The characteristic polynomial

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Abstract

The characteristic polynomial is an nth degree polynomial whose roots correspond to the eigenvalues of an $n \times n$ matrix. In these notes, a number of explicit formulae are derived for the coefficients of the characteristic equation of an $n \times n$ matrix A. One consequence of these formulae is an explicit expression for det$A$ in terms of traces of powers of A.

1. Coefficients of the characteristic polynomial

Consider the eigenvalue problem for an $n \times n$ matrix A,

$$A\vec{\boldsymbol{v}} = \lambda\vec{\boldsymbol{v}}\,, \qquad \vec{\boldsymbol{v}} \neq 0\,, \tag{1}$$

where the trivial solution, $\vec{\boldsymbol{v}} = 0$, is excluded. That is, the zero vector is *not* an eigenvector. The solution to this problem consists of identifying all possible values of λ (called the eigenvalues), and the corresponding non-zero vectors $\vec{\boldsymbol{v}}$ (called the eigenvectors) that satisfy eq. (1). Noting that $\mathbf{I}\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}$, where $\mathbf{I}$ is the $n \times n$ identity matrix, one can rewrite eq. (1) as

$$(A - \lambda\mathbf{I})\vec{\boldsymbol{v}} = 0\,. \tag{2}$$

This is a set of n homogeneous equations, where the n unknowns are the components of the vector $\vec{\boldsymbol{v}} = (v_1, v_2, \ldots v_n)$. If $A - \lambda\mathbf{I}$ is an invertible matrix, then one can simply multiply both sides of eq. (2) by $(A-\lambda\mathbf{I})^{-1}$ to conclude that $\vec{\boldsymbol{v}} = 0$ is the unique solution. Since the zero vector is not an eigenvector, one must demand that $A-\lambda\mathbf{I}$ is not invertible in order to find non-trivial solutions to eq. (2). That is, we demand that,

$$p(\lambda) \equiv \det(A - \lambda\mathbf{I}) = 0\,. \tag{3}$$

Eq. (3) is called the *characteristic equation*. Evaluating the determinant yields an nth order polynomial in λ, called the *characteristic polynomial*, which we have denoted above by $p(\lambda)$. The possible solutions to $p(\lambda) = 0$ yield the eigenvalues of the matrix A.

The determinant in eq. (3) can be evaluated by the usual methods. It takes the form,

$$\begin{aligned} p(\lambda) = \det(A - \lambda\mathbf{I}) &= \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} \\ &= (-1)^n \left[\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n\right]\,, \end{aligned} \tag{4}$$

where $A = [a_{ij}]$. The coefficients c_i are to be computed by evaluating the determinant. Note that we have identified the coefficient of λ^n to be $(-1)^n$. This arises from one term in the determinant that is given by the product of the diagonal elements. Evaluating the determinant via the expansion in cofactors (see, e.g., Section 3 of the class handout entitled *Determinant and the adjugate*), one can quickly verify that $(-1)^n\lambda^n$ is the only term in the characteristic polynomial that is proportional to λ^n. It is then convenient to factor out the $(-1)^n$ before defining the coefficients c_i.

Two of the coefficients are easy to obtain. Note that eq. (4) is valid for any value of λ. If we set $\lambda = 0$, then eq. (4) yields:

$$p(0) = \det A = (-1)^n c_n\,.$$

Noting that $(-1)^n(-1)^n = (-1)^{2n} = +1$ for any integer n, it follows that,

$$\boxed{c_n = (-1)^n \det A\,.} \tag{5}$$

One can also easily work out c_1 by evaluating the determinant in eq. (4) using the cofactor expansion. This yields a characteristic polynomial of the form,[1]

$$p(\lambda) = \det(A - \lambda\mathbf{I}) = (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) + c_2'\lambda^{n-2} + c_3'\lambda^{n-3} + \cdots + c_n'\,. \tag{6}$$

The term $(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$ on the right hand side of eq. (6) is the product of the diagonal elements of $A-\lambda\mathbf{I}$. As explained in footnote 1, *none* of the remaining terms that arise in the cofactor expansion of $\det(A-\lambda\mathbf{I})$ [denoted by $c_2'\lambda^{n-2}+c_3'\lambda^{n-3}+\cdots+c_n'$ in eq. (6)] are proportional to λ^n or λ^{n-1}. Moreover,

$$\begin{aligned}(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda) &= (-\lambda)^n + (-\lambda)^{n+1}\left[a_{11}+a_{22}+\cdots+a_{nn}\right]+\cdots\,,\\ &= (-1)^n\left[\lambda^n - \lambda^{n-1}(\operatorname{Tr} A)+\cdots\right]\,,\end{aligned} \tag{7}$$

where $\cdots$ contains terms that are proportional to λ^p, where $p \leq n-2$. This means that the terms in the characteristic polynomial that are proportional to λ^n and λ^{n-1} arise *solely* from the term $(a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$. As shown in eq. (7), the term proportional to $-(-1)^n\lambda^{n-1}$ is the trace of A, which is defined to be equal to the sum of the diagonal elements of A. Comparing eqs. (4) and (6), it follows that,

$$\boxed{c_1 = -\operatorname{Tr} A\,.} \tag{8}$$

Expressions for $c_2, c_3, \ldots, c_{n-1}$ are more complicated. For example, eqs. (4) and (6) yield

$$c_2 = \sum_{i=1}^{n}\sum_{\substack{j=1\\ i<j}}^{n} a_{ii}a_{jj} + c_2'\,. \tag{9}$$

An explicit expression for c_2' will be given below eq. (C.3).

[1]In computing the cofactor of the ij matrix element of $A-\lambda\mathbf{I}$, one removes row i and column j of $A-\lambda\mathbf{I}$ and evaluates the determinant of the remaining submatrix [multiplied by the sign factor $(-1)^{i+j}$]. Except for the product of diagonal elements, there is always one factor of λ in each of the rows and columns that has been removed. This implies that the maximal power one can achieve outside of the product of diagonal elements of $A-\lambda\mathbf{I}$ is λ^{n-2}.

In conclusion, the general form for the characteristic polynomial, $p(\lambda) \equiv \det(A-\lambda\mathbf{I})$, is given by,

$$p(\lambda) = (-1)^n \left[\lambda^n - \lambda^{n-1}\,\mathrm{Tr}\,A + c_2\lambda^{n-2} + \cdots + (-1)^{n-1}c_{n-1}\lambda + (-1)^n \det\,A\right]. \qquad (10)$$

Explicit formulae for $c_2, c_3, \ldots, c_{n-1}$ in terms of traces of powers of the matrix A are presented in Appendices A and B, and in terms of the matrix elements of A in Appendix C.

By the fundamental theorem of algebra, an nth order polynomial equation of the form $p(\lambda)=0$ possesses precisely n solutions (called roots), which we shall denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$. These are the eigenvalues of A, and they may be real or complex. If a root is non-degenerate (i.e., only one root has a particular numerical value), then we say that the root has multiplicity one—it is called a *simple root*. If a root is degenerate (i.e., more than one root has a particular numerical value), then we say that the root has multiplicity p, where p is the number of roots with that same value—such a root is called a *multiple root*. For example, a double root (as its name implies) arises when precisely two of the roots of $p(\lambda)$ are equal. In the counting of the n roots of $p(\lambda)$, multiple roots are counted according to their multiplicity.

One can always factor a polynomial in terms of its roots.[2] Thus, eq. (4) implies that:

$$p(\lambda) = (-1)^n(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)\,, \qquad (11)$$

where multiple roots appear according to their multiplicity.[3] Multiplying out the n factors above yields,

$$\begin{aligned} p(\lambda) = (-1)^n \Bigg\{ \lambda^n - \lambda^{n-1}\sum_{i=1}^n \lambda_i + \lambda^{n-2}\sum_{\substack{i=1\\ i<j}}^n \sum_{j=1}^n \lambda_i\lambda_j + \ldots \\ +(-1)^k\lambda^{n-k}\sum_{\substack{i_1=1\\ i_1<i_2<\cdots<i_k}}^n \sum_{i_2=1}^n \cdots \sum_{i_k=1}^n \underbrace{\lambda_{i_i}\lambda_{i_2}\cdots\lambda_{i_k}}_{k\ \text{factors}} + \cdots + (-1)^n\,\lambda_1\lambda_2\cdots\lambda_n \Bigg\}. \end{aligned} \qquad (12)$$

Comparing with eq. (10), it immediately follows that:

$$\boxed{\begin{aligned} \mathrm{Tr}\,A &= \sum_{i=1}^n \lambda_i = \lambda_1+\lambda_2+\cdots+\lambda_n\,, && (13a) \\ \det\,A &= \lambda_1\cdot\lambda_2\cdot\lambda_3\cdots\lambda_n\,. && (13b) \end{aligned}}$$

[2]In practice, it may not be possible to explicitly determine the roots algebraically. Indeed, unlike quadratic, cubic, and quartic polynomials, the roots of a general polynomial of degree 5 or higher cannot be obtained algebraically in terms of a finite number of additions, subtractions, multiplications, divisions, and root extractions (due to a famous result known as Abel's impossibility theorem). Of course, one can always determine the roots numerically.

[3]This means that if multiple degenerate roots exist, then some of the λ_i among $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ will be equal, and hence will appear more than once on the right hand side of eq. (11).

The coefficients $c_2, c_3, \ldots, c_{n-1}$ are also determined by the eigenvalues. In general,

$$c_k = (-1)^k \sum_{\substack{i_1=1 \\ i_1<i_2<\cdots<i_k}}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_k=1}^{n} \underbrace{\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}}_{k \text{ factors}}, \qquad \text{for} \quad k = 1, 2, \ldots, n\,. \tag{14}$$

For example, if $k = 2 \leq n$ then eq. (14) yields,

$$c_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \ldots + \lambda_1\lambda_n + \lambda_2\lambda_3 + \ldots + \lambda_2\lambda_n + \ldots + \lambda_{n-1}\lambda_n\,.$$

2. The Cayley-Hamilton Theorem

Theorem: Given an $n \times n$ matrix A, the characteristic polynomial is defined by $p(\lambda) = \det(A - \lambda\mathbf{I}) = (-1)^n \left[\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n\right]$, it follows that[4]

$$p(A) = (-1)^n \left[A^n + c_1A^{n-1} + c_2A^{n-2} + \cdots + c_{n-1}A + c_n\mathbf{I}\right] = \mathbf{0}\,, \tag{15}$$

where $A^0 \equiv \mathbf{I}$ is the $n \times n$ identity matrix and $\mathbf{0}$ is the $n \times n$ zero matrix.

False proof: The characteristic polynomial is $p(\lambda) = \det(A - \lambda\mathbf{I})$. Setting $\lambda = A$, we get $p(A) = \det(A - A\mathbf{I}) = \det(A - A) = \det(\mathbf{0}) = 0$. This "proof" does not make any sense. In particular, $p(A)$ is an $n \times n$ matrix, but in this false proof we obtained $p(A) = 0$ where 0 is a number.

Correct proof: Recall that the adjugate of M, denoted by adj M, is the transpose of the matrix of cofactors. Moreover, using eq. (29) of the class handout entitled *Determinant and the Adjugate*, it follows that $M(\text{adj } M) = \mathbf{I}\det M$ for any matrix M. In particular, setting $M = A - \lambda\mathbf{I}$, it follows that

$$(A - \lambda\mathbf{I})\,\text{adj}(A - \lambda\mathbf{I}) = p(\lambda)\mathbf{I}\,, \tag{16}$$

where $p(\lambda) = \det(A - \lambda\mathbf{I})$. Since $p(\lambda)$ is an nth-order polynomial, it then follows from eq. (16) that $\text{adj}(A - \lambda\mathbf{I})$ is a matrix polynomial of order $n - 1$. Thus, we can write:

$$\text{adj}(A - \lambda\mathbf{I}) = B_0 + B_1\lambda + B_2\lambda^2 + \cdots + B_{n-1}\lambda^{n-1}\,,$$

where $B_0,\ B_1, \ldots, B_{n-1}$ are $n \times n$ matrices (whose explicit forms are not required in these notes). Inserting the above result into eq. (16) and using eq. (4), one obtains:

$$(A - \lambda\mathbf{I})(B_0 + B_1\lambda + B_2\lambda^2 + \cdots + B_{n-1}\lambda^{n-1}) = (-1)^n \left[\lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n\right]\mathbf{I}\,. \tag{17}$$

[4]In the expression for $p(\lambda)$, we interpret c_n to mean $c_n\lambda^0$. Thus, when evaluating $p(A)$, the coefficient c_n multiplies $A^0 \equiv \mathbf{I}$.

Eq. (17) is true for any value of λ. Consequently, the coefficient of λ^k on the left-hand side of eq. (17) must equal the coefficient of λ^k on the right-hand side of eq. (17), for $k = 0, 1, 2, \ldots, n$. As a result, the following $n+1$ equations must be satisfied,

$$AB_0 = (-1)^n c_n \mathbf{I}\,, \tag{18}$$

$$-B_{k-1} + AB_k = (-1)^n c_{n-k}\mathbf{I}\,, \qquad k = 1, 2, \ldots, n-1\,, \tag{19}$$

$$-B_{n-1} = (-1)^n \mathbf{I}\,. \tag{20}$$

Using eqs. (18)–(20), we can evaluate the matrix polynomial $p(A)$ as a telescoping sum,

$$\begin{aligned} p(A) &= (-1)^n \left[A^n + c_1 A^{n-1} + c_2 A^{n-2} + \cdots + c_{n-1}A + c_n \mathbf{I}\right] \\ &= AB_0 + (-B_0 + B_1 A)A + (-B_1 + B_2)A^2 + \cdots + (-B_{n-2} + B_{n-1}A)A^{n-1} - B_{n-1}A^n \\ &= \mathbf{0}\,, \end{aligned}$$

which completes the proof of the Cayley-Hamilton theorem.

It is instructive to illustrate the Cayley-Hamilton theorem for 2×2 matrices. In this case,

$$p(\lambda) = \lambda^2 - \lambda \mathrm{Tr}\ A + \det\ A\,.$$

Hence, by the Cayley-Hamilton theorem,

$$p(A) = A^2 - A\,\mathrm{Tr}\ A + \mathbf{I}\det\ A = 0\,.$$

Let us take the trace of this equation. Since $\mathrm{Tr}\ \mathbf{I} = 2$ for the 2×2 identity matrix,

$$\mathrm{Tr}(A^2) - (\mathrm{Tr}\ A)^2 + 2\det\ A = 0\,.$$

It follows that

$$\boxed{\det\ A = \tfrac{1}{2}\left[(\mathrm{Tr}\ A)^2 - \mathrm{Tr}(A^2)\right]\,, \qquad \text{for any }\ 2\times 2 \text{ matrix } A\,.}$$

One can easily verify the validity of this formula for any 2×2 matrix.

One final application of the Cayley-Hamilton theorem is noteworthy. This theorem provides a new way to evaluate the inverse of a matrix. Assuming that $\det A \neq 0$, then one can multiply both sides of eq. (15) by A^{-1} and solve for A^{-1}. Using $c_n = (-1)^n \det\ A$ [cf. eq. (5)], the end result is

$$A^{-1} = \frac{(-1)^{n+1}}{\det A}\left[A^{n-1} + c_1 A^{n-2} + \ldots + c_{n-2}A + c_{n-1}\mathbf{I}\right]. \tag{21}$$

As a check, one can evaluate eq. (21) in the case of $n = 2$. After employing $c_1 = -\mathrm{Tr}\ A$ [cf. eq. (8)], it follows that for $n = 2$ and $A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$,

$$A^{-1} = -\frac{1}{\det A}\left[A - \mathbf{I}\,\mathrm{Tr}\ A\right] = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}\,, \tag{22}$$

as expected. To evaluate the inverse of an $n\times n$ matrix for $n > 2$, one must evaluate the coefficients $c_2, c_3, \ldots c_{n-1}$. These coefficients can be evaluated in terms of traces of powers of the matrix A, as shown in Appendices A and B.

Appendix A: Identifying the coefficients of the characteristic polynomial of $\boldsymbol{A}$ in terms of traces of powers of $\boldsymbol{A}$

The characteristic polynomial of an $n \times n$ matrix A is given by:

$$p(\lambda) = \det(A - \lambda\mathbf{I}) = (-1)^n \left[\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots + c_{n-1}\lambda + c_n\right] .$$

In Section 1, we identified,

$$c_1 = -\text{Tr}\ A\,, \qquad c_n = (-1)^n \det\ A\,. \tag{A.1}$$

One can also derive expressions for $c_2, c_3, \ldots, c_{n-1}$ in terms of traces of powers of A. In this Appendix, I will exhibit the relevant results without proofs (which can be found in the references at the end of these notes).

Let us introduce the notation:

$$t_k \equiv \text{Tr}(A^k)\,. \tag{A.2}$$

The t_k can be expressed in terms of the eigenvalues of A. In particular, note that if $A\vec{\boldsymbol{v}} = \lambda\vec{\boldsymbol{v}}$, then it follows that $A^k\,\vec{\boldsymbol{v}} = \lambda^k\,\vec{\boldsymbol{v}}$. As a result, eqs. (13a) and (A.2) yield,

$$t_k = \lambda_1^k + \lambda_2^k + \ldots + \lambda_n^k\,. \tag{A.3}$$

Likewise, the c_k can be expressed in terms of the eigenvalues of A as shown explicitly in eq. (14). Then, the following set of recursive equations can be derived,

$$t_1 + c_1 = 0 \quad \text{and} \quad t_k + c_1 t_{k-1} + \cdots + c_{k-1}t_1 + kc_k = 0\,, \qquad k = 2, 3, \ldots, n\,. \tag{A.4}$$

These equations are called *Newton's identities*. Two different proofs of these identities can be found in Refs. [1,2].

The equations exhibited in eq. (A.4) are called recursive, since one can solve for the c_k in terms of the traces $t_1, t_2, \ldots, t_k$ iteratively by starting with $c_1 = -t_1$, and then proceeding step by step by solving the equations with $k = 2, 3, \ldots, n$ in successive order. This recursive procedure yields expressions for the c_k in terms of traces of powers of A,

$$c_1 = -t_1\,, \tag{A.5}$$

$$c_2 = \tfrac{1}{2}(t_1^2 - t_2)\,, \tag{A.6}$$

$$c_3 = -\tfrac{1}{6}t_1^3 + \tfrac{1}{2}t_1 t_2 - \tfrac{1}{3}t_3\,, \tag{A.7}$$

$$c_4 = \tfrac{1}{24}t_1^4 - \tfrac{1}{4}t_1^2 t_2 + \tfrac{1}{3}t_1 t_3 + \tfrac{1}{8}t_2^2 - \tfrac{1}{4}t_4\,, \tag{A.8}$$

and so on for $k > 4$. The results above can be summarized by the following equation (see, e.g., Ref. [3]),

$$c_k = -\frac{t_k}{k} + \frac{1}{2!}\sum_{\substack{i=1 \\ i+j=k}}^{k-1}\sum_{j=1}^{k-1}\frac{t_i t_j}{ij} - \frac{1}{3!}\sum_{\substack{i=1 \\ i+j+\ell=k}}^{k-2}\sum_{j=1}^{k-2}\sum_{\ell=1}^{k-2}\frac{t_i t_j t_\ell}{ij\ell} + \cdots + \frac{(-1)^k t_1^k}{k!}\,, \qquad k = 1, 2, \ldots, n\,. \tag{A.9}$$

Note that by using $c_n = (-1)^n \det A$, one obtains a general expression for the determinant in terms of traces of powers of A [cf. eq. (A.2)],

$$\det A = (-1)^n \left[-\frac{t_n}{n} + \frac{1}{2!}\sum_{\substack{i=1 \\ i+j=n}}^{n-1}\sum_{j=1}^{n-1} \frac{t_i t_j}{ij} - \frac{1}{3!}\sum_{\substack{i=1 \\ i+j+k=n}}^{n-2}\sum_{j=1}^{n-2}\sum_{k=1}^{n-2} \frac{t_i t_j t_k}{ijk} + \cdots + \frac{(-1)^n t_1^n}{n!} \right]. \tag{A.10}$$

For example, using eq. (A.10), one obtains the following explicit expressions for $n = 2$ and $n = 3$, respectively,

$$\det A = \tfrac{1}{2}\left[(\mathrm{Tr}\ A)^2 - \mathrm{Tr}(A^2)\right], \qquad \text{for any } \ 2\times 2 \text{ matrix},$$

$$\det A = \tfrac{1}{6}\left[(\mathrm{Tr}\ A)^3 - 3(\mathrm{Tr}\ A)\,\mathrm{Tr}(A^2) + 2\,\mathrm{Tr}(A^3)\right], \qquad \text{for any } \ 3\times 3 \text{ matrix}.$$

B: The coefficients of the characteristic polynomial revisited

One can derive another closed-form expression for the c_k. To see how to do this, let us write out the Newton identities explicitly. Eq. (A.4) for $k = 1, 2, \ldots, n$ yields the following n equations,

$$\begin{aligned} c_1 &= -t_1\,, \\ t_1 c_1 + 2c_2 &= -t_2\,, \\ t_2 c_1 + t_1 c_2 + 3c_3 &= -t_3\,, \\ \vdots \qquad & \qquad \vdots \\ t_{k-1}c_1 + t_{k-2}c_2 + \cdots + t_1 c_{k-1} + kc_k &= -t_k\,, \\ \vdots \qquad & \qquad \vdots \\ t_{n-1}c_1 + t_{n-2}c_2 + \cdots + t_1 c_{n-1} + nc_n &= -t_n\,, \end{aligned}$$

where $t_k \equiv \mathrm{Tr}(A^k)$. Consider the first k equations above (for any value of $k = 1, 2, \ldots, n$). This is a system of linear equations for $c_1, c_2, \ldots, c_k$, which can be written in matrix form:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ t_1 & 2 & 0 & \cdots & 0 & 0 \\ t_2 & t_1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{k-2} & t_{k-3} & t_{k-4} & \cdots & k-1 & 0 \\ t_{k-1} & t_{k-2} & t_{k-3} & \cdots & t_1 & k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{k-1} \\ c_k \end{pmatrix} = \begin{pmatrix} -t_1 \\ -t_2 \\ -t_3 \\ \vdots \\ -t_{k-1} \\ -t_k \end{pmatrix}.$$

Applying Cramer's rule, we can solve for c_k in terms of $t_1, t_2, \ldots, t_k$ [3]:

$$c_k = \frac{\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & -t_1 \\ t_1 & 2 & 0 & \cdots & 0 & -t_2 \\ t_2 & t_1 & 3 & \cdots & 0 & -t_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{k-2} & t_{k-3} & t_{k-4} & \cdots & k-1 & -t_{k-1} \\ t_{k-1} & t_{k-2} & t_{k-3} & \cdots & t_1 & -t_k \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ t_1 & 2 & 0 & \cdots & 0 & 0 \\ t_2 & t_1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{k-2} & t_{k-3} & t_{k-4} & \cdots & k-1 & 0 \\ t_{k-1} & t_{k-2} & t_{k-3} & \cdots & t_1 & k \end{vmatrix}} . \tag{B.1}$$

Note that the denominator is the determinant of a lower triangular matrix, which is equal to the product of its diagonal elements (i.e., it is equal to $k!$).

From eq. (B.1), it then follows that,

$$c_k = \frac{1}{k!} \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & -t_1 \\ t_1 & 2 & 0 & \cdots & 0 & -t_2 \\ t_2 & t_1 & 3 & \cdots & 0 & -t_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ t_{k-2} & t_{k-3} & t_{k-4} & \cdots & k-1 & -t_{k-1} \\ t_{k-1} & t_{k-2} & t_{k-3} & \cdots & t_1 & -t_k \end{vmatrix} .$$

It is convenient to multiply the kth column of the above matrix by -1, and then move the kth column over to the first column (which requires a series of $k-1$ interchanges of adjacent columns). These operations multiply the determinant by (-1) and $(-1)^{k-1}$ respectively, leading to an overall sign change of $(-1)^k$. Hence, our final result is:[5]

$$c_k = \frac{(-1)^k}{k!} \begin{vmatrix} t_1 & 1 & 0 & 0 & \cdots & 0 \\ t_2 & t_1 & 2 & 0 & \cdots & 0 \\ t_3 & t_2 & t_1 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{k-1} & t_{k-2} & t_{k-3} & t_{k-4} & \cdots & k-1 \\ t_k & t_{k-1} & t_{k-2} & t_{k-3} & \cdots & t_1 \end{vmatrix} , \qquad k = 1, 2, \ldots, n \,, \tag{B.2}$$

which is equivalent to eq. (A.9).

[5] Eq. (B.2) is derived in section 4.1 on p. 20 of Ref. [4]. However, the determinantal expression given in Ref. [4] for $\sigma_k \equiv (-1)^k c_k$ contains a typographical error—the diagonal series of integers, $1, 1, 1, \ldots, 1$, appearing just above the main diagonal of σ_k should be replaced by $1, 2, 3, \ldots, k-1$.

One can test this formula by evaluating the first three cases $k = 1, 2, 3$:

$$c_1 = -t_1\,, \qquad\qquad c_2 = \frac{1}{2!}\begin{vmatrix} t_1 & 1 \\ t_2 & t_1 \end{vmatrix} = \tfrac{1}{2}(t_1^2 - t_2)\,,$$

$$c_3 = -\frac{1}{3!}\begin{vmatrix} t_1 & 1 & 0 \\ t_2 & t_1 & 2 \\ t_3 & t_2 & t_1 \end{vmatrix} = \tfrac{1}{6}\left[-t_1^3 + 3t_1t_2 - 2t_3\right]\,,$$

which coincide with the previously stated results [cf. eqs. (A.5)–(A.7)]. Finally, setting $k = n$ in eq. (B.2) yields $c_n = (-1)^n \det A$, which provides a formula for the determinant of the $n \times n$ matrix A in terms of traces of powers of A [cf. eq. (A.2)],

$$\det\, A = \frac{1}{n!}\begin{vmatrix} t_1 & 1 & 0 & 0 & \cdots & 0 \\ t_2 & t_1 & 2 & 0 & \cdots & 0 \\ t_3 & t_2 & t_1 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & t_{n-2} & t_{n-3} & t_{n-4} & \cdots & n-1 \\ t_n & t_{n-1} & t_{n-2} & t_{n-3} & \cdots & t_1 \end{vmatrix}\,, \qquad \text{(B.3)}$$

which is equivalent to eq. (A.10). Indeed, one can check that our previous results for the determinants of a 2×2 matrix and a 3×3 matrix, exhibited below eq. (A.10), are recovered.

Appendix C: Identifying the coefficients of the characteristic polynomial of A in terms of its matrix elements

Given an $n \times n$ matrix $A = [a_{ij}]$, eqs. (5) and (8) provide expressions for c_1 and c_n in terms of the trace and determinant of A, respectively, which again are repeated here,

$$c_1 = -\mathrm{Tr}\, A\,, \qquad\qquad c_n = (-1)^n \det A\,. \qquad \text{(C.1)}$$

Related expressions exist for the c_k ($k = 2, 3, \ldots, n-1$). In particular, Ref. [5] shows that $(-1)^k c_k$ is equal to the sum of the determinants of the $k \times k$ principal submatrices[6] of A in all possible ways [there are $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ terms in the sum]. That is,

$$c_k = (-1)^k \sum_{\substack{i_1=1 \\ i_1<i_2<\cdots<i_k}}^{n} \sum_{i_2=1}^{n} \cdots \sum_{i_k=1}^{n} \det \begin{pmatrix} a_{i_1i_1} & a_{i_1i_2} & \cdots & a_{i_1i_k} \\ a_{i_2i_1} & a_{i_2i_2} & \cdots & a_{i_2i_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_ki_1} & a_{i_ki_2} & \cdots & a_{i_ki_k} \end{pmatrix}\,, \qquad \text{for} \quad k = 1, 2, \ldots, n\,. \qquad \text{(C.2)}$$

[6] A $k \times k$ principal submatrix of the $n \times n$ matrix A is obtained by removing $n-k$ rows and columns of A such that the set of row indices that remain is the same as the set of column indices that remain.

It is easy to verify that the $k = 1$ and $k = n$ cases of eq. (C.2) reduce to the results given in eq. (C.1). As a more nontrivial example, note that for $k = 2$ we obtain,

$$c_2 = \sum_{\substack{i=1 \\ i<j}}^{n} \sum_{j=1}^{n} \det \begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix} = \sum_{\substack{i=1 \\ i<j}}^{n} \sum_{j=1}^{n} (a_{ii}a_{jj} - a_{ij}a_{ji}) \,. \tag{C.3}$$

Comparing with eq. (9), we can identify $c_2' = -\sum_{1\leq i<j\leq n} a_{ij}a_{ji}$. Indeed a careful analysis of $\det(A - \lambda\mathbf{I})$ will yield the quoted result, where c_2' is defined in eq. (6).

Finally, it is straightforward but tedious to check that eq. (C.3) is equivalent to

$$c_2 = \tfrac{1}{2}\left[(\operatorname{Tr} A)^2 - \operatorname{Tr}(A^2)\right] ,$$

which was previously obtained in eq. (A.6).

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