^2}\right)-F\left(\epsilon\,,\,2\,;\,3-\epsilon\,;\,\frac{p^2}{m^2}\right)\right]-\frac{2p^2}{m^2}F\left(1+\epsilon\,,\,3\,;\,4-\epsilon\,;\,\frac{p^2}{m^2}\right)=0\,.$$

Hence, it follows that

$$\frac{\partial B}{\partial p^2}-\frac{\partial A}{\partial p^2}=\frac{\alpha}{2\pi m^2}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}(4\pi)^\epsilon\,\frac{\Gamma(1+\epsilon)}{2-\epsilon}\left\{\left[\frac{\frac{1}{2}(3+\xi)-\epsilon}{1-\epsilon}-\frac{\xi m^2}{p^2}\right]F\left(1+\epsilon\,,\,2\,;\,3-\epsilon\,;\,\frac{p^2}{m^2}\right)\right.$$
$$\left.+\frac{\xi m^2}{p^2}F\left(\epsilon\,,\,2\,;\,3-\epsilon\,;\,\frac{p^2}{m^2}\right)\right\}$$
$$=\frac{\alpha}{4\pi m^2}\left(\frac{m^2}{\mu^2}\right)^{-\epsilon}(4\pi)^\epsilon\,\frac{\Gamma(1+\epsilon)}{(1-\epsilon)(2-\epsilon)}\left[3-\xi-2\epsilon(1-\xi)+\xi\left(1-\frac{m^2}{p^2}\right)\right]F\left(1+\epsilon\,,\,2\,;\,3-\epsilon\,;\,\frac{p^2}{m^2}\right)$$
$$\left.+\frac{2\xi(1-\epsilon)m^2}{p^2}F\left(\epsilon\,,\,2\,;\,3-\epsilon\,;\,\frac{p^2}{m^2}\right)\right\}.$$

In the case of $\xi = 3$, we can take the $\epsilon \to 0$ limit when $p^2 = m^2$ without encountering an infrared divergence in δZ_2,

$$\lim_{p^2 \to m^2} \left(\frac{\partial B}{\partial p^2} - \frac{\partial A}{\partial p^2} \right)_{\xi=3} = \frac{\alpha}{8\pi m^2} \left\{ 6 + \lim_{\epsilon \to 0} 4\epsilon F(1+\epsilon\,,\, 2\,;\, 3-\epsilon\,;\, 1) \right\} = \frac{\alpha}{4\pi m^2}\,.$$

That is, in light of eq. (30), we recover eq. (41) in the Yennie gauge,

$$(\delta Z_2^{\rm OS} - \delta Z_2^{\overline{\rm MS}})_{\xi=3} = -\frac{\alpha}{\pi} \left[1 - \tfrac{3}{4} \ln \left(\frac{m^2}{\mu^2} \right) \right]\,.$$

References

[1] Victor Ilisie, *Concepts in Quantum Field Theory–A Practitioner's Toolkit* (Springer International Publishing, Cham, Switzerland, 2016).

[2] John Collins, *Renormalization* (Cambridge University Press, Cambridge, MA, 1984).

[3] Alexey A. Petrov and Andrew E. Blechman, *Effective Field Theories* (World Scientific, Singapore, 2016).