# The complex inverse trigonometric and hyperbolic functions

Howard E. Haber

Santa Cruz Institute for Particle Physics

University of California, Santa Cruz, CA 95064, USA

October 23, 2019

**Abstract**

In these notes, we examine the inverse trigonometric and hyperbolic functions, where the arguments of these functions can be complex numbers. The multivalued functions are defined in terms of the complex logarithm. We also carefully define the corresponding single-valued principal values of the inverse trigonometric and hyperbolic functions following the conventions employed by the computer algebra software system, Mathematica.

## 1 Introduction

The inverse trigonometric and hyperbolic functions evaluated in the complex plane are multivalued functions (see e.g. Refs. 1 and 2). In many applications, it is convenient to define the corresponding single-valued functions, called the *principal values* of the inverse trigonometric and hyperbolic functions, according to some convention. Different conventions appear in various reference books. In these notes, we shall follow the conventions employed by the computer algebra software system, Mathematica, which are outlined in section 2.2.5 of Ref. 3.

The principal value of a multivalued complex function $f(z)$ of the complex variable $z$, which we denote by $F(z)$, is continuous in all regions of the complex plane, except on a specific line (or lines) called branch cuts. The function $F(z)$ has a discontinuity when $z$ crosses a branch cut. Branch cuts end at a branch point, which is unambiguous for each function $F(z)$. But the choice of branch cuts is a matter of convention. Thus, if mathematics software is employed to evaluate the function $F(z)$, you need to know the conventions of the software for the location of the branch cuts. The mathematical software needs to precisely define the principal value of $f(z)$ in order that it can produce a unique answer when the user types in $F(z)$ for a particular complex number $z$. There are often different possible candidates for $F(z)$ that differ only in the values assigned to them when $z$ lies on the branch cut(s). These notes provide a careful discussion of these issues as they apply to the complex inverse trigonometric and hyperbolic functions.

The simplest example of a multivalued function is the argument of a complex number $z$, denoted by $\arg z$. In these notes, the principal value of the argument of the complex number $z$, denoted by $\operatorname{Arg} z$, is defined to lie in the interval $-\pi < \operatorname{Arg} z \leq \pi$. That is, $\operatorname{Arg} z$ is single-valued and is continuous at all points in the complex plane excluding a branch cut along the negative real axis. In Appendix A, a detailed review of the properties of $\arg z$ and $\operatorname{Arg} z$ are provided.

The properties of $\operatorname{Arg} z$ determine the location of the branch cuts of the principal values of the logarithm the square root functions. The complex logarithm and generalized power functions are reviewed in Appendix B. If $f(z)$ is expressible in terms of the logarithm the square root functions, then the definition of the principal value of $F(z)$ is not unique. However given a specific definition of $F(z)$ in terms of the principal values of the logarithm the square root functions, the locations of the branch cuts of $F(z)$ are inherited from that of $\operatorname{Arg} z$ and are thus uniquely determined.

## 2 The inverse trigonometric functions: arctan and arccot

We begin by examining the solution to the equation

$$z = \tan w = \frac{\sin w}{\cos w} = \frac{1}{i}\left(\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}}\right) = \frac{1}{i}\left(\frac{e^{2iw} - 1}{e^{2iw} + 1}\right) .$$

We now solve for $e^{2iw}$,

$$iz = \frac{e^{2iw} - 1}{e^{2iw} + 1} \qquad \Longrightarrow \qquad e^{2iw} = \frac{1 + iz}{1 - iz} .$$

Taking the complex logarithm of both sides of the equation, we can solve for $w$,

$$w = \frac{1}{2i} \ln\left(\frac{1 + iz}{1 - iz}\right) .$$

The solution to $z = \tan w$ is $w = \arctan z$. Hence,

$$\boxed{\arctan z = \frac{1}{2i} \ln\left(\frac{1 + iz}{1 - iz}\right)} \tag{1}$$

Since the complex logarithm is a multivalued function, it follows that the arctangent function is also a multivalued function. Using the definition of the multivalued complex logarithm,

$$\arctan z = \frac{1}{2i} \operatorname{Ln}\left|\frac{1 + iz}{1 - iz}\right| + \tfrac{1}{2}\left[\operatorname{Arg}\left(\frac{1 + iz}{1 - iz}\right) + 2\pi n\right] , \qquad n = 0\,, \pm 1\,, \pm 2\,, \ldots\,, \tag{2}$$

where Arg is the principal value of the argument function.

Similarly,

$$z = \cot w = \frac{\cos w}{\sin w} = \left(\frac{i(e^{iw} + e^{-iw}}{e^{iw} - e^{-iw}}\right) = \left(\frac{i(e^{2iw} + 1}{e^{2iw} - 1}\right) .$$

Again, we solve for $e^{2iw}$,

$$-iz = \frac{e^{2iw} + 1}{e^{2iw} - 1} \qquad \Longrightarrow \qquad e^{2iw} = \frac{iz - 1}{iz + 1} .$$

Taking the complex logarithm of both sides of the equation, we conclude that

$$w = \frac{1}{2i} \ln\left(\frac{iz-1}{iz+1}\right) = \frac{1}{2i} \ln\left(\frac{z+i}{z-i}\right) ,$$

after multiplying numerator and denominator by $-i$ to get a slightly more convenient form. The solution to $z = \cot w$ is $w = \text{arccot}z$. Hence,

$$\boxed{\text{arccot}z = \frac{1}{2i} \ln\left(\frac{z+i}{z-i}\right)} \tag{3}$$

Thus, the arccotangent function is a multivalued function,

$$\text{arccot}z = \frac{1}{2i} \,\text{Ln}\left|\frac{z+i}{z-i}\right| + \tfrac{1}{2}\left[\text{Arg}\left(\frac{z+i}{z-i}\right) + 2\pi n\right] , \qquad n = 0\,, \pm 1\,, \pm 2\,, \ldots\,, \tag{4}$$

Using the definitions given by eqs. (1) and (3), the following relation is easily derived:

$$\text{arccot}(z) = \arctan\left(\frac{1}{z}\right) . \tag{5}$$

Note that eq. (5) can be used as the *definition* of the arccotangent function. It is instructive to derive another relation between the arctangent and arccotangent functions. First, we first recall the property of the multivalued complex logarithm,

$$\ln(z_1 z_2) = \ln(z_1) + \ln(z_2) , \tag{6}$$

as a set equality [cf. eq. (B.15)]. It is convenient to define a new variable,

$$v = \frac{i-z}{i+z} , \qquad \Longrightarrow \qquad -\frac{1}{v} = \frac{z+i}{z-i} . \tag{7}$$

It follows that:

$$\arctan z + \text{arccot}\, z = \frac{1}{2i}\left[\ln v + \ln\left(-\frac{1}{v}\right)\right] = \frac{1}{2i} \ln\left(\frac{-v}{v}\right) = \frac{1}{2i} \ln(-1) .$$

Since $\ln(-1) = i(\pi + 2\pi n)$ for $n = 0, \pm 1, \pm 2 \ldots$, we conclude that

$$\boxed{\arctan z + \text{arccot}\, z = \tfrac{1}{2}\pi + \pi n , \qquad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots} \tag{8}$$

Finally, we mention two equivalent forms for the multivalued complex arctangent and arccotangent functions. Recall that the complex logarithm satisfies

$$\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2 , \tag{9}$$

where this equation is to be viewed as a set equality [cf. eq. (B.16). Thus, the multi-valued arctangent and arccotangent functions given in eqs. (1) and (5), respectively, are equivalent to

$$\arctan z = \frac{1}{2i}\Big[\ln(1+iz) - \ln(1-iz)\Big]\,, \tag{10}$$

$$\operatorname{arccot} z = \frac{1}{2i}\left[\ln\left(1+\frac{i}{z}\right) - \ln\left(1-\frac{i}{z}\right)\right]\,, \tag{11}$$

## 3 The principal values Arctan and Arccot

It is convenient to define principal values of the inverse trigonometric functions, which are single-valued functions, which will necessarily exhibit a discontinuity across some appropriately chosen line in the complex plane. In Mathematica, the principal values of the complex arctangent and arccotangent functions, denoted by Arctan and Arccot respectively (with an upper case A), are defined by employing the principal values of the complex logarithms in eqs. (10) and (11),

$$\boxed{\operatorname{Arctan} z = \frac{1}{2i}\Big[\operatorname{Ln}(1+iz) - \operatorname{Ln}(1-iz)\Big]\,, \qquad z \neq \pm i} \tag{12}$$

and

$$\boxed{\operatorname{Arccot} z = \operatorname{Arctan}\left(\frac{1}{z}\right) = \frac{1}{2i}\left[\operatorname{Ln}\left(1+\frac{i}{z}\right) - \operatorname{Ln}\left(1-\frac{i}{z}\right)\right]\,, \quad z \neq \pm i\,,\, z \neq 0} \tag{13}$$

One useful feature of these definitions is that they satisfy:

$$\begin{aligned} \operatorname{Arctan}(-z) &= -\operatorname{Arctan} z\,, \quad \text{for } z \neq \pm i, \\ \operatorname{Arccot}(-z) &= -\operatorname{Arccot} z\,, \quad \text{for } z \neq \pm i \text{ and } z \neq 0. \end{aligned} \tag{14}$$

Because the principal value of the complex logarithm Ln does not satisfy eq. (9) in all regions of the complex plane, it follows that the definitions of the complex arctangent and arccotangent functions adopted by Mathematica do not coincide with some alternative definitions employed by some of the well known mathematical reference books [for further details, see Appendix C]. Note that the points $z = \pm i$ are excluded from the above definitions, as the arctangent and arccotangent are divergent at these two points. The definition of the principal value of the arccotangent given in eq. (13) is deficient in one respect since it is not well-defined at $z = 0$. We shall address this problem shortly.

First, we shall identify the location of the discontinuity of the principal values of the complex arctangent and arccotangent functions in the complex plane. The principal value of the complex arctangent function is single-valued for all $z \neq \pm i$. These two points, called *branch points*, must be excluded as the arctangent function is singular there. Moreover, the the principal-valued logarithms, $\text{Ln}\,(1 \pm iz)$ are discontinuous as $z$ crosses the lines $1 \pm iz < 0$, respectively. We conclude that $\text{Arctan}\, z$ must be discontinuous when $z = x + iy$ crosses lines on the imaginary axis such that

$$x = 0 \quad \text{and} \quad -\infty < y < -1 \quad \text{and} \quad 1 < y < \infty\,. \tag{15}$$

These two lines that lie along the imaginary axis are called the *branch cuts* of $\text{Arctan}\, z$.

Note that $\text{Arctan}\, z$ is single-valued on the branch cut itself, since it inherits this property from the principal value of the complex logarithm. In particular, for values of $z = iy$ ($|y| > 1$) that lie on the branch cut of $\text{Arctan}\, z$, eq. (12) yields,

$$\text{Arctan}(iy) = \begin{cases} \frac{1}{2i}\text{Ln}\left(\frac{y-1}{y+1}\right) - \frac{1}{2}\pi\,, & \text{for} \quad -\infty < y < -1\,, \\ \frac{1}{2i}\text{Ln}\left(\frac{y-1}{y+1}\right) + \frac{1}{2}\pi\,, & \text{for} \quad 1 < y < \infty\,. \end{cases} \tag{16}$$

Likewise, the principal value of the complex arccotangent function is single-valued for all complex $z$ excluding the branch points $z \neq \pm i$. Moreover, the the principal-valued logarithms, $\text{Ln}\left(1 \pm \frac{i}{z}\right)$ are discontinuous as $z$ crosses the lines $1 \pm \frac{i}{z} < 0$, respectively. We conclude that $\text{Arccot}\, z$ must be discontinuous when $z = x + iy$ crosses the branch cuts located on the imaginary axis such that

$$x = 0 \quad \text{and} \quad -1 < y < 1\,. \tag{17}$$

In particular, due to the presence of the branch cut,

$$\lim_{x \to 0^-} \text{Arccot}(x + iy) \neq \lim_{x \to 0^+} \text{Arccot}(x + iy)\,, \qquad \text{for } -1 < y < 1\,,$$

for real values of $x$, where $0^+$ indicates that the limit is approached from positive real axis and $0^-$ indicates that the limit is approached from negative real axis. If $z \neq 0$, eq. (13) provides unique values for $\text{Arccot}\, z$ for all $z \neq \pm i$ in the complex plane, including on the branch cut. Using eq. (13), one can easily show that if $z$ is a nonzero complex number infinitesimally close to 0, then it follow that,

$$\text{Arccot}\, z \underset{z \to 0\,,\, z \neq 0}{=} \begin{cases} \frac{1}{2}\pi\,, & \text{for Re } z > 0\,, \\ \frac{1}{2}\pi\,, & \text{for Re } z = 0 \text{ and Im } z < 0\,, \\ -\frac{1}{2}\pi\,, & \text{for Re } z < 0\,, \\ -\frac{1}{2}\pi\,, & \text{for Re } z = 0 \text{ and Im } z > 0\,. \end{cases} \tag{18}$$

It is now apparent why the point $z = 0$ is problematical in eq. (13), since there is no well defined way of defining Arccot(0). Indeed, for values of $z = iy$ ($-1 < y < 1$) that lie on the branch cut of $\text{Arccot} z$, eq. (13) yields,

$$\text{Arccot}(iy) = \begin{cases} \frac{1}{2i}\text{Ln}\left(\frac{1+y}{1-y}\right) + \frac{1}{2}\pi\,, & \text{for} \quad -1 < y < 0\,, \\ \frac{1}{2i}\text{Ln}\left(\frac{1+y}{1-y}\right) - \frac{1}{2}\pi\,, & \text{for} \quad 0 < y < 1\,. \end{cases} \tag{19}$$

Mathematica supplements the definition of the principal value of the complex arccotangent given in eq. (13) by declaring that

$$\text{Arccot}(0) = \tfrac{1}{2}\pi\,. \tag{20}$$

With the definitions given in eqs. (12), (13) and (20), Arctan $z$ and Arccot $z$ are single-valued functions in the entire complex plane, excluding the branch points $z = \pm i$ and are continuous functions as long as the complex number $z$ does not cross the branch cuts specified in eqs. (15) and (17), respectively.

Having defined precisely the principal values of the complex arctangent and arccotangent functions, let us check that they reduce to the conventional definitions when $z$ is real. First consider the principal value of the real arctangent function, which satisfies

$$-\tfrac{1}{2}\pi \leq \text{Arctan}\, x \leq \tfrac{1}{2}\pi\,, \qquad \text{for } -\infty \leq x \leq \infty\,, \tag{21}$$

where $x$ is a real variable. The definition given by eq. (12) does reduce to the conventional definition of the principal value of the real-valued arctangent function when $z$ is real. In particular, for real values of $x$,

$$\text{Arctan}\, x = \frac{1}{2i}\Big[\text{Ln}(1+ix) - \text{Ln}(1-ix)\Big] = \tfrac{1}{2}\Big[\text{Arg}(1+ix) - \text{Arg}(1-ix)\Big]\,, \tag{22}$$

after noting that $\text{Ln}|1+ix| = \text{Ln}|1-ix| = \tfrac{1}{2}\text{Ln}(1+x^2)$. Geometrically, the quantity $\text{Arg}(1+ix) - \text{Arg}(1-ix)$ is the angle between the complex numbers $1+ix$ and $1-ix$ viewed as vectors lying in the complex plane. This angle varies between $-\pi$ and $\pi$ over the range $-\infty < x < \infty$. Moreover, the values $\pm\pi$ are achieved in the limit as $x \to \pm\infty$, respectively. Hence, we conclude that the principal interval of the real-valued arctangent function is indeed given by eq. (21). For all possible values of $x$ excluding $x = -\infty$, one can check that it is permissible to subtract the two principal-valued logarithms (or equivalently the two Arg functions) using eq. (9). In the case of $x \to -\infty$, eq. (B.23) yields $\text{Arg}(1+ix) - \text{Arg}(1-ix) \to -\pi$, corresponding to $N_- = -1$ in the notation of eq. (A.21). Hence, an extra term appears when combining the two logarithms that is equal to $2\pi i N_- = -2\pi i$. The end result is,

$$\text{Arctan}(-\infty) = \frac{1}{2i}\left[\ln(-1) - 2\pi i\right] = -\tfrac{1}{2}\pi\,,$$

as required. As a final check, we can use the results of Tables 1 and 2 given in Appendix A to conclude that $\text{Arg}(a+bi) = \text{Arctan}(b/a)$ for $a > 0$. Setting $a = 1$ and $b = x$ then yields:

$$\text{Arg}\,(1+ix) = \text{Arctan}\, x\,, \qquad \text{Arg}\,(1-ix) = \text{Arctan}(-x) = -\text{Arctan}\, x\,.$$

Subtracting these two results yields eq. (22).

In contrast to the real arctangent function, there is no generally agreed definition for the principal range of the real-valued arccotangent function. However, a growing consensus among computer scientists has led to the following choice for the principal range of the real-valued arccotangent function,[1]

$$-\tfrac{1}{2}\pi < \operatorname{Arccot} x \leq \tfrac{1}{2}\pi\,, \qquad \text{for } -\infty \leq x \leq \infty\,, \tag{23}$$

where $x$ is a real variable. Note that the principal value of the arccotangent function does not include the endpoint $-\frac{1}{2}\pi$ [contrast this with eq. (21) for Arctan]. The reason for this behavior is that $\operatorname{Arccot} x$ is *discontinuous* at $x = 0$. In particular,

$$\lim_{x\to 0^-} \operatorname{Arccot} x = -\tfrac{1}{2}\pi\,, \qquad \lim_{x\to 0^+} \operatorname{Arccot} x = \tfrac{1}{2}\pi\,, \tag{24}$$

as a consequence of eq. (18). In particular, eq. (23) corresponds to the convention in which $\operatorname{Arccot}(0) = \frac{1}{2}\pi$ [cf. eq. (20)]. Thus, as $x$ increases from negative to positive values, $\operatorname{Arccot} x$ never reaches $-\frac{1}{2}\pi$ but jumps discontinuously to $\frac{1}{2}\pi$ at $x = 0$.

Finally, we examine the the analog of eq. (8) for the corresponding principal values. Employing the Mathematica definitions for the principal values of the complex arctangent and arccotangent functions, we find that

$$\operatorname{Arctan} z + \operatorname{Arccot} z = \begin{cases} \tfrac{1}{2}\pi\,, & \text{for } \operatorname{Re} z > 0\,, \\ \tfrac{1}{2}\pi\,, & \text{for } \operatorname{Re} z = 0\,, \text{ and } \operatorname{Im} z > 1 \text{ or } -1 < \operatorname{Im} z \leq 0\,, \\ -\tfrac{1}{2}\pi\,, & \text{for } \operatorname{Re} z < 0\,, \\ -\tfrac{1}{2}\pi\,, & \text{for } \operatorname{Re} z = 0\,, \text{ and } \operatorname{Im} z < -1 \text{ or } 0 < \operatorname{Im} z < 1\,. \end{cases} \tag{25}$$

The derivation of this result will be given in Appendix D. In Mathematica, one can confirm eq. (25) with many examples.

The relations between the single-valued and multivalued functions is summarized by:

$$\arctan z = \operatorname{Arctan} z + n\pi\,, \qquad n = 0\,, \pm 1\,, \pm 2\,, \cdots\,,$$
$$\operatorname{arccot} z = \operatorname{Arccot} z + n\pi\,, \qquad n = 0\,, \pm 1\,, \pm 2\,, \cdots\,.$$

These relations can be used along with eq. (25) to confirm the result obtained in eq. (8).

## 4 The inverse trigonometric functions: arcsin and arccos

The arcsine function is the solution to the equation:

$$z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}\,.$$

Defining $v \equiv e^{iw}$ and multiplying the resulting equation by $v$ yields the quadratic equation,

$$v^2 - 2izv - 1 = 0\,. \tag{26}$$

[1]The reader is warned that in some reference books (see, e.g., Ref. 4), the principal range of the real-valued arccotangent function is taken as $0 \leq \operatorname{Arccot} x \leq \pi$, for $-\infty \leq x \leq \infty$. For further details, see the cautionary remarks at the end of Appendix C. We do *not* adopt this convention in this section.

The solution to eq. (26) is:

$$v = iz + (1 - z^2)^{1/2} \,. \tag{27}$$

Since $z$ is a complex variable, $(1-z^2)^{1/2}$ is the complex square-root function. This is a multivalued function with two possible values that differ by an overall minus sign. Hence, we do not explicitly write out the $\pm$ sign in eq. (27). To avoid ambiguity, we shall write

$$\begin{aligned} v &= iz + (1-z^2)^{1/2} = iz + e^{\frac{1}{2}\ln(1-z^2)} = iz + e^{\frac{1}{2}\left[\mathrm{Ln}|1-z^2| + i\arg(1-z^2)\right]} \\ &= iz + |1-z^2|^{1/2} e^{\frac{i}{2}\arg(1-z^2)} \,. \end{aligned}$$

In particular, note that

$$e^{\frac{i}{2}\arg(1-z^2)} = e^{\frac{i}{2}\mathrm{Arg}(1-z^2)} e^{in\pi} = \pm e^{\frac{i}{2}\mathrm{Arg}(1-z^2)} \,, \qquad \text{for } n = 0, 1 \,,$$

which exhibits the two possible sign choices.

By definition, $v \equiv e^{iw}$, from which it follows that

$$w = \frac{1}{i}\ln v = \frac{1}{i}\ln\left(iz + |1-z^2|^{1/2} e^{\frac{i}{2}\arg(1-z^2)}\right) \,.$$

The solution to $z = \sin w$ is $w = \arcsin z$. Hence,

$$\boxed{\arcsin z = \frac{1}{i}\ln\left(iz + |1-z^2|^{1/2} e^{\frac{i}{2}\arg(1-z^2)}\right)}$$

The arccosine function is the solution to the equation:

$$z = \cos w = \frac{e^{iw} + e^{-iw}}{2} \,.$$

Letting $v \equiv e^{iw}$ , we solve the equation

$$v + \frac{1}{v} = 2z \,.$$

Multiplying by $v$, one obtains a quadratic equation for $v$,

$$v^2 - 2zv + 1 = 0 \,. \tag{28}$$

The solution to eq. (28) is:

$$v = z + (z^2 - 1)^{1/2} \,.$$

Following the same steps as in the analysis of arcsine, we write

$$w = \arccos z = \frac{1}{i}\ln v = \frac{1}{i}\ln\left[z + (z^2-1)^{1/2}\right] \,, \tag{29}$$

where $(z^2-1)^{1/2}$ is the multivalued square root function. More explicitly,

$$\arccos z = \frac{1}{i}\ln\left(z + |z^2-1|^{1/2} e^{\frac{i}{2}\arg(z^2-1)}\right) \,. \tag{30}$$

It is sometimes more convenient to rewrite eq. (30) in a slightly different form. Recall that

$$\arg(z_1 z_2) = \arg z + \arg z_2\,, \tag{31}$$

as a *set equality* [cf. eq. (A.10). We now substitute $z_1 = z$ and $z_2 = -1$ into eq. (31) and note that $\arg(-1) = \pi + 2\pi n$ (for $n = 0, \pm 1, \pm 2, \ldots$) and $\arg z = \arg z + 2\pi n$ as a set equality. It follows that

$$\arg(-z) = \pi + \arg z\,,$$

as a set equality. Thus,

$$e^{\frac{i}{2}\arg(z^2-1)} = e^{i\pi/2}\,e^{\frac{i}{2}\arg(1-z^2)} = ie^{\frac{i}{2}\arg(1-z^2)}\,,$$

and we can rewrite eq. (29) as follows:

$$\arccos z = \frac{1}{i}\ln\left(z + i\sqrt{1-z^2}\right)\,, \tag{32}$$

which is equivalent to the more explicit form,

$$\boxed{\arccos z = \frac{1}{i}\ln\left(z + i|1-z^2|^{1/2}e^{\frac{i}{2}\arg(1-z^2)}\right)}$$

The arcsine and arccosine functions are related in a very simple way. Using eq. (27),

$$\frac{i}{v} = \frac{i}{iz + \sqrt{1-z^2}} = \frac{i(-iz + \sqrt{1-z^2})}{(iz + \sqrt{1-z^2})(-iz + \sqrt{1-z^2})} = z + i\sqrt{1-z^2}\,,$$

which we recognize as the argument of the logarithm in the definition of the arccosine [cf. eq. (32)]. Using eq. (6), it follows that

$$\arcsin z + \arccos z = \frac{1}{i}\left[\ln v + \ln\left(\frac{i}{v}\right)\right] = \frac{1}{i}\ln\left(\frac{iv}{v}\right) = \frac{1}{i}\ln i\,.$$

Since $\ln i = i(\frac{1}{2}\pi + 2\pi n)$ for $n = 0, \pm 1, \pm 2 \ldots$, we conclude that

$$\boxed{\arcsin z + \arccos z = \tfrac{1}{2}\pi + 2\pi n\,, \qquad \text{for}\quad n = 0, \pm 1, \pm 2, \ldots} \tag{33}$$

## 5 The principal values Arcsin and Arccos

In Mathematica, the principal value of the arcsine function is obtained by employing the principal value of the logarithm and the principle value of the square-root function (which corresponds to employing the principal value of the argument). Thus,

$$\text{Arcsin}\, z = \frac{1}{i}\text{Ln}\left(iz + |1-z^2|^{1/2}e^{\frac{i}{2}\text{Arg}(1-z^2)}\right)\,. \tag{34}$$

It is convenient to introduce some notation for the the principle value of the square-root function. Consider the multivalued square root function, denoted by $z^{1/2}$. Henceforth, we shall employ the symbol $\sqrt{z}$ to denote the single-valued function,

$$\sqrt{z} = \sqrt{|z|}\, e^{\frac{1}{2}\text{Arg } z}\,, \tag{35}$$

where $\sqrt{|z|}$ denotes the unique positive squared root of the real number $|z|$. In this notation, eq. (34) is rewritten as:

$$\boxed{\text{Arcsin}\, z = \frac{1}{i}\text{Ln}\left(iz + \sqrt{1-z^2}\right)} \tag{36}$$

One noteworthy property of the principal value of the arcsine function is

$$\text{Arcsin}(-z) = -\text{Arcsin}\, z\,. \tag{37}$$

To prove this result, it is convenient to define:

$$v = iz + \sqrt{1-z^2}\,, \qquad\qquad \frac{1}{v} = \frac{1}{iz+\sqrt{1-z^2}} = -iz + \sqrt{1-z^2}\,. \tag{38}$$

Then,

$$\text{Arcsin}\, z = \frac{1}{i}\,\text{Ln}\, v\,, \qquad\qquad \text{Arcsin}(-z) = \frac{1}{i}\,\text{Ln}\left(\frac{1}{v}\right)\,.$$

The second logarithm above can be simplified by making use of eq. (B.25),

$$\text{Ln}(1/z) = \begin{cases} -\text{Ln}(z) + 2\pi i\,, & \text{if } z \text{ is real and negative}\,, \\ -\text{Ln}(z)\,, & \text{otherwise}\,. \end{cases} \tag{39}$$

In Appendix E, we prove that $v$ can never be real and negative. Hence it follows from eq. (39) that

$$\text{Arcsin}(-z) = \frac{1}{i}\,\text{Ln}\left(\frac{1}{v}\right) = -\frac{1}{i}\,\text{Ln}\, v = -\text{Arcsin}\, z\,,$$

as asserted in eq. (37).

We now examine the principal value of the arcsine for real-valued arguments such that $-1 \leq x \leq 1$. Setting $z = x$, where $x$ is real and $|x| \leq 1$,

$$\begin{aligned} \text{Arcsin}\, x = \frac{1}{i}\text{Ln}\left(ix + \sqrt{1-x^2}\right) &= \frac{1}{i}\left[\text{Ln}\left|ix + \sqrt{1-x^2}\,\right| + i\text{Arg}\left(ix + \sqrt{1-x^2}\right)\right] \\ &= \text{Arg}\left(ix + \sqrt{1-x^2}\right)\,, \qquad\quad \text{for } |x| \leq 1\,, \end{aligned} \tag{40}$$

since $ix + \sqrt{1-x^2}$ is a complex number with magnitude equal to 1 when $x$ is real with $|x| \leq 1$. Moreover, $ix + \sqrt{1-x^2}$ lives either in the first or fourth quadrant of the complex plane, since $\text{Re}(ix + \sqrt{1-x^2}) \geq 0$. It follows that:

$$-\frac{\pi}{2} \leq \text{Arcsin}\, x \leq \frac{\pi}{2}\,, \qquad \text{for } \ |x| \leq 1\,.$$

In Mathematica, the principal value of the arccosine is defined by:

$$\mathrm{Arccos}\, z = \tfrac{1}{2}\pi - \mathrm{Arcsin}\, z\,. \tag{41}$$

We demonstrate below that this definition is equivalent to choosing the principal value of the complex logarithm and the principal value of the square root in eq. (32). That is,

$$\boxed{\mathrm{Arccos}\, z = \frac{1}{i}\mathrm{Ln}\left(z + i\sqrt{1-z^2}\right)} \tag{42}$$

To verify that eq. (41) is a consequence of eq. (42), we employ the notation of eq. (38) to obtain:

$$\begin{aligned}\mathrm{Arcsin}\, z + \mathrm{Arccos}\, z &= \frac{1}{i}\left[\mathrm{Ln}|v| + \mathrm{Ln}\left(\frac{1}{|v|}\right) + i\mathrm{Arg}\, v + i\mathrm{Arg}\left(\frac{i}{v}\right)\right] \\ &= \mathrm{Arg}\ v + \mathrm{Arg}\left(\frac{i}{v}\right)\,. \end{aligned}\tag{43}$$

It is straightforward to check that:

$$\mathrm{Arg}\ v + \mathrm{Arg}\left(\frac{i}{v}\right) = \tfrac{1}{2}\pi\,, \qquad \text{for } \mathrm{Re}\ v \geq 0\,.$$

However in Appendix F, we prove that $\mathrm{Re}\ v \equiv \mathrm{Re}\ (iz + \sqrt{1-z^2}) \geq 0$ for all complex numbers $z$. Hence, eq. (43) yields:

$$\mathrm{Arcsin}\, z + \mathrm{Arccos}\, z = \tfrac{1}{2}\pi\,,$$

as claimed.

We now examine the principal value of the arccosine for real-valued arguments such that $-1 \leq x \leq 1$. Setting $z = x$, where $x$ is real and $|x| \leq 1$,

$$\begin{aligned}\mathrm{Arccos}\, x = \frac{1}{i}\mathrm{Ln}\left(x + i\sqrt{1-x^2}\right) &= \frac{1}{i}\left[\mathrm{Ln}\left|x + i\sqrt{1-x^2}\right| + i\mathrm{Arg}\left(x + i\sqrt{1-x^2}\right)\right] \\ &= \mathrm{Arg}\left(x + i\sqrt{1-x^2}\right)\,, \qquad \text{for } |x| \leq 1\,, \end{aligned}\tag{44}$$

since $x + i\sqrt{1-x^2}$ is a complex number with magnitude equal to 1 when $x$ is real with $|x| \leq 1$. Moreover, $x+i\sqrt{1-x^2}$ lives either in the first or second quadrant of the complex plane, since $\mathrm{Im}(x + i\sqrt{1-x^2}) \geq 0$. It follows that:

$$0 \leq \mathrm{Arccos}\, x \leq \pi\,, \qquad \text{for } |x| \leq 1\,.$$

The principal value of the complex arcsine and arccosine functions are single-valued for all complex $z$. The choice of branch cuts for $\mathrm{Arcsin}\, z$ and $\mathrm{Arccos}\, z$ must coincide in light of eq. (41). Moreover, due to the standard branch cut of the principal value square root function,[2] it follows that $\mathrm{Arcsin}\,z$ is discontinuous when $z = x + iy$ crosses lines on the real axis such that[3]

$$y = 0 \quad \text{and} \quad -\infty < x < -1 \quad \text{and} \quad 1 < x < \infty\,. \tag{45}$$

These two lines comprise the branch cuts of $\mathrm{Arcsin}\,z$ and $\mathrm{Arccos}\,z$; each branch cut ends at a branch point located at $x = -1$ and $x = 1$, respectively (although the square root function is not divergent at these points).[4]

To obtain the relations between the single-valued and multivalued functions, we first notice that the multivalued nature of the logarithms imply that $\arcsin z$ can take on the values $\mathrm{Arcsin}\,z + 2\pi n$ and $\arccos z$ can take on the values $\mathrm{Arccos}\,z + 2\pi n$, where $n$ is any integer. However, we must also take into account the fact that $(1-z^2)^{1/2}$ can take on two values, $\pm\sqrt{1-z^2}$. In particular,

$$\begin{aligned}\arcsin z &= \frac{1}{i}\ln(iz \pm \sqrt{1-z^2}) = \frac{1}{i}\ln\left(\frac{-1}{iz \mp \sqrt{1-z^2}}\right) = \frac{1}{i}\left[\ln(-1) - \ln(iz \mp \sqrt{1-z^2})\right] \\ &= -\frac{1}{i}\ln(iz \mp \sqrt{1-z^2}) + (2n+1)\pi\,,\end{aligned}$$

where $n$ is any integer. Likewise,

$$\arccos z = \frac{1}{i}\ln(z \pm i\sqrt{1-z^2}) = \frac{1}{i}\ln\left(\frac{1}{z \mp i\sqrt{1-z^2}}\right) = -\frac{1}{i}\ln(z \mp i\sqrt{1-z^2}) + 2\pi n\,,$$

where $n$ is any integer. Hence, it follows that

$$\arcsin z = (-1)^n \mathrm{Arcsin}\,z + n\pi\,, \qquad n = 0\,,\ \pm 1\,,\ \pm 2\,,\ \cdots\,, \tag{46}$$

$$\arccos z = \pm \mathrm{Arccos}\,z + 2n\pi\,, \qquad n = 0\,,\ \pm 1\,,\ \pm 2\,,\ \cdots\,, \tag{47}$$

where either $\pm\mathrm{Arccos}\,z$ can be employed to obtain a possible value of $\arccos z$. In particular, the choice of $n = 0$ in eq. (47) implies that:

$$\arccos z = -\arccos z\,, \tag{48}$$

which should be interpreted as a set equality. Note that one can use eqs. (46) and (47) along with eq. (41) to confirm the result obtained in eq. (33).

---

[2] One can check that the branch cut of the Ln function in eq. (36) is never encountered for any finite value of $z$. For example, in the case of $\mathrm{Arcsin}\,z$, the branch cut of Ln can only be reached if $iz + \sqrt{1-z^2}$ is real and negative. But this never happens since if $iz + \sqrt{1-z^2}$ is real then $z = iy$ for some real value of $y$, in which case $iz + \sqrt{1-z^2} = -y + \sqrt{1+y^2} > 0$.

[3] Note that for real $w$, we have $|\sin w| \leq 1$ and $|\cos w| \leq 1$. Hence, for both the functions $w = \mathrm{Arcsin}\,z$ and $w = \mathrm{Arccos}\,z$, it is desirable to choose the branch cuts to lie outside the interval on the real axis where $|\mathrm{Re}\ z| \leq 1$.

[4] The functions Arcsin $z$ and Arccos $z$ also possess a branch point at the point of infinity (which is defined more precisely in footnote 5). This can be verified by demonstrating that $\mathrm{Arcsin}(1/z)$ and $\mathrm{Arccos}(1/z)$ possess a branch point at $z = 0$. For further details, see e.g. Section 58 of Ref 3.

## 6 The inverse hyperbolic functions: arctanh and arccoth

Consider the solution to the equation

$$z = \tanh w = \frac{\sinh w}{\cosh w} = \left(\frac{e^w - e^{-w}}{e^w + e^{-w}}\right) = \left(\frac{e^{2w} - 1}{e^{2w} + 1}\right) .$$

We now solve for $e^{2w}$,

$$z = \frac{e^{2w} - 1}{e^{2w} + 1} \quad \Longrightarrow \quad e^{2w} = \frac{1+z}{1-z} .$$

Taking the complex logarithm of both sides of the equation, we can solve for $w$,

$$w = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right) .$$

The solution to $z = \tanh w$ is $w = \text{arctanh} z$. Hence,

$$\boxed{\text{arctanh}\, z = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right)} \tag{49}$$

Similarly, by considering the solution to the equation

$$z = \coth w = \frac{\cosh w}{\sinh w} = \left(\frac{e^w + e^{-w}}{e^w - e^{-w}}\right) = \left(\frac{e^{2w} + 1}{e^{2w} - 1}\right) .$$

we end up with:

$$\boxed{\text{arccoth} z = \frac{1}{2} \ln\left(\frac{z+1}{z-1}\right)} \tag{50}$$

The above results then yield:

$$\text{arccoth}(z) = \text{arctanh}\left(\frac{1}{z}\right) ,$$

as a set equality.

Finally, we note the relation between the inverse trigonometric and the inverse hyperbolic functions:

$$\text{arctanh}\, z = i \arctan(-iz) ,$$
$$\text{arccoth}\, z = i\, \text{arccot}(iz) .$$

As in the discussion at the end of Section 1, one can rewrite eqs. (49) and (50) in an equivalent form:

$$\text{arctanh}\, z = \tfrac{1}{2}\left[\ln(1+z) - \ln(1-z)\right] , \tag{51}$$

$$\text{arccoth}\, z = \tfrac{1}{2}\left[\ln\left(1 + \frac{1}{z}\right) - \ln\left(1 - \frac{1}{z}\right)\right] . \tag{52}$$

## 7 The principal values Arctanh and Arccoth

Mathematica defines the principal values of the inverse hyperbolic tangent and inverse hyperbolic cotangent, Arctanh and Arccoth, by employing the principal value of the complex logarithms in eqs. (51) and (52). We can define the principal value of the inverse hyperbolic tangent function by employing the principal value of the logarithm,

$$\boxed{\text{Arctanh}\, z = \tfrac{1}{2}\left[\text{Ln}(1+z) - \text{Ln}(1-z)\right]} \tag{53}$$

and

$$\boxed{\text{Arccoth}\, z = \text{Arctanh}\left(\frac{1}{z}\right) = \tfrac{1}{2}\left[\text{Ln}\left(1+\frac{1}{z}\right) - \text{Ln}\left(1-\frac{1}{z}\right)\right]} \tag{54}$$

Note that the branch points at $z = \pm 1$ are excluded from the above definitions, as $\text{Arctanh}\, z$ and $\text{Arccoth}\, z$ are divergent at these two points. The definition of the principal value of the inverse hyperbolic cotangent given in eq. (54) is deficient in one respect since it is not well-defined at $z = 0$. For this special case, Mathematica defines

$$\text{Arccoth}(0) = \tfrac{1}{2} i\pi\,. \tag{55}$$

Of course, this discussion parallels that of Section 2. Moreover, alternative definitions of $\text{Arctanh}\, z$ and $\text{Arccoth}\, z$ analogous to those defined in Appendix C for the corresponding inverse trigonometric functions can be found in Refs. 1, 2 and 5. There is no need to repeat the analysis of Section 2 since a comparison of eqs. (12) and (13) with eqs. (53) and (54) shows that the inverse trigonometric and inverse hyperbolic tangent and cotangent functions are related by:

$$\text{Arctanh}\, z = i\text{Arctan}(-iz)\,, \tag{56}$$

$$\text{Arccoth}\, z = i\,\text{Arccot}(iz)\,. \tag{57}$$

Using these results, all other properties of the inverse hyperbolic tangent and cotangent functions can be easily derived from the properties of the corresponding arctangent and arccotangent functions.

For example the branch cuts of these functions are easily obtained from eqs. (15) and (17). $\text{Arctanh}\, z$ is discontinuous when $z = x + iy$ crosses the branch cuts located on the real axis such that[5]

$$y = 0 \quad \text{and} \quad -\infty < x < -1 \quad \text{and} \quad 1 < x < \infty\,. \tag{58}$$

$\text{Arccoth}\, z$ is discontinuous when $z = x + iy$ crosses the branch cuts located on the real axis such that

$$y = 0 \quad \text{and} \quad -1 < x < 1\,. \tag{59}$$

[5]Note that for real $w$, we have $|\tanh w| \leq 1$ and $|\coth w| \geq 1$. Hence, for $w = \text{Arctanh}\, z$ it is desirable to choose the branch cut to lie outside the interval on the real axis where $|\text{Re}\ z| \leq 1$. Likewise, for $w = \text{Arccoth}\, z$ it is desirable to choose the branch cut to lie outside the interval on the real axis where $|\text{Re}\ z| \geq 1$.

The relations between the single-valued and multivalued functions can be summarized by:

$$\begin{aligned}
\text{arctanh} z &= \text{Arctanh}\, z + in\pi\,, \qquad n = 0\,,\ \pm 1\,,\ \pm 2\,,\ \cdots\,, \\
\text{arccoth}\, z &= \text{Arccoth}\, z + in\pi\,, \qquad n = 0\,,\ \pm 1\,,\ \pm 2\,,\ \cdots\,.
\end{aligned}$$

## 8 The inverse hyperbolic functions: arcsinh and arccosh

The inverse hyperbolic sine function is the solution to the equation:

$$z = \sinh w = \frac{e^w - e^{-w}}{2}\,.$$

Letting $v \equiv e^w$ , we solve the equation

$$v - \frac{1}{v} = 2z\,.$$

Multiplying by $v$, one obtains a quadratic equation for $v$,

$$v^2 - 2zv - 1 = 0\,. \tag{60}$$

The solution to eq. (60) is:

$$v = z + (1 + z^2)^{1/2}\,. \tag{61}$$

Since $z$ is a complex variable, $(1 + z^2)^{1/2}$ is the complex square-root function. This is a multivalued function with two possible values that differ by an overall minus sign. Hence, we do not explicitly write out the $\pm$ sign in eq. (61). To avoid ambiguity, we shall write

$$\begin{aligned}
v &= z + (1 + z^2)^{1/2} = z + e^{\frac{1}{2}\ln(1+z^2)} = z + e^{\frac{1}{2}\left[\text{Ln}|1+z^2| + i\arg(1+z^2)\right]} \\
&= z + |1 + z^2|^{1/2} e^{\frac{i}{2}\arg(1+z^2)}\,.
\end{aligned}$$

By definition, $v \equiv e^w$, from which it follows that

$$w = \ln v = \ln\left(z + |1 + z^2|^{1/2} e^{\frac{i}{2}\arg(1+z^2)}\right)\,.$$

The solution to $z = \sinh w$ is $w = \text{arcsinh} z$. Hence,

$$\boxed{\text{arcsinh}\, z = \ln\left(z + |1 + z^2|^{1/2} e^{\frac{i}{2}\arg(1+z^2)}\right)} \tag{62}$$

The inverse hyperbolic cosine function is the solution to the equation:

$$z = \cos w = \frac{e^w + e^{-w}}{2}\,.$$

Letting $v \equiv e^w$ , we solve the equation

$$v + \frac{1}{v} = 2z\,.$$

Multiplying by $v$, one obtains a quadratic equation for $v$,

$$v^2 - 2zv + 1 = 0\,. \tag{63}$$

The solution to eq. (63) is:

$$v = z + (z^2-1)^{1/2}\,.$$

Following the same steps as in the analysis of inverse hyperbolic sine function, we write

$$w = \operatorname{arccosh} z = \ln v = \ln\left[z + (z^2-1)^{1/2}\right]\,, \tag{64}$$

where $(z^2-1)^{1/2}$ is the multivalued square root function. More explicitly,

$$\boxed{\operatorname{arccosh} z = \ln\left(z + |z^2-1|^{1/2} e^{\frac{i}{2}\arg(z^2-1)}\right)}$$

The multivalued square root function satisfies:

$$(z^2-1)^{1/2} = (z+1)^{1/2}(z-1)^{1/2}\,.$$

Hence, an equivalent form for the multivalued inverse hyperbolic cosine function is:

$$\operatorname{arccosh} z = \ln\left[z + (z+1)^{1/2}(z-1)^{1/2}\right]\,,$$

or equivalently,

$$\operatorname{arccosh} z = \ln\left(z + |z^2-1|^{1/2} e^{\frac{i}{2}\arg(z+1)} e^{\frac{i}{2}\arg(z-1)}\right)\,. \tag{65}$$

Finally, we note the relations between the inverse trigonometric and the inverse hyperbolic functions:

$$\operatorname{arcsinh} z = i \arcsin(-iz)\,, \tag{66}$$

$$\operatorname{arccosh} z = \pm i \,\arccos z\,, \tag{67}$$

where the equalities in eqs. (66) and (67) are interpreted as set equalities for the multivalued functions. The $\pm$ in eq. (67) indicates that both signs are employed in determining the members of the set of all possible $\operatorname{arccosh} z$ values. In deriving eq. (67), we have employed eqs. (29) and (64). In particular, the origin of the two possible signs in eq. (67) is a consequence of eq. (48) [and its hyperbolic analog, eq. (76)].

## 9 The principal values Arcsinh and Arccosh

The principal value of the inverse hyperbolic sine function, $\text{Arcsinh}\, z$, is defined by Mathematica 8 by replacing the complex logarithm and argument functions of eq. (62) by their principal values. That is,

$$\boxed{\text{Arcsinh}\, z = \text{Ln}\left(z + \sqrt{1+z^2}\right)} \tag{68}$$

For the principal value of the inverse hyperbolic cosine function $\text{Arccosh}\, z$, Mathematica chooses eq. (65) with the complex logarithm and argument functions replaced by their principal values. That is,

$$\boxed{\text{Arccosh}\, z = \text{Ln}\left(z + \sqrt{z+1}\sqrt{z-1}\,\right)} \tag{69}$$

In eqs. (68) and (69), the principal values of the square root functions are employed following the notation of eq. (35).

The relation between the principal values of the inverse trigonometric and the inverse hyperbolic sine functions is given by

$$\text{Arcsinh}\, z = i\text{Arcsin}(-iz)\,, \tag{70}$$

as one might expect in light of eq. (66). A comparison of eqs. (42) and (69) reveals that

$$\text{Arccosh}\, z = \begin{cases} i\text{Arccos}\, z\,, & \text{for either Im } z > 0 \text{ or for Im } z = 0 \text{ and Re } z \leq 1\,, \\ -i\text{Arccos}\, z\,, & \text{for either Im } z < 0 \text{ or for Im } z = 0 \text{ and Re } z \geq 1\,. \end{cases} \tag{71}$$

The existence of two possible signs in eq. (71) is not surprising in light of the $\pm$ that appears in eq. (67). Note that either choice of sign is valid in the case of Im $z = 0$ and Re $z = 1$, since for this special point, $\text{Arccosh}(1) = \text{Arccos}(1) = 0$. For a derivation of eq. (71), see Appendix F.

The principal value of the inverse hyperbolic sine and cosine functions are single-valued for all complex $z$. Moreover, due to the branch cut of the principal value square root function,[6] it follows that $\text{Arcsinh}\, z$ is discontinuous when $z = x + iy$ crosses lines on the imaginary axis such that

$$x = 0 \quad \text{and} \quad -\infty < y < -1 \quad \text{and} \quad 1 < y < \infty\,. \tag{72}$$

These two lines comprise the branch cuts of $\text{Arcsinh}\, z$, and each branch cut ends at a branch point located at $z = -i$ and $z = i$, respectively, due to the square root function in

[6] One can check that the branch cut of the Ln function in eq. (68) is never encountered for any value of $z$. In particular, the branch cut of Ln can only be reached if $z + \sqrt{1+z^2}$ is real and negative. But this never happens since if $z + \sqrt{1+z^2}$ is real then $z$ is also real. But for any real value of $z$, we have $z + \sqrt{1+z^2} > 0$.

eq. (68), although the square root function is not divergent at these points. The function Arcsinh $z$ also possesses a branch point at the point of infinity, which can be verified by examining the behavior of Arcsinh$(1/z)$ at the point $z = 0$.[7]

The branch cut for Arccosh $z$ derives from the standard branch cuts of the square root function and the branch cut of the complex logarithm. In particular, for real $z$ satisfying $|z| < 1$, we have a branch cut due to $(z+1)^{1/2}(z-1)^{1/2}$, whereas for real $z$ satisfying $-\infty < z \leq -1$, the branch cut of the complex logarithm takes over. Hence, it follows that Arccosh $z$ is discontinuous when $z = x + iy$ crosses lines on the real axis such that[8]

$$y = 0 \quad \text{and} \quad -\infty < x < 1\,. \tag{73}$$

In particular, there are branch points at $z = \pm 1$ due to the square root functions in eq. (69) and a branch point at the point of infinity due to the logarithm [cf. footnote 5]. As a result, eq. (73) actually represents two branch cuts made up of a branch cut from $z = 1$ to $z = -1$ followed by a second branch cut from $z = -1$ to the point of infinity.[9]

The relations between the single-valued and multivalued functions can be obtained by following the same steps used to derive eqs. (46) and (47). Alternatively, we can make use of these results along with those of eqs. (66), (67), (70) and (71). The end result is:

$$\text{arcsinh}\, z = (-1)^n \text{Arcsinh}\, z + in\pi\,, \qquad n = 0\,, \pm 1\,, \pm 2\,, \cdots\,, \tag{74}$$

$$\text{arccosh}\, z = \pm \text{Arccosh}\, z + 2in\pi\,, \qquad n = 0\,, \pm 1\,, \pm 2\,, \cdots\,, \tag{75}$$

where either $\pm$Arccosh $z$ can be employed to obtain a possible value of arccosh $z$. In particular, the choice of $n = 0$ in eq. (75) implies that:

$$\text{arccosh}\, z = -\text{arccosh}\, z\,, \tag{76}$$

which should be interpreted as a set equality.

This completes our survey of the multivalued complex inverse trigonometric and hyperbolic functions and their single-valued principal values.

---

[7]In the complex plane, the behavior of the complex function $F(z)$ at the point of infinity, $z = \infty$, corresponds to the behavior of $F(1/z)$ at the origin of the complex plane, $z = 0$ [cf. footnote 3]. Since the argument of the complex number 0 is undefined, the argument of the point of infinity is likewise undefined. This means that the point of infinity (sometimes called *complex infinity*) actually corresponds to $|z| = \infty$, independently of the direction in which infinity is approached in the complex plane. Geometrically, the complex plane plus the point of infinity can be mapped onto a surface of a sphere by stereographic projection. Place the sphere on top of the complex plane such that the origin of the complex plane coincides with the south pole. Consider a straight line from any complex number in the complex plane to the north pole. Before it reaches the north pole, this line intersects the surface of the sphere at a unique point. Thus, every complex number in the complex plane is uniquely associated with a point on the surface of the sphere. In particular, the north pole itself corresponds to complex infinity. For further details, see Chapter 5 of Ref. 3.

[8]Note that for real $w$, we have $\cosh w \geq 1$. Hence, for $w =$ Arccosh $z$ it is desirable to choose the branch cut to lie outside the interval on the real axis where Re $z \geq 1$.

[9]Given that the branch cuts of Arccosh $z$ and $i$Arccos $z$ are different, it is not surprising that the relation Arccosh $z = i$Arccos $z$ cannot be respected for all complex numbers $z$.

# APPENDIX A: The argument of a complex number

In this Appendix, we examine the *argument* of a non-zero complex number $z$. Any complex number can be written as, we write:

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}\,, \tag{A.1}$$

where $x = \text{Re } z$ and $y = \text{Im } z$ are real numbers. The *complex conjugate* of $z$, denoted by $z^*$, is given by

$$z^* = x - iy = re^{-i\theta}\,.$$

The argument of $z$ is denoted by $\theta$, which is measured in radians. However, there is an ambiguity in definition of the argument. The problem is that

$$\sin(\theta + 2\pi) = \sin\theta\,, \qquad \cos(\theta + 2\pi) = \cos\theta\,,$$

since the sine and the cosine are periodic functions of $\theta$ with period $2\pi$. Thus $\theta$ is defined only up to an additive integer multiple of $2\pi$. It is common practice to establish a convention in which $\theta$ is defined to lie within an interval of length $2\pi$. This defines the so-called *principal value* of the argument, which we denote by $\theta \equiv \text{Arg } z$ (note the upper case A). The most common convention, which we adopt in these notes, is to define the single-valued function Arg $z$ such that:

$$-\pi < \text{Arg } z \leq \pi\,. \tag{A.2}$$

In many applications, it is convenient to define a multivalued argument function,

$$\arg z \equiv \text{Arg } z + 2\pi n = \theta + 2\pi n\,, \qquad n = 0, \pm 1, \pm 2, \pm 3, \ldots\,. \tag{A.3}$$

This is a multivalued function because for a given complex number $z$, the number $\arg z$ represents an infinite number of possible values.

### A.1. The principal value of the argument function

Any non-zero complex number $z$ can be written in polar form

$$z = |z|e^{i\arg z}\,, \tag{A.4}$$

where $\arg z$ is a multivalued function defined in eq. (A.3) It is convenient to have an explicit formula for Arg $z$ in terms of $\arg z$. First, we introduce some notation: $[x]$ means the largest integer less than or equal to the real number $x$. That is, $[x]$ is the unique integer that satisfies the inequality

$$x - 1 < [x] \leq x\,, \qquad \text{for real } x \text{ and integer } [x]\,. \tag{A.5}$$

For example, $[1.5] = [1] = 1$ and $[-0.5] = -1$. With this notation, one can write Arg $z$ in terms of $\arg z$ as follows:

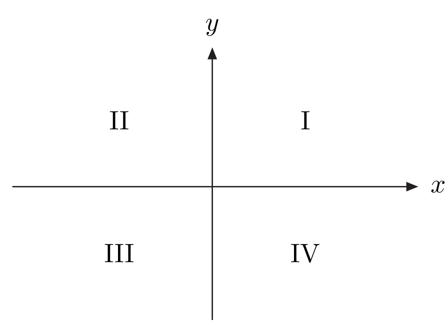
$$\text{Arg } z = \arg z + 2\pi \left[ \frac{1}{2} - \frac{\arg z}{2\pi} \right] , \tag{A.6}$$

where [ ] denotes the bracket (or greatest integer) function introduced above. It is straightforward to check that Arg $z$ as defined by eq. (A.6) does indeed fall inside the principal interval, $-\pi < \theta \leq \pi$.

A more useful equation for Arg $z$ can be obtained as follows. Using the polar representation of $z = x + iy$ given in eq. (A.1), it follows that $x = r\cos\theta$ and $y = r\sin\theta$. From these two results, one easily derives,

$$|z| = r = \sqrt{x^2 + y^2}\,, \qquad \tan\theta = \frac{y}{x}\,. \tag{A.7}$$

We identify $\theta =$ Arg $z$ in the convention where $-\pi < \theta \leq \pi$. In light of eq. (A.7), it is tempting to identify Arg $z$ with $\arctan(y/x)$. However, the real function $\arctan x$ is a multivalued function for real values of $x$. It is conventional to introduce a single-valued real arctangent function, called the principal value of the arctangent,[10] which is denoted by Arctan $x$ and satisfies $-\frac{1}{2}\pi \leq \text{Arctan } x \leq \frac{1}{2}\pi$. Since $-\pi < \text{Arg } z \leq \pi$, it follows that Arg $z$ *cannot* be identified with Arctan$(y/x)$ in all regions of the complex plane. The correct relation between these two quantities is easily ascertained by considering the four quadrants of the complex plane separately. The quadrants of the complex plane (called regions I, II, III and IV) are illustrated in the figure below:




The principal value of the argument of $z = x + iy$ is given in Table 1, depending on in which of the four quadrants of the complex plane $z$ resides. Note that the relation Arg $z =$ Arctan$(y/x)$ is valid only in quadrants I and IV. If $z$ resides in quadrant II then $y/x < 0$, in which case $-\frac{1}{2}\pi < \text{Arctan}(y/x) < 0$. Thus if $z$ lies in quadrant II, then one must add $\pi$ to Arctan$(y/x)$ to ensure that $\frac{1}{2}\pi < \text{Arg } z < \pi$. Likewise, if $z$ resides in quadrant III then $y/x > 0$, in which case $0 < \text{Arctan}(y/x) < \frac{1}{2}\pi$. Thus if $z$ lies in quadrant III, then one must subtract $\pi$ from Arctan$(y/x)$ in order to ensure that $-\pi < \text{Arg } z < -\frac{1}{2}\pi$.

[10] In defining the principal value of the arctangent, we follow the conventions of Keith B. Oldham, Jan Myland and Jerome Spanier, *An Atlas of Functions* (Springer Science, New York, 2009), Chapter 35.

Table 1: Formulae for the argument of a complex number $z = x + iy$. The range of $\text{Arg}\, z$ is indicated for each of the four quadrants of the complex plane. For example, in quadrant I, the notation $(0, \frac{1}{2}\pi)$ means that $0 < \text{Arg}\, z < \frac{1}{2}\pi$, etc. By convention, the principal value of the real arctangent function lies in the range $-\frac{1}{2}\pi \leq \text{Arctan}(y/x) \leq \frac{1}{2}\pi$.

| Quadrant | Sign of $x$ and $y$ | range of $\text{Arg}\, z$ | $\text{Arg}\, z$ |
|---|---|---|---|
| I | $x > 0\,,\ y > 0$ | $(0, \frac{1}{2}\pi)$ | $\text{Arctan}(y/x)$ |
| II | $x < 0\,,\ y > 0$ | $(\frac{1}{2}\pi, \pi)$ | $\pi + \text{Arctan}(y/x)$ |
| III | $x < 0\,,\ y < 0$ | $(-\pi, -\frac{1}{2}\pi)$ | $-\pi + \text{Arctan}(y/x)$ |
| IV | $x > 0\,,\ y < 0$ | $(-\frac{1}{2}\pi, 0)$ | $\text{Arctan}(y/x)$ |

Table 2: Formulae for the argument of a complex number $z = x + iy$ when $z$ is real or pure imaginary. By convention, the principal value of the argument satisfies $-\pi < \text{Arg}\ z \leq \pi$.

| Quadrant border | type of complex number $z$ | Conditions on $x$ and $y$ | Arg $z$ |
|---|---|---|---|
| IV/I | real and positive | $x > 0\,,\ y = 0$ | $0$ |
| I/II | pure imaginary with Im $z > 0$ | $x = 0\,,\ y > 0$ | $\frac{1}{2}\pi$ |
| II/III | real and negative | $x < 0\,,\ y = 0$ | $\pi$ |
| III/IV | pure imaginary with Im $z < 0$ | $x = 0\,,\ y < 0$ | $-\frac{1}{2}\pi$ |
| origin | zero | $x = y = 0$ | undefined |

Cases where $z$ lies on the border between two adjacent quadrants are considered separately in Table 2. To derive these results, note that for the borderline cases, the principal value of the arctangent is given by

$$\text{Arctan}(y/x) = \begin{cases} 0\,, & \text{if } y = 0 \text{ and } x \neq 0\,, \\ \frac{1}{2}\pi\,, & \text{if } x = 0 \text{ and } y > 0\,, \\ -\frac{1}{2}\pi\,, & \text{if } x = 0 \text{ and } y < 0\,, \\ \text{undefined}\,, & \text{if } x = y = 0\,. \end{cases}$$

In particular, note that the argument of zero is undefined. Since $z = 0$ if and only if $|z| = 0$, eq. (A.4) remains valid despite the fact that $\arg 0$ is not defined. When studying the properties of $\arg\, z$ and $\text{Arg}\, z$ below, we shall always assume implicitly that $z \neq 0$.

## 2. Properties of the multivalued argument function

We can view a multivalued function $f(z)$ evaluated at $z$ as a set of values, where each element of the set corresponds to a different choice of some integer $n$. For example, given the multivalued function $\arg z$ whose principal value is $\text{Arg } z \equiv \theta$, then $\arg z$ consists of the set of values:

$$\arg z = \{\theta\,,\ \theta + 2\pi\,,\ \theta - 2\pi\,,\ \theta + 4\pi\,,\ \theta - 4\pi\,,\ \cdots\}\,. \tag{A.8}$$

Consider the case of two multivalued functions of the form, $f(z) = F(z) + 2\pi n$ and $g(z) = G(z) + 2\pi n$, where $F(z)$ and $G(z)$ are the principal values of $f(z)$ and $g(z)$ respectively. Then, $f(z) = g(z)$ if and only if for each point $z$, the corresponding set of values of $f(z)$ and $g(z)$ precisely coincide:

$$\{F(z)\,,\ F(z) + 2\pi\,,\ F(z) - 2\pi\,,\ \cdots\} = \{G(z)\,,\ G(z) + 2\pi\,,\ G(z) - 2\pi\,,\ \cdots\}\,. \tag{A.9}$$

Sometimes, one refers to the equation $f(z) = g(z)$ as a *set equality* since all the distinct elements of the two sets in eq. (A.9) must coincide. We add two additional rules to the concept of set equality. First, the ordering of terms within the set is unimportant. Second, we only care about the distinct elements of each set. That is, if our list of set elements has repeated entries, we omit all duplicate elements.

To see how the set equality of two multivalued functions works, let us consider the multivalued function $\arg z$. One can prove that:

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2\,, \qquad \text{for } z_1,\ z_2 \neq 0, \tag{A.10}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2\,, \qquad \text{for } z_1,\ z_2 \neq 0, \tag{A.11}$$

$$\arg\left(\frac{1}{z}\right) = \arg z^* = -\arg z\,, \qquad \text{for } z \neq 0. \tag{A.12}$$

To prove eq. (A.10), consider $z_1 = |z_1|e^{i\arg z_1}$ and $z_2 = |z_2|e^{i\arg z_2}$. The arguments of these two complex numbers are: $\arg z_1 = \text{Arg } z_1 + 2\pi n_1$ and $\arg z_2 = \text{Arg } z_2 + 2\pi n_2$, where $n_1$ and $n_2$ are arbitrary integers. [One can also write $\arg z_1$ and $\arg z_2$ in set notation as in eq. (A.8).] Thus, one can also write $z_1 = |z_1|e^{i\text{Arg } z_1}$ and $z_2 = |z_2|e^{i\text{Arg } z_2}$, since $e^{2\pi i n} = 1$ for any integer $n$. It then follows that

$$z_1 z_2 = |z_1 z_2| e^{i(\text{Arg } z_1 + \text{Arg } z_2)}\,,$$

where we have used $|z_1||z_2| = |z_1 z_2|$. Thus, $\arg(z_1 z_2) = \text{Arg } z_1 + \text{Arg } z_2 + 2\pi n_{12}$, where $n_{12}$ is also an arbitrary integer. Therefore, we have established that:

$$\arg z_1 + \arg z_2 = \text{Arg } z_1 + \text{Arg } z_2 + 2\pi(n_1 + n_2)\,,$$

$$\arg(z_1 z_2) = \text{Arg } z_1 + \text{Arg } z_2 + 2\pi n_{12}\,,$$

where $n_1$, $n_2$ and $n_{12}$ are arbitrary integers. Thus, $\arg z_1 + \arg z_2$ and $\arg(z_1 z_2)$ coincide as sets, and so eq. (A.10) is confirmed. One can easily prove eqs. (A.11) and (A.12) by a similar method. In particular, if one writes $z = |z|e^{i\arg z}$ and employs the definition of the complex conjugate (which yields $z^* = |z|e^{-i\arg z}$ and $|z^*| = |z|$), then it follows that $\arg(1/z) = \arg z^* = -\arg z$. As an instructive example, consider the last relation in the case of $z = -1$. It then follows that

$$\arg(-1) = -\arg(-1)\,,$$

as a set equality. This is not paradoxical, since the sets,

$$\arg(-1) = \{\pm\pi\,,\, \pm 3\pi\,,\, \pm 5\pi\,,\, \ldots\} \qquad \text{and} \qquad -\arg(-1) = \{\mp\pi\,,\, \mp 3\pi\,,\, \mp 5\pi\,,\, \ldots\}\,,$$

coincide, as they possess precisely the same list of elements.

Now, for a little surprise:

$$\arg z^2 \neq 2\arg z\,. \tag{A.13}$$

To see why this statement is surprising, consider the following false proof. Use eq. (A.10) with $z_1 = z_2 = z$ to derive:

$$\arg z^2 = \arg z + \arg z \stackrel{?}{=} 2\arg z\,, \qquad \text{[FALSE!!]}. \tag{A.14}$$

The false step is the one indicated by the symbol $\stackrel{?}{=}$ above. Given $z = |z|e^{i\arg z}$, one finds that $z^2 = |z|^2 e^{2i(\text{Arg } z + 2\pi n)} = |z|^2 e^{2i\text{Arg } z}$, and so the possible values of $\arg(z^2)$ are:

$$\arg(z^2) = \{2\text{Arg } z\,,\, 2\text{Arg } z + 2\pi\,,\, 2\text{Arg } z - 2\pi\,,\, 2\text{Arg } z + 4\pi\,,\, 2\text{Arg } z - 4\pi\,,\, \cdots\}\,,$$

whereas the possible values of $2\arg z$ are:

$$\begin{aligned} 2\arg z &= \{2\text{Arg } z\,,\, 2(\text{Arg } z + 2\pi)\,,\, 2(\text{Arg } z - 2\pi)\,,\, 2(\text{Arg } z + 4\pi)\,,\, \cdots\} \\ &= \{2\text{Arg } z\,,\, 2\text{Arg } z + 4\pi\,,\, 2\text{Arg } z - 4\pi\,,\, 2\text{Arg } z + 8\pi\,,\, 2\text{Arg } z - 8\pi\,,\, \cdots\}\,. \end{aligned}$$

Thus, $2\arg z$ is a *subset* of $\arg(z^2)$, but half the elements of $\arg(z^2)$ are missing from $2\arg z$. These are therefore unequal sets, as indicated by eq. (A.13). Now, you should be able to see what is wrong with the statement:

$$\arg z + \arg z \stackrel{?}{=} 2\arg z\,. \tag{A.15}$$

When you add $\arg\ z$ as a set to itself, the element you choose from the first $\arg z$ need not be the same as the element you choose from the second $\arg z$. In contrast, $2\arg z$ means take the set $\arg z$ and multiply each element by two. The end result is that $2\arg z$ contains only half the elements of $\arg z + \arg z$ as shown above. Similarly, for any non-negative integer $n = 2, 3, 4, \ldots$,

$$\arg z^n = \underbrace{\arg z + \arg z + \cdots \arg z}_{n} \neq n\ \arg z\,. \tag{A.16}$$

In light of eq. (A.12), if we replace $z$ with $z^*$ above, we obtain the following generalization of eq. (A.16),

$$\arg z^n \neq n \arg z\,, \qquad \text{for any integer } n \neq 0\,,\ \pm 1\,. \tag{A.17}$$

Here is one more example of an incorrect proof. Consider eq. (A.11) with $z_1 = z_2 \equiv z$. Then, you might be tempted to write:

$$\arg\left(\frac{z}{z}\right) = \arg(1) = \arg z - \arg z \stackrel{?}{=} 0\,.$$

This is clearly wrong since $\arg(1) = 2\pi n$, where $n$ is the set of integers. Again, the error occurs with the step:

$$\arg z - \arg z \stackrel{?}{=} 0\,. \tag{A.18}$$

The fallacy of this statement is the same as above. When you subtract $\arg z$ as a set from itself, the element you choose from the first $\arg z$ need not be the same as the element you choose from the second $\arg z$.

### 3. Properties of the principal value of the argument

The properties of the principal value Arg $z$ are not as simple as those given in eqs. (A.10)–(A.12), since the range of Arg $z$ is restricted to lie within the principal range $-\pi < \text{Arg } z \leq \pi$. Instead, the following relations are satisfied, assuming $z_1,\ z_2 \neq 0$,

$$\text{Arg }(z_1 z_2) = \text{Arg } z_1 + \text{Arg } z_2 + 2\pi N_+\,, \tag{A.19}$$

$$\text{Arg }(z_1/z_2) = \text{Arg } z_1 - \text{Arg } z_2 + 2\pi N_-\,, \tag{A.20}$$

where the integers $N_\pm$ are determined as follows:

$$N_\pm = \begin{cases} -1\,, & \text{if} \quad \text{Arg } z_1 \pm \text{Arg } z_2 > \pi\,, \\ \ \ 0\,, & \text{if} \quad -\pi < \text{Arg } z_1 \pm \text{Arg } z_2 \leq \pi\,, \\ \ \ 1\,, & \text{if} \quad \text{Arg } z_1 \pm \text{Arg } z_2 \leq -\pi\,. \end{cases} \tag{A.21}$$

Eq. (A.21) is really two separate equations for $N_+$ and $N_-$, respectively. To obtain the equation for $N_+$, one replaces the $\pm$ sign with a plus sign wherever it appears on the right hand side of eq. (A.21). To obtain the equation for $N_-$, one replaces the $\pm$ sign with a minus sign wherever it appears on the right hand side of eq. (A.21).

If we set $z_1 = 1$ in eq. (A.20), we find that

$$\text{Arg}(1/z) = \text{Arg } z^* = \begin{cases} \ \ \text{Arg } z\,, & \text{if Im } z = 0 \ \text{ and } z \neq 0\,, \\ -\text{Arg } z\,, & \text{if Im } z \neq 0\,. \end{cases} \tag{A.22}$$

Note that for $z$ real, both $1/z$ and $z^*$ are also real so that in this case $z = z^*$ and $\text{Arg}(1/z) = \text{Arg } z^* = \text{Arg } z$.

Finally,

$$\text{Arg}(z^n) = n\text{Arg } z + 2\pi N_n\,, \tag{A.23}$$

where the integer $N_n$ is given by:

$$N_n = \left[\frac{1}{2} - \frac{n}{2\pi}\text{Arg } z\right]\,, \tag{A.24}$$

and [ ] is the greatest integer bracket function introduced in eq. (A.5). It is straightforward to verify eqs. (A.19)–(A.22) and eq. (A.23). These formulae follow from the corresponding properties of $\arg z$, taking into account the requirement that Arg $z$ must lie within the principal interval, $-\pi < \theta \leq \pi$.

# APPENDIX B: The complex logarithm and its principal value

## B.1. Definition of the complex logarithm

In order to define the complex logarithm, one must solve the complex equation:

$$z = e^w\,, \tag{B.1}$$

for $w$, where $z$ is any non-zero complex number. In eq. (B.1), the complex exponential function is defined via its power series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}\,,$$

where $z$ is any complex number. Using this power series definition, one can verify that:

$$e^{z_1+z_2} = e^{z_1}e^{z_2}\,, \qquad \text{for all complex } z_1 \text{ and } z_2\,. \tag{B.2}$$

In particular, if $z = x + iy$ where $x$ and $y$ are real, then it follows that

$$e^z = e^{x+iy} = e^x\, e^{iy} = e^x(\cos y + i\sin y)\,.$$

If we write $w = u + iv$, then eq. (B.1) can be written as

$$e^u e^{iv} = |z|e^{i\arg z}\,. \tag{B.3}$$

Eq. (B.3) implies that:

$$|z| = e^u\,, \qquad v = \arg z\,.$$

The equation $|z| = e^u$ is a real equation, so we can write $u = \ln|z|$, where $\ln|z|$ is the ordinary logarithm evaluated with positive real number arguments. Thus,

$$w = u + iv = \ln|z| + i\arg z = \ln|z| + i(\text{Arg } z + 2\pi n)\,, \tag{B.4}$$

where $n$ is an integer. We call $w$ the complex logarithm and write $w = \ln z$. This is a somewhat awkward notation since in eq. (B.4) we have already used the symbol ln for the real logarithm. We shall finesse this notational quandary by denoting the real logarithm in eq. (B.4) by the symbol Ln. That is, $\mathrm{Ln}|z|$ shall denote the ordinary real logarithm of $|z|$. With this notational convention, we rewrite eq. (B.4) as:

$$\ln z = \mathrm{Ln}|z| + i \arg z = \mathrm{Ln}|z| + i(\mathrm{Arg}\ z + 2\pi n)\,, \quad n = 0\,, \pm 1\,, \pm 2\,, \pm 3\,, \ldots\,, \tag{B.5}$$

for any non-zero complex number $z$.

Clearly, $\ln z$ is a multivalued function (as its value depends on the integer $n$). It is useful to define a single-valued function *complex* function, $\mathrm{Ln}\, z$, called the principal value of $\ln z$ as follows:

$$\mathrm{Ln}\, z = \mathrm{Ln}\,|z| + i\mathrm{Arg}\ z\,, \qquad -\pi < \mathrm{Arg}\ z \leq \pi\,, \tag{B.6}$$

which extends the definition of Ln $z$ to the entire complex plane (excluding the origin, $z = 0$, where the logarithmic function is singular). In particular, eq. (B.6) implies that $\mathrm{Ln}(-1) = i\pi$. Note that for real positive $z$, we have $\mathrm{Arg}\ z = 0$, so that eq. (B.6) simply reduces to the usual real logarithmic function in this limit. The relation between $\ln z$ and its principal value is simple:

$$\ln z = \mathrm{Ln}\ z + 2\pi i n\,, \quad n = 0\,, \pm 1\,, \pm 2\,, \pm 3\,, \ldots\,.$$

## B.2. Properties of the real and complex logarithm

The properties of the real logarithmic function (whose argument is a positive real number) are well known:

$$e^{\ln x} = x\,, \tag{B.7}$$

$$\ln(e^a) = a\,, \tag{B.8}$$

$$\ln(xy) = \ln(x) + \ln(y)\,, \tag{B.9}$$

$$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)\,, \tag{B.10}$$

$$\ln\left(\frac{1}{x}\right) = -\ln(x)\,, \tag{B.11}$$

for positive real numbers $x$ and $y$ and arbitrary real number $a$. Repeated use of eqs. (B.9) and (B.10) then yields

$$\ln x^n = n \ln x\,, \qquad \text{for arbitrary integer } n\,. \tag{B.12}$$

We now consider which of the properties given in eqs. (B.7)–(B.12) apply to the complex logarithm. Since we have defined the multi-value function $\ln z$ and the single-valued function $\text{Ln}\ z$, we should examine the properties of both these functions. We begin with the multivalued function $\ln z$. First, we examine eq. (B.7). Using eq. (B.5), it follows that:

$$e^{\ln z} = e^{\text{Ln}|z|} e^{i\text{Arg }z} e^{2\pi i n} = |z| e^{i\text{Arg }z} = z \,. \tag{B.13}$$

Thus, eq. (B.7) is satisfied. Next, we examine eq. (B.8) for $z = x + iy$:

$$\ln(e^z) = \text{Ln}|e^z| + i(\arg\ e^z) = \text{Ln}(e^x) + i(y + 2\pi k) = x + iy + 2\pi i k = z + 2\pi i k \,,$$

where $k$ is an arbitrary integer. In deriving this result, we used the fact that $e^z = e^x e^{iy}$, which implies that $\arg(e^z) = y + 2\pi k$.[11] Thus,

$$\ln(e^z) = z + 2\pi i k \neq z \,, \qquad \text{unless } k = 0 \,. \tag{B.14}$$

This is not surprising, since $\ln(e^z)$ is a multivalued function, which cannot be equal to the single-valued function $z$. Indeed eq. (B.8) is false for the multivalued complex logarithm.

As a check, let us compute $\ln(e^{\ln z})$ in two different ways. First, using eq. (B.13), it follows that $\ln(e^{\ln z}) = \ln z$. Second, using eq. (B.14), $\ln(e^{\ln z}) = \ln z + 2\pi i k$. This seems to imply that $\ln z = \ln z + 2\pi i k$. In fact, the latter is completely valid as a *set equality* in light of eq. (B.5).

We now consider the properties exhibited in eqs. (B.9)–(B.11). Using the definition of the multivalued complex logarithms and the properties of $\arg\ z$ given in eqs. (A.10)–(A.12), it follows that eqs. (B.9)–(B.12) are satisfied as set equalities:

$$\ln(z_1 z_2) = \ln z_1 + \ln z_2 \,, \tag{B.15}$$

$$\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2 \,. \tag{B.16}$$

$$\ln\left(\frac{1}{z}\right) = -\ln z \,. \tag{B.17}$$

However, one must be careful in employing these results. One should not make the mistake of writing, for example, $\ln z + \ln z \stackrel{?}{=} 2\ln z$ or $\ln z - \ln z \stackrel{?}{=} 0$. Both these latter statements are false for the same reasons that eqs. (A.14) and (A.18) are *not* identities under set equality. In general, the multivalued complex logarithm does *not* satisfy eq. (B.12) as a set equality. In particular, the multivalued complex logarithm does *not* satisfy eq. (B.12) when $p$ is an integer $n$:

$$\ln z^n = \underbrace{\ln z + \ln z + \cdots + \ln z}_{n} \neq n \ln z \,, \tag{B.18}$$

[11] Note that $\text{Arg } e^z = y + 2\pi N$, where $N$ is chosen such that $-\pi < y + 2\pi N \leq \pi$. Moreover, eq. (A.3) implies that $\arg e^z = \text{Arg } e^z + 2\pi n$, where $n = 0, \pm 1, \pm 2, \ldots$. Hence, $\arg(e^z) = y + 2\pi k$, where $k = n + N$ is still some integer.

which follows from eq. (A.16). If $p$ is not an integer, then $z^p$ is a complex multivalued function, and one needs further analysis to determine whether eq. (B.12) is valid. In Appendix B.3, we will prove [see eq. (B.28)] that eq. (B.12) is satisfied by the complex logmarithm only if $p = 1/n$ where $n$ is a nonzero integer. In this case,

$$\ln(z^{1/n}) = \frac{1}{n}\ln z\,, \qquad n = \pm 1\,,\ \pm 2\,,\ \pm 3\,,\ \ldots\,. \tag{B.19}$$

We next examine the properties of the single-valued function $\operatorname{Ln} z$. Again, we examine the six properties given by eqs. (B.7)–(B.12). First, eq. (B.7) is trivially satisfied since

$$e^{\operatorname{Ln} z} = e^{\operatorname{Ln}|z|}e^{i\operatorname{Arg} z} = |z|e^{i\operatorname{Arg} z} = z\,. \tag{B.20}$$

However, eq. (B.8) is generally false. In particular, for $z = x + iy$

$$\begin{aligned}\operatorname{Ln}(e^z) &= \operatorname{Ln}|e^z| + i(\operatorname{Arg} e^z) = \operatorname{Ln}(e^x) + i(\operatorname{Arg} e^{iy}) = x + i\operatorname{Arg}(e^{iy})\\ &= x + i\arg(e^{iy}) + 2\pi i\left[\frac{1}{2} - \frac{\arg(e^{iy})}{2\pi}\right] = x + iy + 2\pi i\left[\frac{1}{2} - \frac{y}{2\pi}\right]\\ &= z + 2\pi i\left[\frac{1}{2} - \frac{\operatorname{Im} z}{2\pi}\right]\,,\end{aligned} \tag{B.21}$$

after using eq. (A.6), where [ ] is the greatest integer bracket function defined in eq. (A.5). Thus, eq. (B.8) is satisfied only when $-\pi < y \leq \pi$. For values of $y$ outside the principal interval, eq. (B.8) contains an additive correction term as shown in eq. (B.21).

As a check, let us compute $\operatorname{Ln}(e^{\operatorname{Ln} z})$ in two different ways. First, using eq. (B.20), it follows that $\operatorname{Ln}(e^{\operatorname{Ln} z}) = \operatorname{Ln} z$. Second, using eq. (B.21),

$$\operatorname{Ln}(e^{\operatorname{Ln} z}) = \ln z + 2\pi i\left[\frac{1}{2} - \frac{\operatorname{Im}\operatorname{Ln} z}{2\pi}\right] = \operatorname{Ln} z + 2\pi i\left[\frac{1}{2} - \frac{\operatorname{Arg} z}{2\pi}\right] = \operatorname{Ln} z\,,$$

where we have used $\operatorname{Im}\operatorname{Ln} z = \operatorname{Arg} z$ [see eq. (B.6)]. In the last step, we noted that

$$0 \leq \frac{1}{2} - \frac{\operatorname{Arg}(z)}{2\pi} < 1\,,$$

due to eq. (A.2), which implies that the integer part of $\frac{1}{2} - \operatorname{Arg} z/(2\pi)$ is zero. Thus, the two computations agree.

We now consider the properties exhibited in eqs. (B.9)–(B.12). $\operatorname{Ln} z$ may not satisfy any of these properties due to the fact that the principal value of the complex logarithm must lie in the interval $-\pi < \operatorname{Im}\operatorname{Ln} z \leq \pi$. Using the results of eqs. (A.19)–(A.24), it follows that

$$\operatorname{Ln}(z_1 z_2) = \operatorname{Ln} z_1 + \operatorname{Ln} z_2 + 2\pi i N_+\,, \tag{B.22}$$

$$\operatorname{Ln}(z_1/z_2) = \operatorname{Ln} z_1 - \operatorname{Ln} z_2 + 2\pi i N_-\,, \tag{B.23}$$

$$\operatorname{Ln}(z^n) = n\operatorname{Ln} z + 2\pi i N_n \qquad (\text{integer } n)\,, \tag{B.24}$$

where the integers $N_\pm = -1$, 0 or +1 and $N_n$ are determined by eqs. (A.21) and (A.24), respectively, and

$$\mathrm{Ln}(1/z) = \begin{cases} -\mathrm{Ln}(z) + 2\pi i\,, & \text{if } z \text{ is real and negative}\,, \\ -\mathrm{Ln}(z)\,, & \text{otherwise (with } z \neq 0)\,. \end{cases} \tag{B.25}$$

Note that eq. (B.9) is satisfied if $\mathrm{Re}\; z_1 > 0$ and $\mathrm{Re}\; z_2 > 0$ (in which case $N_+ = 0$). In other cases, $N_+ \neq 0$ and eq. (B.9) fails. Similar considerations also apply to eqs. (B.10) and (B.11). For example, eq. (B.11) is satisfied by $\mathrm{Ln}\; z$ unless $\mathrm{Arg}\; z = \pi$ (equivalently for negative real values of $z$), as indicated by eq. (B.25). In particular, one may use eq. (B.24) to verify that:

$$\mathrm{Ln}[(-1)^{-1}] = -\mathrm{Ln}(-1) + 2\pi i = -\pi i + 2\pi i = \pi i = \mathrm{Ln}(-1)\,,$$

as expected, since $(-1)^{-1} = -1$.

## B.3. Properties of the generalized power function

The generalized complex power function is *defined* via the following equation:

$$w = z^c = e^{c \ln z}\,, \qquad z \neq 0\,. \tag{B.26}$$

Note that due to the multivalued nature of $\ln z$, it follows that $w = z^c = e^{c \ln z}$ is also multivalued for any non-integer value of $c$, with a branch point at $z = 0$:

$$w = z^c = e^{c \ln z} = e^{c \,\mathrm{Ln}\, z} e^{2\pi i n c}\,, \qquad n = 0\,, \pm 1\,, \pm 2\,, \pm 3\,, \cdots\,. \tag{B.27}$$

If $c$ is a rational number, then it can always be expressed in the form $c = m/k$, where $m$ is an integer and $k$ is a positive integer such that $m$ and $k$ possess no common divisor. One can then assume that $n = 0, 1, 2, \ldots, k-1$ in eq. (B.27), since other values of $n$ will not produce any new values of $z^{m/k}$. It follows that the multivalued function $w = z^{m/k}$ has precisely $k$ distinct branches. If $c$ is irrational or complex, then the number of branches is infinite (with one branch for each possible choice of integer $n$).

Having defined the multivalued complex power function, we are now able to compute $\ln(z^c)$ for arbitrary complex number $c$ and complex variable $z$,

$$\begin{aligned} \ln(z^c) &= \ln(e^{c \ln z}) = \ln(e^{c\,(\mathrm{Ln}\, z + 2\pi i m)}) = \ln(e^{c \,\mathrm{Ln}\, z} e^{2\pi i m c}) \\ &= \ln(e^{c\,\mathrm{Ln}\, z}) + \ln(e^{2\pi i m c}) = c\,(\mathrm{Ln}\, z + 2\pi i m) + 2\pi i k \\ &= c\,\ln z + 2\pi i k = c\left(\ln z + \frac{2\pi i k}{c}\right)\,, \end{aligned} \tag{B.28}$$

where $k$ and $m$ are arbitrary integers. Thus, $\ln(z^c) = c \ln z$ in the sense of *set equality* (in which case the sets corresponding to $\ln z$ and $\ln z + 2\pi i k/c$ coincide) if and only if $k/c$ is an integer for all values of $k$. The only possible way to satisfy this latter requirement is to take $c = 1/n$, where $n$ is an integer. Thus, eq. (B.19) is now verified.

We now explore the properties of the multivalued complex power function. First, it is tempting to write:

$$z^a z^b = e^{a\ln z}e^{b\ln z} = e^{a\ln z+b\ln z} \stackrel{?}{=} e^{(a+b)\ln z} = z^{a+b}\,. \tag{B.29}$$

However, consider the case of non-integer $a$ and $b$ where $a+b$ is an integer. In this case, eq. (B.29) cannot be correct since it would equate a multivalued function $z^a z^b$ with a single-valued function $z^{a+b}$. In fact, the questionable step in eq. (B.29) is false:

$$a\ln z + b\ln z \stackrel{?}{=} (a+b)\ln z \qquad \text{[FALSE!!]}. \tag{B.30}$$

We previously noted that eq. (B.30) is false in the case of $a=b=1$ [cf. eq. (A.14)]. A more careful computation yields:

$$\begin{aligned} z^a z^b &= e^{a\ln z}e^{b\ln z} = e^{a(\mathrm{Ln}\,z+2\pi i n)}e^{b(\mathrm{Ln}\,z+2\pi i k)} = e^{(a+b)\mathrm{Ln}\,z}e^{2\pi i(na+kb)}\,, \\ z^{a+b} &= e^{(a+b)\ln z} = e^{(a+b)(\mathrm{Ln}\,z+2\pi i k)} = e^{(a+b)\mathrm{Ln}\,z}e^{2\pi i k(a+b)}\,, \end{aligned} \tag{B.31}$$

where $k$ and $n$ are arbitrary integers. Hence, $z^{a+b}$ is a subset of $z^a z^b$. Whether the set of values for $z^a z^b$ and $z^{a+b}$ does or does not coincide depends on $a$ and $b$. However, in general, the relation $z^a z^b = z^{a+b}$ does not hold.

Similarly,

$$\begin{aligned} \frac{z^a}{z^b} &= \frac{e^{a\ln z}}{e^{b\ln z}} = \frac{e^{a(\mathrm{Ln}\,z+2\pi i n)}}{e^{b(\mathrm{Ln}\,z+2\pi i k)}} = e^{(a-b)\mathrm{Ln}\,z}e^{2\pi i(na-kb)}\,, \\ z^{a-b} &= e^{(a-b)\ln z} = e^{(a-b)(\mathrm{Ln}\,z+2\pi i k)} = e^{(a-b)\mathrm{Ln}\,z}e^{2\pi i k(a-b)}\,, \end{aligned} \tag{B.32}$$

where $k$ and $n$ are arbitrary integers. Hence, $z^{a-b}$ is a subset of $z^a/z^b$. Whether the set of values $z^a/z^b$ and $z^{a-b}$ does or does not coincide depends on $a$ and $b$. However, in general, the relation $z^a/z^b = z^{a-b}$ does not hold. Setting $a=b$ in eq. (B.32) yields the expected result:

$$z^0 = 1\,, \qquad z\neq 0$$

for any non-zero complex number $z$. Setting $a=0$ in eq. (B.32) yields the set equality:

$$z^{-b} = \frac{1}{z^b}\,, \tag{B.33}$$

i.e., the set of values for $z^{-b}$ and $1/z^b$ coincide. However, note that

$$z^a z^{-a} = e^{a\ln z}e^{-a\ln z} = e^{a(\ln z-\ln z)} = e^{a\ln 1} = e^{2\pi i k a}\,,$$

where $k$ is an arbitrary integer. Hence, if $a$ is a non-integer, then $z^a z^{-a}\neq 1$ for $k\neq 0$. This is not in conflict with the set equality given in eq. (B.33) since there always exists at least one value of $k$ (namely $k=0$) for which $z^a z^{-a}=1$.

To show that the realtion $(z^a)^b = z^{ab}$ can fail, we use eqs. (B.2), (B.14) and (B.26) to conclude that

$$(z^a)^b = (e^{a\ln z})^b = e^{b\ln(e^{a\ln z})} = e^{b(a\ln z+2\pi i k)} = e^{ba\ln z}e^{2\pi i b k} = z^{ab}e^{2\pi i b k}\,, \tag{B.34}$$

where $k$ is an arbitrary integer. Thus, $z^{ab}$ is a subset of $(z^a)^b$.

For example, if $ab = 1$ then $z^{ab} = 1$ whereas $(z^a)^{1/a} = ze^{2\pi ibk}$ ($k = 0, \pm 1, \pm 2, \ldots$), which differs from $z$ if $b$ is not an integer. Another instructive example is provided by $z = a = b = i$, in which case $z^{ab} = i^{-1} = -i$, whereas eq. (B.34) yields,

$$(i^i)^i = i^{i\cdot i}e^{-2\pi k} = i^{-1}e^{-2\pi k} = -ie^{-2\pi k}\,, \qquad k = 0\,,\, \pm 1\,,\, \pm 2\,,\, \cdots\,. \tag{B.35}$$

However, it is easy to construct examples in which the elements of $z^{ab}$ and $(z^a)^b$ coincide; e.g., if $a = \pm 1$ and/or $b$ is an integer.[12]

On the other hand, the multivalued power function satisfies the relations,

$$(z_1 z_2)^a = e^{a\ln(z_1 z_2)} = e^{a(\ln z_1 + \ln z_2)} = e^{a\ln z_1}e^{a\ln z_2} = z_1^a z_2^a\,, \tag{B.36}$$

$$\left(\frac{z_1}{z_2}\right)^a = e^{a\ln(z_1/z_2)} = e^{a(\ln z_1 - \ln z_2)} = e^{a\ln z_1}e^{-a\ln z_2} = z_1^a z_2^{-a}\,. \tag{B.37}$$

We can define a single-valued power function by selecting the principal value of $\ln z$ in eq. (B.26). Consequently, the *principal value* of $z^c$ is defined by

$$Z^c = e^{c\,\mathrm{Ln}\,z}\,, \qquad z \neq 0\,.$$

For a lack of a better notation, I will indicate the principal value by employing the upper case $Z$ as above. The principal value definition of $z^c$ can lead to some unexpected results. For example, consider the principal value of the cube root function $w = Z^{1/3} = e^{\mathrm{Ln}(z)/3}$. Then, for $z = -1$, the principal value of

$$\sqrt[3]{-1} = e^{\mathrm{Ln}(-1)/3} = e^{\pi i/3} = \tfrac{1}{2}\left(1 + i\sqrt{3}\right)\,.$$

This may have surprised you, if you were expecting that $\sqrt[3]{-1} = -1$. To obtain the latter result would require a different choice of the principal interval in the definition of the principal value of $z^{1/3}$.

Employing the principal value for the complex logarithm and complex power function, it follows that

$$\mathrm{Ln}(Z^c) = \mathrm{Ln}(e^{c\,\mathrm{Ln}\,z}) = c\,\mathrm{Ln}\,z + 2\pi i N_c\,, \tag{B.38}$$

after using eq. (B.21), where $N_c$ is an integer determined by

$$N_c \equiv \left[\frac{1}{2} - \frac{\mathrm{Im}\;(c\,\mathrm{Ln}\,z)}{2\pi}\right]\,, \tag{B.39}$$

and [ ] is the greatest integer bracket function defined in eq. (A.5). $N_c$ can be evaluated by noting that:

$$\mathrm{Im}\;(c\,\mathrm{Ln}\,z) = \mathrm{Im}\;\{c\,(\mathrm{Ln}|z| + i\mathrm{Arg}\;z)\} = \mathrm{Arg}\;z\,\mathrm{Re}\;c + \mathrm{Ln}|z|\,\mathrm{Im}\;c\,.$$

[12]Many other special cases exist in which the elements of $z^{ab}$ and $(z^a)^b$ coincide. For example, if $a = 3/2$ and $b = 1/2$, then one can check that allowing for all possible integer values of $k$ in eq. (B.34) yields $(z^a)^b = z^{ab}$, where both sets of this set equality contain precisely the same four elements.

Note that if $c = n$ where $n$ is an integer, then eq. (B.38) simply reduces to eq. (B.24), as expected. Hence, eq. (B.12) is generally false both for the multivalued complex logarithm and its principal value.

The properties of the generalized power function and its principal value follow from the corresponding properties of the complex logarithm derived in Appendix B.2. For example, the single-valued power function satisfies:

$$Z^a Z^b = e^{a\mathrm{Ln}\, z} e^{b\mathrm{Ln}\, z} = e^{(a+b)\mathrm{Ln}\, z} = Z^{a+b} \,, \tag{B.40}$$

$$\frac{Z^a}{Z^b} = \frac{e^{a\mathrm{Ln}\, z}}{e^{b\mathrm{Ln}\, z}} = e^{(a-b)\mathrm{Ln}\, z} = Z^{a-b} \,, \tag{B.41}$$

$$Z^a Z^{-a} = e^{a\mathrm{Ln}\, z} e^{-a\mathrm{Ln}\, z} = 1 \,. \tag{B.42}$$

Setting $a = b$ in eq. (B.41) yields $Z^0 = 1$ (for $Z \neq 0$) as expected.

The principal value of the complex power function satisfies a relation similar to that of eq. (B.34),

$$(Z^c)^b = (e^{c\mathrm{Ln}\, z})^b = e^{b\,\mathrm{Ln}(e^{c\mathrm{Ln}\, z})} = e^{b(c\mathrm{Ln}\, z + 2\pi i N_c)} = e^{bc\,\mathrm{Ln}\, z}\, e^{2\pi i b N_c} = Z^{cb}\, e^{2\pi i b N_c} \,,$$

where $N_c$ is an integer determined by eq. (B.39). As an example, if $z = b = c = i$, eq. (B.39) gives $N_c = 0$, which yields the principal value of $(i^i)^i = i^{i\cdot i} = i^{-1} = -i$. However, in general $N_c \neq 0$ is possible in which case $(Z^c)^b \neq Z^{cb}$ unless $bN_c$ is an integer. For example, if $z$ is real and negative and $c = -1$, then $N_c = 1$ and $(Z^{-1})^b = Z^{-b} e^{2\pi i b}$. That is, if $z$ is real and negative then $(Z^{-1})^b \neq Z^{-b}$ unless $b$ is an integer.

In contrast to eqs. (B.36) and (B.37), the corresponding relations satisfied by the principal value are more complicated,

$$(Z_1 Z_2)^a = e^{a\mathrm{Ln}(z_1 z_2)} = e^{a(\mathrm{Ln}\, z_1 + \mathrm{Ln}\, z_2 + 2\pi i N_+)} = Z_1^a Z_2^a\, e^{2\pi i a N_+} \,, \tag{B.43}$$

$$\left(\frac{Z_1}{Z_2}\right)^a = e^{a\mathrm{Ln}(z_1/z_2)} = e^{a(\mathrm{Ln}\, z_1 - \mathrm{Ln}\, z_2 + 2\pi i N_-)} = \frac{Z_1^a}{Z_2^a}\, e^{2\pi i a N_-} \,, \tag{B.44}$$

where the integers $N_\pm$ are determined from eq. (A.21).

## APPENDIX C: Alternative definitions for Arctan and Arccot

The well-known reference book for mathematical functions by Abramowitz and Stegun (see Ref. 1) and the more recent NIST Handbook of Mathematical Functions (see Ref. 2) define the principal values of the complex arctangent and arccotangent functions as follows,

$$\mathrm{Arctan}\, z = \tfrac{1}{2} i\, \mathrm{Ln}\left(\frac{1 - iz}{1 + iz}\right) \,, \tag{C.1}$$

$$\mathrm{Arccot}\, z = \mathrm{Arctan}\left(\frac{1}{z}\right) = \tfrac{1}{2} i\, \mathrm{Ln}\left(\frac{z - i}{z + i}\right) \,. \tag{C.2}$$

With these definitions, the branch cuts are still given by eqs. (15) and (17), respectively. Comparing the above definitions with those of eqs. (12) and (13), one can check that the two definitions differ only on the branch cuts. One can use eqs. (C.1) and (C.2) to define the single-valued functions by employing the standard conventions for evaluating the complex logarithm on its branch cut [namely, by defining $\mathrm{Arg}(-x) = \pi$ for any real positive number $x$].[13] For example, for values of $z = iy$ ($|y| > 1$) that lie on the branch cut of $\mathrm{Arctan}\, z$, eq. (C.1) yields,[14]

$$\mathrm{Arctan}(iy) = \frac{i}{2}\mathrm{Ln}\left(\frac{y+1}{y-1}\right) - \tfrac{1}{2}\pi\,, \quad \text{for } |y| > 1\,. \tag{C.3}$$

This result differs from eq. (16) when $1 < y < \infty$.

It is convenient to define a new variable,

$$v = \frac{1-iz}{1+iz} = \frac{i+z}{i-z}\,, \qquad \Longrightarrow \qquad -\frac{1}{v} = \frac{z-i}{z+i}\,. \tag{C.4}$$

Then, we can write:

$$\begin{aligned}
\mathrm{Arctan}\, z + \mathrm{Arccot}\, z &= \frac{i}{2}\left[\mathrm{Ln}\, v + \mathrm{Ln}\left(-\frac{1}{v}\right)\right] \\
&= \frac{i}{2}\left[\mathrm{Ln}|v| + \mathrm{Ln}\left(\frac{1}{|v|}\right) + i\mathrm{Arg}\, v + i\mathrm{Arg}\left(-\frac{1}{v}\right)\right] \\
&= -\frac{1}{2}\left[\mathrm{Arg}\, v + \mathrm{Arg}\left(-\frac{1}{v}\right)\right]\,.
\end{aligned} \tag{C.5}$$

It is straightforward to check that for any nonzero complex number $v$,

$$\mathrm{Arg}\, v + \mathrm{Arg}\left(-\frac{1}{v}\right) = \begin{cases} \pi\,, & \text{for Im } v \geq 0\,, \\ -\pi\,, & \text{for Im } v < 0\,. \end{cases} \tag{C.6}$$

Using eq. (C.4), we can evaluate Im $v$ by computing

$$\frac{i+z}{i-z} = \frac{(i+z)(-i-z^*)}{(i-z)(-i-z^*)} = \frac{1-|z|^2 - 2i\,\mathrm{Re}\ z}{|z|^2 + 1 - 2\,\mathrm{Im}\ z}\,.$$

Writing $|z|^2 = (\mathrm{Re}\ z)^2 + (\mathrm{Im}\ z)^2$ in the denominator,

$$\frac{i+z}{i-z} = \frac{1-|z|^2 - 2i\,\mathrm{Re}\ z}{(\mathrm{Re}\ z)^2 + (1-\mathrm{Im}\ z)^2}\,.$$

[13] Ref. 2 does not assign a unique value to Arctan or Arccot for values of $z$ that lie on the branch cut. However, computer programs such as Mathematica do not have this luxury since it must return a unique value for the corresponding functions evaluated at any complex number $z$.

[14] In light of footnote 13, the result obtained in eq. (4.23.27) of Ref. 2 for $\mathrm{Arctan}(iy)$ is not single-valued, in contrast to eq. (C.3).

Hence,

$$\text{Im } v \equiv \text{Im}\left(\frac{i+z}{i-z}\right) = \frac{-2\,\text{Re } z}{(\text{Re } z)^2 + (1-\text{Im } z)^2}\,.$$

We conclude that

$$\text{Im } v \geq 0 \quad \Longrightarrow \quad \text{Re } z \leq 0\,, \qquad\qquad \text{Im } v < 0 \quad \Longrightarrow \quad \text{Re } z > 0\,.$$

Therefore, eqs. (C.5) and (C.6) yield:

$$\text{Arctan}\, z + \text{Arccot}\, z = \begin{cases} -\tfrac{1}{2}\pi\,, & \text{for Re } z \leq 0 \text{ and } z \neq \pm i\,, \\ \tfrac{1}{2}\pi\,, & \text{for Re } z > 0 \text{ and } z \neq \pm i\,. \end{cases} \tag{C.7}$$

which differs from eq. (25) when $z$ lives on one of the branch cuts, for Re $z = 0$ and $z \neq \pm i$. Moreover, there is no longer any ambiguity in how to define Arccot(0). Indeed, for values of $z = iy$ $(-1 < y < 1)$ that lie on the branch cut of Arccot $z$, eq. (C.2) yields,

$$\text{Arccot}(iy) = \frac{i}{2}\text{Ln}\left(\frac{1-y}{1+y}\right) - \tfrac{1}{2}\pi\,, \quad \text{for } |y| < 1, \tag{C.8}$$

which differs from the result of eq. (19) when $-1 < y < 0$. That is, by employing the definition of the principal value of the arccotangent function given by eq. (C.2), Arccot$(iy)$ is a continuous function of $y$ on the branch cut. In particular, plugging $z = 0$ into eq. (C.2) yields,

$$\text{Arccot}(0) = \tfrac{1}{2}i\,\text{Ln}(-1) = -\tfrac{1}{2}\pi\,. \tag{C.9}$$

Unfortunately, this result is the negative of the convention proposed in eq. (20).

One disadvantage of the definition of the principal value of the arctangent given by eq. (C.1) concerns the value of Arctan$(-\infty)$. In particular, if $z = x$ is real,

$$\left|\frac{1-ix}{1+ix}\right| = 1\,. \tag{C.10}$$

Since Ln 1 = 0, it would follow from eq. (C.1) that for all real $x$,

$$\text{Arctan}\, x = -\tfrac{1}{2}\text{Arg}\left(\frac{1-ix}{1+ix}\right)\,. \tag{C.11}$$

Indeed, eq. (C.11) is correct for all finite real values of $x$. It also correctly implies that Arctan $(-\infty) = -\tfrac{1}{2}\text{Arg}(-1) = -\tfrac{1}{2}\pi$, as expected. However, if we take $x \to \infty$ in eq. (C.11), we would also get Arctan $(\infty) = -\tfrac{1}{2}\text{Arg}(-1) = -\tfrac{1}{2}\pi$, in contradiction with the conventional definition of the principal value of the real-valued arctangent function, where Arctan $(\infty) = \tfrac{1}{2}\pi$. This slight inconsistency is not surprising, since the principal value of the argument of any complex number $z$ must lie in the range $-\pi < \text{Arg}\, z \leq \pi$. Consequently, eq. (C.11) implies that $-\tfrac{1}{2}\pi \leq \text{Arctan}\, x < \tfrac{1}{2}\pi$, which is not quite consistent with eq. (21) as the endpoint at $\tfrac{1}{2}\pi$ is missing.

Some authors finesse this defect by defining the value of $\mathrm{Arctan}\,(\infty)$ as the limit of $\mathrm{Arctan}\,(x)$ as $x \to \infty$. Note that

$$\lim_{x\to\infty} \mathrm{Arg}\left(\frac{1-ix}{1+ix}\right) = -\pi\,,$$

since for any finite real value of $x > 1$, the complex number $(1-ix)/(1+ix)$ lies in Quadrant III[15] and approaches the negative real axis as $x \to \infty$. Hence, eq. (C.11) yields

$$\lim_{x\to\infty} \mathrm{Arctan}\,(x) = \tfrac{1}{2}\pi\,.$$

With this interpretation, eq. (C.1) is consistent with the definition for the principal value of the real arctangent function.[16]

It is instructive to consider the difference of the two definitions of $\mathrm{Arctan}\,z$ given by eqs. (12) and (C.1). Using eqs. (A.21) and (B.23), it follows that

$$\mathrm{Ln}\left(\frac{1-iz}{1+iz}\right) - [\mathrm{Ln}(1-iz) - \mathrm{Ln}(1+iz)] = 2\pi i N_-\,,$$

where

$$N_- = \begin{cases} -1\,, & \text{if} \quad \mathrm{Arg}(1-iz) - \mathrm{Arg}(1+iz) > \pi\,, \\ \;\;0\,, & \text{if} \quad -\pi < \mathrm{Arg}(1-iz) - \mathrm{Arg}(1+iz) \leq \pi\,, \\ \;\;1\,, & \text{if} \quad \mathrm{Arg}(1-iz) - \mathrm{Arg}(1+iz) \leq -\pi\,. \end{cases} \tag{C.12}$$

To evaluate $N_-$ explicitly, we must examine the quantity $\mathrm{Arg}(1-iz) - \mathrm{Arg}(1+iz)$ as a function of the complex number $z = x+iy$. Hence, we shall focus on the quantity $\mathrm{Arg}(1+y-ix) - \mathrm{Arg}(1-y+ix)$ as a function of $x$ and $y$. If we plot the numbers $1+y-ix$ and $1-y+ix$ in the complex plane, it is evident that for finite values of $x$ and $y$ and $x \neq 0$ then

$$-\pi < \mathrm{Arg}(1+y-ix) - \mathrm{Arg}(1-y+ix) < \pi\,.$$

The case of $x=0$ is easily treated separately, and we find that

$$\mathrm{Arg}(1+y) - \mathrm{Arg}(1-y) = \begin{cases} -\pi\,, & \text{if } y > 1\,, \\ \;\;0\,, & \text{if } -1 < y < 1\,, \\ \;\;\pi\,, & \text{if } y < -1\,. \end{cases}$$

[15]This is easily verified. We write:

$$z \equiv \frac{1-ix}{1+ix} = \frac{1-ix}{1+ix}\cdot\frac{1-ix}{1-ix} = \frac{1-x^2-2ix}{1+x^2}\,.$$

Thus, for real values of $x > 1$, it follows that $\mathrm{Re}\,z < 0$ and $\mathrm{Im}\,z < 0$, i.e. the complex number $z$ lies in Quadrant III. Moreover, as $x \to \infty$, we see that $\mathrm{Re}\,z \to -1$ and $\mathrm{Im}\,z \to 0^-$, where $0^-$ indicates that one is approaching 0 from the negative side. Some authors write $\lim_{x\to\infty}(1-ix)/(1+ix) = -1 - i0$ to indicate this behavior, and then define $\mathrm{Arg}(-1-i0) = -\pi$.

[16]This is strategy adopted in Ref. 2 since this reference does not assign a unique value to $\mathrm{Arctan}\,z$ and $\mathrm{Arccot}\,z$ on their respective branch cuts.

Note that we have excluded the points $x = 0$, $y = \pm 1$, which correspond to the branch points where the arctangent function diverges. Hence, it follows that in the finite complex plane excluding the branch points at $z = \pm i$,

$$N_- = \begin{cases} 1\,, & \text{if Re } z = 0 \text{ and Im } z > 1\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

This means that in the finite complex plane, the two possible definitions for the principal value of the arctangent function given by eqs. (12) and (C.1) differ only on the branch cut along the positive imaginary axis above $z = i$. That is, for finite $z \neq \pm i$,

$$\tfrac{1}{2} i \,\text{Ln}\left(\frac{1 - iz}{1 + iz}\right) = \begin{cases} -\pi + \tfrac{1}{2} i \left[\text{Ln}(1 - iz) - \text{Ln}(1 + iz)\right]\,, & \text{if Re } z = 0 \text{ and Im } z > 1\,, \\ \\ \tfrac{1}{2} i \left[\text{Ln}(1 - iz) - \text{Ln}(1 + iz)\right]\,, & \text{otherwise}\,. \end{cases} \tag{C.13}$$

Additional discrepancies between the two definitions can arise when $x$ and/or $y$ become infinite. For example, since $\text{Arg}(a + i\infty) = \frac{1}{2}\pi$ and $\text{Arg}(a - i\infty) = -\frac{1}{2}\pi$ for any real number $a$, it follows that $N_- = 1$ for $x = \infty$.

Likewise one can determine the difference of the two definitions of $\text{Arccot}\, z$ given by eqs. (13) and (C.2). Using the relation $\text{Arccot}\, z = \text{Arctan}(1/z)$ [which holds for both sets of definitions], eq. (C.13) immediately yields:

$$\tfrac{1}{2} i \,\text{Ln}\left(\frac{z - i}{z + i}\right) = \begin{cases} -\pi + \dfrac{i}{2}\left[\text{Ln}\left(1 - \dfrac{i}{z}\right) - \text{Ln}\left(1 + \dfrac{i}{z}\right)\right]\,, & \text{if Re } z = 0 \text{ and } -1 < \text{Im } z < 0\,, \\ \dfrac{i}{2}\left[\text{Ln}\left(1 - \dfrac{i}{z}\right) - \text{Ln}\left(1 + \dfrac{i}{z}\right)\right]\,, & \text{otherwise}\,. \end{cases} \tag{C.14}$$

It follows that the two possible definitions for the principal value of the arccotangent function given by eqs. (13) and (C.2) differ only on the branch cut along the negative imaginary axis above $z = -i$.

A similar set of issues arise in the definitions of the principal values of the inverse hyperbolic tangent and cotangent functions. It is most convenient to define these functions in terms of the corresponding principal values of the arctangent and arccotangent functions following eqs. (56) and (57),

$$\text{Arctanh}\, z = i\text{Arctan}(-iz)\,, \tag{C.15}$$

$$\text{Arccoth}\, z = i\text{Arccot}(-iz)\,. \tag{C.16}$$

So which set of conventions is best? Of course, there is no one right or wrong answer to this question. As a practical matter, I always employ the Mathematica definitions, as this is a program that I often use in my research. In contrast, the authors of Refs. 6–8 argue for choosing eq. (12) to define the principal value of the arctangent but use a slight variant of eq. (C.2) to define the principal value of the arccotangent function,[17]

$$\text{Arccot}\, z = \frac{1}{2i}\text{Ln}\left(\frac{z+i}{z-i}\right)\,. \tag{C.17}$$

This new definition has the benefit of ensuring that $\text{Arccot}(0) = \frac{1}{2}\pi$ [in contrast to eq. (C.9)]. But, adopting eq. (C.17) will lead to modifications of $\text{Arccot}\, z$ (compared to alternative definitions previously considered) when evaluated on the branch cut, $\text{Re}\ z = 0$ and $|\text{Im}\ z| < 1$. For example, with the definitions of $\text{Arctan}\, z$ and $\text{Arccot}\, z$ given by eqs. (12) and (C.17), respectively, it is straightforward to show that a number of relations, such as $\text{Arccot}\, z = \text{Arctan}(1/z)$, are modified.

For example, one can easily derive,

$$\text{Arccot}\, z = \begin{cases} \pi + \text{Arctan}\left(\frac{1}{z}\right)\,, & \text{if Re } z = 0 \text{ and } 0 < \text{Im } z < 1\,, \\ \text{Arctan}\left(\frac{1}{z}\right)\,, & \text{otherwise}\,, \end{cases}$$

excluding the branch points $z = \pm i$ where $\text{Arctan}\, z$ and $\text{Arccot}\, z$ both diverge. Likewise, the expression for $\text{Arctan}\, z + \text{Arccot}\, z$ previously obtained will also be modified,

$$\text{Arctan}\, z + \text{Arccot}\, z = \begin{cases} \frac{1}{2}\pi\,, & \text{for Re } z > 0\,, \\ \frac{1}{2}\pi\,, & \text{for Re } z = 0\,, \text{ and Im } z > -1\,, \\ -\frac{1}{2}\pi\,, & \text{for Re } z < 0\,, \\ -\frac{1}{2}\pi\,, & \text{for Re } z = 0\,, \text{ and Im } z < -1\,. \end{cases} \tag{C.18}$$

Other modifications of the results of this Appendix in the case where eq. (C.17) is adopted as the definition of the principal value of the arccotangent function are left as an exercise for the reader.

CAUTION!!

The principal value of the arccotangent is given in terms the principal value of the arctangent,

$$\text{Arccot}\, z = \text{Arctan}\left(\frac{1}{z}\right)\,, \tag{C.19}$$

for both the Mathematica definition [eq. (13)] or the alternative definition presented in eq. (C.2). However, some reference books (see, e.g., Ref. 4) define the principal value of the arccotangent differently via the relation,

$$\text{Arccot}\, z = \tfrac{1}{2}\pi - \text{Arctan}\, z\,. \tag{C.20}$$

This relation should be compared with the corresponding relations, eqs. (25), (C.7) and (C.18), which are satisfied with the definitions of the principal value of the arccotangent introduced in eqs. (13), (C.2) and (C.17), respectively. Indeed, eq. (C.20) has been

[17] The right hand side of eq. (C.17) can be identified with $-\text{Arccot}(-z)$ in the convention where $\text{Arccot}\, z$ is defined by eq. (C.2).

adopted by the Maple computer algebra system (see Ref. 8), which is one of the main competitors of Mathematica.

The main motivation for eq. (C.20) is that the principal value of the real cotangent function satisfies

$$0 \leq \operatorname{Arccot} x \leq \pi\,, \qquad \text{for } -\infty \leq x \leq +\infty,$$

instead of the interval quoted in eq. (23). One advantage of this latter definition is that for real values of $x$, $\operatorname{Arccot} x$ is continuous at $x = 0$, in contrast to eq. (C.19) which exhibits a discontinuity at $x = 0$. Note that if one adopts eq. (C.20) as the the definition of the principal value of the arccotangent, then the branch cuts of $\operatorname{Arccot} z$ are the same as those of $\operatorname{Arctan} z$ [cf. eq. (15)]. The disadvantages of the definition given in eq. (C.20) are discussed in detail in Refs. 6 and 9.

Which convention does your calculator and/or your favorite mathematics software use? Try evaluating Arccot(−1). In the convention of eq. (13) or eq. (C.2), we have $\operatorname{Arccot}(-1) = -\frac{1}{4}\pi$, whereas in the convention of eq. (C.20), we have $\operatorname{Arccot}(-1) = \frac{3}{4}\pi$.

## APPENDIX D: Derivation of eq. (25)

To derive eq. (25), we will make use of the computations provided in Appendix C. Start from eq. (C.7), which is based on the definitions of the principal values of the arctangent and arccotangent given in eqs. (C.1) and (C.2), respectively. We then use eqs. (C.13) and (C.14) which allow us to translate between the definitions of eqs. (C.1) and (C.2) and the Mathematica definitions of the principal values of the arctangent and arccotangent given in eqs. (12) and (13), respectively. Eqs. (C.13) and (C.14) imply that the result for $\operatorname{Arctan} z + \operatorname{Arccot} z$ does not change if $\operatorname{Re} z \neq 0$. For the case of $\operatorname{Re} z = 0$, $\operatorname{Arctan} z + \operatorname{Arccot} z$ changes from $\frac{1}{2}\pi$ to $-\frac{1}{2}\pi$ if $0 < \operatorname{Im} z < 1$ or $\operatorname{Im} z < -1$. This is precisely what is exhibited in eq. (25).

## APPENDIX E: Proof that $\operatorname{Re}\left(\pm iz + \sqrt{1-z^2}\right) > 0$

It is convenient to define:

$$v = iz + \sqrt{1-z^2}\,, \qquad\qquad \frac{1}{v} = \frac{1}{iz + \sqrt{1-z^2}} = -iz + \sqrt{1-z^2}\,.$$

In this Appendix, we shall prove that $\operatorname{Re} v \geq 0$ and $\operatorname{Re}\,(1/v) \geq 0$.

Using the fact that $\operatorname{Re}\,(\pm iz) = \mp \operatorname{Im} z$ for any complex number $z$,

$$\operatorname{Re} v = -\operatorname{Im} z + |1-z^2|^{1/2} \cos\left[\tfrac{1}{2}\operatorname{Arg}(1-z^2)\right]\,, \tag{E.1}$$

$$\operatorname{Re}\left(\frac{1}{v}\right) = \operatorname{Im} z + |1-z^2|^{1/2} \cos\left[\tfrac{1}{2}\operatorname{Arg}(1-z^2)\right]\,. \tag{E.2}$$

One can now prove that Re $v \geq 0$ and Re $(1/v) \geq 0$ for any finite complex number $z$ by considering separately the cases of Im $z < 0$, Im $z = 0$ and Im $z > 0$. The case of Im $z = 0$ is the simplest, since in this case Re $v = 0$ for $|z| \geq 1$ and Re $v > 0$ for $|z| < 1$ (since the principal value of the square root of a positive number is always positive). In the case of Im $z \neq 0$, we first note that that $-\pi < \text{Arg}(1 - z^2) \leq \pi$ implies that $\cos\left[\frac{1}{2}\text{Arg}(1 - z^2)\right] \geq 0$. Thus if Im $z < 0$, then it immediately follows from eq. (E.1) that Re $v > 0$. Likewise, if Im $z > 0$, then it immediately follows from eq. (E.2) that Re $(1/v) > 0$. However, the sign of the real part of any complex number $z$ is the *same* as the sign of the real part of $1/z$, since

$$\frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}\,.$$

Hence, it follows that both Re $v \geq 0$ and Re $(1/v) \geq 0$, as required.

## APPENDIX F: Derivation of eq. (71)

We begin with the definitions given in eqs. (42) and (69),[18]

$$i\text{Arccos}\, z = \text{Ln}\left(z + i\sqrt{1 - z^2}\right)\,, \tag{F.1}$$

$$\text{Arccosh}\, z = \text{Ln}\left(z + \sqrt{z + 1}\sqrt{z - 1}\right)\,, \tag{F.2}$$

where the principal values of the square root functions are employed following the notation of eq. (35). Our first task is to relate $\sqrt{z+1}\sqrt{z-1}$ to $\sqrt{z^2 - 1}$. Of course, these two quantities are equal for all real numbers $z \geq 1$. But, as these quantities are principal values of the square roots of complex numbers, one must be more careful in the general case. Employing eq. (B.44) to the principal value of the complex square root function yields:[19]

$$\sqrt{z_1 z_2} = e^{\frac{1}{2}\text{Ln}(z_1 z_2)} = e^{\frac{1}{2}(\text{Ln}\, z_1 + \text{Ln}\, z_2 + 2\pi i N_+)} = \sqrt{z_1}\sqrt{z_2}\, e^{\pi i N_+}\,,$$

where

$$N_+ = \begin{cases} -1\,, & \text{if} \quad \text{Arg}\, z_1 + \text{Arg}\, z_2 > \pi\,, \\ 0\,, & \text{if} \quad -\pi < \text{Arg}\, z_1 + \text{Arg}\, z_2 \leq \pi\,, \\ 1\,, & \text{if} \quad \text{Arg}\, z_1 + \text{Arg}\, z_2 \leq -\pi\,. \end{cases}$$

[18]We caution the reader that some authors employ different choices for the definitions of the principal values of $\arccos z$ and $\text{arccosh}\, z$ and their branch cuts. The most common alternative definitions are:

$$\text{Arccosh}\, z = i\,\text{Arccos}\, z = \text{Ln}(z + \sqrt{z^2 - 1})\,,$$

which differ from the definitions, eqs. (F.1) and (F.2), employed by Mathematica and these notes. In particular, with the alternative definitions given above, Arccos $z$ now possesses the same set of branch cuts as Arccosh $z$ given by eq. (73), in contrast to eq. (45). Moreover, Arccos $z$ no longer satisfies eq. (41) if either (Re $z$)(Im $z$) $< 0$ or if $|\text{Re}\, z| > 1$ and Im $z = 0$ [cf. eq. (F.6)]. Other disadvantages of the alternative definitions of Arccos $z$ and Arccosh $z$ are discussed in Ref. 5.

[19]Following the convention established above eq. (35), we employ the ordinary square root sign ($\sqrt{\quad}$) to designate the principal value of the complex square root function.

That is,

$$\sqrt{z_1 z_2} = \varepsilon \sqrt{z_1}\sqrt{z_2}\,, \qquad \varepsilon = \pm 1\,, \tag{F.3}$$

where the choice of sign is determined by:

$$\varepsilon = \begin{cases} +1\,, & \text{if } -\pi < \text{Arg } z_1 + \text{Arg } z_2 \leq \pi\,, \\ -1\,, & \text{otherwise.} \end{cases}$$

Thus, we must determine in which interval the quantity $\text{Arg}(z+1) + \text{Arg}(z-1)$ lies as a function of $z$. The special cases of $z = \pm 1$ must be treated separately, since $\text{Arg}\,0$ is not defined. By plotting the complex points $z+1$ and $z-1$ in the complex plane, one can easily show that for $z \neq \pm 1$,

$$-\pi < \text{Arg }(z+1) + \text{Arg }(z-1) \leq \pi\,, \qquad \text{if } \begin{cases} \text{Im } z > 0 \text{ and } \text{Re } z \geq 0\,, \\ \qquad\text{or} \\ \text{Im } z = 0 \text{ and } \text{Re } z > -1\,, \\ \qquad\text{or} \\ \text{Im } z < 0 \text{ and } \text{Re } z > 0\,. \end{cases}$$

If the above conditions do not hold, then $\text{Arg}(z+1) + \text{Arg}(z-1)$ lies outside the range of the principal value of the argument function. Hence, we conclude that if $z_1 = z+1$ and $z_2 = z-1$ then if $\text{Im } z \neq 0$ then $\varepsilon$ in eq. (F.3) is given by:

$$\varepsilon = \begin{cases} +1\,, & \text{if Re } z > 0\,, \text{Im } z \neq 0 \text{ or } \text{Re } z = 0\,, \text{Im } z > 0\,, \\ -1\,, & \text{if Re } z < 0\,, \text{Im } z \neq 0 \text{ or } \text{Re } z = 0\,, \text{Im } z < 0\,. \end{cases}$$

In the case of $\text{Im } z = 0$, we must exclude the points $z = \pm 1$, in which case we also have

$$\varepsilon = \begin{cases} +1\,, & \text{if Im } z = 0 \text{ and } \text{Re } z > -1 \text{ with } \text{Re } z \neq 1\,, \\ -1\,, & \text{if Im } z = 0 \text{ and } \text{Re } z < -1\,. \end{cases}$$

It follow that $\text{Arccosh}\, z = \text{Ln}(z \pm \sqrt{z^2-1})$, where the sign is identified with $\varepsilon$ above. Noting that $z - \sqrt{z^2-1} = [z + \sqrt{z^2-1}]^{-1}$, where $z + \sqrt{z^2-1}$ is real and negative if and only if $\text{Im } z = 0$ and $\text{Re } z \leq -1$,[20] one finds after applying eq. (39) that:

$$\text{Ln}(z - \sqrt{z^2-1}) = \begin{cases} 2\pi i - \text{Ln}(z + \sqrt{z^2-1})\,, & \text{for Im } z = 0 \text{ and Re } z \leq -1\,, \\ -\text{Ln}(z + \sqrt{z^2-1})\,, & \text{otherwise}\,. \end{cases}$$

To complete this part of the analysis, we must consider separately the points $z = \pm 1$. At these two points, eq. (F.2) yields $\text{Arccosh}(1) = 0$ and $\text{Arccosh}(-1) = \text{Ln}(-1) = \pi i$.

[20] Let $w = z + \sqrt{z^2-1}$, and assume that $\text{Im } w = 0$ and $\text{Re } w \neq 0$. That is, $w$ is real and nonzero, in which case $\text{Im } w^2 = 0$. But

$$0 = \text{Im } w^2 = \text{Im } \left[2z^2 - 1 + 2z\sqrt{z^2-1}\right] = \text{Im }(2zw - 1) = 2w\text{Im } z\,,$$

which confirms that $\text{Im } z = 0$, i.e. $z$ must be real. If we require in addition that that $\text{Re } w < 0$, then we also must have $\text{Re } z \leq -1$.

Collecting all of the above results then yields:

$$\text{Arccosh}\,z = \begin{cases} \text{Ln}(z+\sqrt{z^2-1})\,, & \text{if Im } z>0\,,\ \text{Re } z\geq 0 \ \text{ or } \ \text{Im } z=0\,,\ \text{Re } z\geq -1 \\ & \qquad \text{or } \ \text{Im } z<0, \text{Re } z>0\,, \\ -\text{Ln}(z+\sqrt{z^2-1})\,, & \text{if Im } z>0\,,\ \text{Re } z<0 \ \text{ or } \ \text{Im } z<0\,,\ \text{Re } z\leq 0\,, \\ 2\pi i-\text{Ln}(z+\sqrt{z^2-1})\,, & \text{if Im } z=0, \text{Re } z\leq -1\,. \end{cases} \tag{F.4}$$

Note that the cases of $z = \pm 1$ are each covered twice in eq. (F.4) but in both respective cases the two results are consistent.

Our second task is to relate $i\sqrt{1-z^2}$ to $\sqrt{z^2-1}$. We first note that for any non-zero complex number $z$, the principal value of the argument of $-z$ is given by:

$$\text{Arg}(-z) = \begin{cases} \text{Arg}\, z-\pi\,, & \text{if Arg}\, z>0\,, \\ \text{Arg}\, z+\pi\,, & \text{if Arg}\, z\leq 0\,. \end{cases} \tag{F.5}$$

This result is easily checked by considering the locations of the complex numbers $z$ and $-z$ in the complex plane. Hence, by making use of eqs. (35) and (F.5) along with $i = e^{i\pi/2}$, it follows that:

$$i\sqrt{1-z^2} = \sqrt{|z^2-1|}e^{\frac{1}{2}\left[\pi+\text{Arg}(1-z^2)\right]} = \eta\sqrt{z^2-1}\,, \qquad \eta=\pm 1\,,$$

where the sign $\eta$ is determined by:

$$\eta = \begin{cases} +1\,, & \text{if Arg}(1-z^2)\leq 0\,, \\ -1\,, & \text{if Arg}(1-z^2)>0\,, \end{cases}$$

assuming that $z \neq \pm 1$. If we put $z = x + iy$, then $1 - z^2 = 1 - x^2 + y^2 - 2ixy$, and we deduce that

$$\text{Arg}(1-z^2) \text{ is } \quad \begin{cases} \text{positive}\,, & \text{either if } xy<0 \quad \text{or} \quad \text{if } y=0 \text{ and } |x|>1\,, \\ \text{zero}\,, & \text{either if } x=0 \quad \text{or} \quad \text{if } y=0 \text{ and } |x|<1\,, \\ \text{negative}\,, & \text{if } xy>0\,. \end{cases}$$

We exclude the points $z = \pm 1$ (corresponding to $y = 0$ and $x = \pm 1$) where $\text{Arg}(1-z^2)$ is undefined. Treating these two points separately, eq. (F.1) yields $\text{Arccos}(1) = 0$ and $i\text{Arccos}(-1) = \text{Ln}(-1) = \pi i$. Collecting all of the above results then yields:

$$i\text{Arccos}z = \begin{cases} \text{Ln}(z+\sqrt{z^2-1})\,, & \text{if Im } z>0\,,\ \text{Re } z\geq 0 \ \text{ or } \ \text{Im } z<0\,,\ \text{Re } z\leq 0 \\ & \qquad \text{or } \ \text{Im } z=0\,,\ |\text{Re } z|\leq 1\,, \\ -\text{Ln}(z+\sqrt{z^2-1})\,, & \text{if Im } z>0\,,\ \text{Re } z<0 \ \text{ or } \ \text{Im } z<0\,,\ \text{Re } z>0\,, \\ & \qquad \text{or } \ \text{Im } z=0\,,\ \text{Re } z\geq 1\,, \\ 2\pi i-\text{Ln}(z+\sqrt{z^2-1})\,, & \text{if Im } z=0\,,\ \text{Re } z\leq -1\,. \end{cases} \tag{F.6}$$

Note that the cases of $z = \pm 1$ are each covered twice in eq. (F.6) but in both respective cases the two results are consistent.

Comparing eqs. (F.4) and (F.6) established eq. (71) and our proof is complete.

# References


1. A comprehensive treatment of the properties of the inverse trigonometric and inverse hyperbolic functions can be found in Milton Abramowitz and Irene A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, Inc., New York, 1972).

2. Ref. 1 has now been superseded by the *NIST Handbook of Mathematical Functions*, edited by Frank W.K. Olver, Daniel W. Lozier, Ronald F. Boisvert and Charles W. Clark (Cambridge University Press, Cambridge, UK, 2010). Note that in this reference, the first letter of the multivalued functions is in uppercase, whereas a lower case letter is used in the corresponding principal value (which is opposite to the naming convention used in these notes).

3. Michael Trott, *The Mathematica Guidebook for Programming* (Springer Science, Inc., New York, NY, 2004). In particular, see section 2.2.5. In particular, see section 2.2.5, pp. 188–189.

4. I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series and Products*, 8th Edition, edited by Daniel Zwillinger and Victor Moll (Academic Press, Waltham, MA, 2015).

5. A.I. Markushevich, *Theory of Functions of a Complex Variable*, Part I (AMS Chelsea Publishing, Providence, RI, 2005)

6. R.M. Corless, D.J. Jeffrey, S.M. Watt and J.H. Davenport, "*According to Abramowitz and Stegun or arccoth needn't be uncouth*," ACM SIGSAM Bulletin **34**, 58–65 (2000); J.H. Davenport, "*According to Abramowitz & Stegun (II)*," October 18, 2006 [available from `http://staff.bath.ac.uk/masjhd/2Nov-2.pdf`].

7. R. Bradford, R.M. Corless, J.H. Davenport, D.J. Jeffrey and Stephen M. Watt, "*Reasoning about the elementary functions of complex analysis*," Annals of Mathematics and Artificial Intelligence **36**, 303–318 (2002).

8. M. England, E. Cheb-Terrab, R. Bradford, J.H. Davenport and D. Wilson, "*Branch Cuts in Maple 17*," ACM Communications in Computer Algebra, **48**, 24 (2014).

9. W. Kahan, "*Branch Cuts for Complex Elementary Functions*," in *The State of Art in Numerical Analysis*, edited by A. Iserles and M.J.D. Powell (Clarendon Press, Oxford, UK, 1987) pp. 165–211.