What is the group of conjugate symplectic matrices?

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Abstract

The group of complex symplectic matrices, $\mathrm{Sp}(n,\mathbb{C})$ is defined as the set of $2n \times 2n$ complex matrices, $\{M \in \mathrm{GL}(2n,\mathbb{C})\,|\,M^{\mathsf{T}}JM = J\}$, where $J \equiv \left(\begin{smallmatrix} 0 & \mathbf{I_n} \\ -\mathbf{I_n} & 0 \end{smallmatrix}\right)$, and $\mathbf{I_n}$ is the $n \times n$ identity matrix. If the transpose is replaced by hermitian conjugate, then one obtains the group of conjugate symplectic matrices, $\{M \in \mathrm{GL}(2n,\mathbb{C})\,|\,M^{\dagger}JM = J\}$. In these notes, we demonstrate that the group of conjugate symplectic matrices is isomorphic to $\mathrm{U}(n,n) = \{M \in \mathrm{GL}(2n,\mathbb{C})\,|\,M^{\dagger}I_{n,n}M = I_{n,n}\}$, where $I_{n,n} \equiv \left(\begin{smallmatrix} \mathbf{I_n} & 0 \\ 0 & -\mathbf{I_n} \end{smallmatrix}\right)$.

The group of complex symplectic matrices, $\mathrm{Sp}(n,\mathbb{C})$ is defined as the set of $2n \times 2n$ complex matrices,[1]

$$\mathrm{Sp}(n,\mathbb{C}) \equiv \{M \in \mathrm{GL}(2n,\mathbb{C})\,|\,M^{\mathsf{T}}JM = J\}\,, \tag{1}$$

where the matrix J is defined in block form by,

$$J \equiv \begin{pmatrix} 0 & \mathbf{I_n} \\ -\mathbf{I_n} & 0 \end{pmatrix}\,, \tag{2}$$

with 0 denoting the $n \times n$ zero matrix and $\mathbf{I_n}$ the $n \times n$ identity matrix. $\mathrm{Sp}(n,\mathbb{C})$ is a complex simple Lie group. Moreover, the determinant of any complex symplectic matrix is 1.[2]

If the transpose is replaced by hermitian conjugate in eq. (1), then one obtains the group of conjugate symplectic matrices,

$$G = \{M \in \mathrm{GL}(2n,\mathbb{C})\,|\,M^{\dagger}JM = J\}\,, \tag{3}$$

following the terminology or Refs. [1, 2]. Indeed, the group of conjugate symplectic matrices is a Lie group that is neither semi-simple nor complex. First, note that taking the determinant of the condition $M^{\dagger}JM = J$ implies that $\det M$ is a complex phase of unit magnitude. This group of phases constitutes a normal U(1) subgroup of G, which implies that G is not semi-simple. Second, G not a complex Lie group since it is defined by a condition, $M^{\dagger}JM = J$, that is not holomorphic. Nevertheless, it should be possible to identify this group as being isomorphic to one of the real forms of $\mathrm{GL}(2n,\mathbb{C})$. In these notes, I shall identify the real Lie group in question.

[1] The reader should be aware that in the literature, the group defined by eq. (1) is often denoted by $\mathrm{Sp}(2n,\mathbb{C})$.

[2] Taking the determinant of the defining relation $M^{\mathsf{T}}JM = J$ yields $\det M = \pm 1$. However, since all elements of $\mathrm{Sp}(n,\mathbb{C})$ are connected to the identity, it follows that $\det M = 1$. Another proof of this fact, which employs the properties of the pfaffian, is provided in Ref. [3].

We begin by diagonalizing the matrix J defined in eq. (2),

$$\frac{1}{2}\begin{pmatrix}\mathbf{I_n} & -i\mathbf{I_n}\\ \mathbf{I_n} & i\mathbf{I_n}\end{pmatrix}\begin{pmatrix}0 & \mathbf{I_n}\\ -\mathbf{I_n} & 0\end{pmatrix}\begin{pmatrix}\mathbf{I_n} & \mathbf{I_n}\\ i\mathbf{I_n} & -i\mathbf{I_n}\end{pmatrix} = i\begin{pmatrix}\mathbf{I_n} & 0\\ 0 & -\mathbf{I_n}\end{pmatrix}. \tag{4}$$

It is convenient to introduce the notation for the $(p+q)\times(p+q)$ diagonal matrix,

$$I_{p,q} \equiv \mathrm{diag}(\underbrace{1,1,\ldots,1}_{p},\underbrace{-1,-1,\ldots,-1}_{q}). \tag{5}$$

In particular, note that

$$I_{n,n} \equiv \begin{pmatrix}\mathbf{I_n} & 0\\ 0 & -\mathbf{I_n}\end{pmatrix}. \tag{6}$$

Employing the unitary $2n\times 2n$ matrix,

$$U \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}\mathbf{I_n} & \mathbf{I_n}\\ i\mathbf{I_n} & -i\mathbf{I_n}\end{pmatrix}, \tag{7}$$

allows us to rewrite eq. (4) as

$$U^\dagger J U = iI_{n,n}. \tag{8}$$

We now introduce the matrix N defined by

$$N \equiv U^\dagger M U, \tag{9}$$

where M is a conjugate symplectic matrix defined in eq. (3). Using eq. (8) and the fact that U is unitary, it follows that the condition $M^\dagger J M = J$ yields,

$$N^\dagger I_{n,n} N = I_{n,n}. \tag{10}$$

We now recognize the Lie group [4],

$$\mathrm{U}(p,q) \equiv \{V \in \mathrm{GL}(n,\mathbb{C})\,|\,V^\dagger I_{p,q} V = I_{p,q}\}, \quad \text{where } p+q=n. \tag{11}$$

Note that the Lie groups $\mathrm{U}(p\,,n-p)$ are real forms[3] of the complex Lie group $\mathrm{GL}(n,\mathbb{C})$. In light of eq. (10), it follows that $N\in\mathrm{U}(n,n)$.

Thus, eq. (9) establishes a bijective map from the group of conjugate symplectic matrices to the Lie group $\mathrm{U}(n,\,n)$ that takes $M \longmapsto N$, which implies that the following is an isomorphism,

$$\mathrm{U}(n,n) \cong \{M \in \mathrm{GL}(2n,\mathbb{C})\,|\,M^\dagger J M = J\}. \tag{12}$$

[3]That is, the complexification of $\mathrm{U}(p\,,n-p)$, for any value of $p=0,1,2,\ldots,n$, yields the complex Lie group $\mathrm{GL}(n,\mathbb{C})$.

References

[1] A. Bunse-Gerstner, R. Byers, V. Mehrmann, "A chart of numerical methods for structured eigenvalue problems," SIAM J. Matrix Anal. Appl. 13, 419 (1992).

[2] D.S. Mackey, N. Mackey, F. Tisseur, "Structured tools for structured matrices," Electron. J. Linear Algebra 10, 106 (2003).

[3] H.E. Haber, "Notes on antisymmetric matrices and the pfaffian," available on the Web via `http://scipp.ucsc.edu/~haber/webpage/pfaffian2.pdf`.

[4] See, e.g., Table 8.1 on p. 150 of J.F. Cornwell, *Group Theory in Physics—An Introduction* (Academic Press, San Diego, CA, 1997).