Evaluating a logarithmic integral arising at one-loop

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Abstract

In these notes, we provide an explicit calculation of the finite part of the integral $B_0(p^2, m_1^2, m_2^2)$ that arises in the calculation of one-loop self-energy diagrams.

1. The Passarino-Veltman function $B_0(p^2, m_1^2, m_2^2)$

In Ref. [1], Passarino and Veltman introduced the following function that arises in the calculation of one-loop self-energy diagrams,[1]

$$B_0(p^2; m_1^2, m_2^2) \equiv -16\pi^2(\mu^2)^{2-\frac{1}{2}n}\, i \int \frac{d^n q}{(2\pi)^n}\, \frac{1}{(q^2 - m_1^2 + i\varepsilon)[(q+p)^2 - m_2^2 + i\varepsilon]}\,, \tag{1}$$

where m_1 and m_2 are real nonnegative quantities and $\mu^2 > 0$.

To evaluate $B_0(p^2; m_1^2, m_2^2)$, we first employ Feynman's trick to write

$$\begin{aligned}\frac{1}{(q^2 - m_1^2 + i\varepsilon)[(q+p)^2 - m_2^2 + i\varepsilon]} &= \int_0^1 \frac{dx}{\big[(1-x)(q^2 - m_1^2) + x[(q+p_)^2 - m_2^2] + i\varepsilon\big]^2} \\ &= \int_0^1 \frac{dx}{\big[q^2 + 2xq\cdot p + (p^2 + m_1^2 - m_2^2)x - m_1^2 + i\varepsilon\big]^2}\,. \end{aligned} \tag{2}$$

Plugging this result into eq. (1), interchanging the order of integration and employing the well-known formula of dimensional regularization,

$$\int \frac{d^n q}{(2\pi)^n}\, \frac{1}{(q^2 + 2q\cdot p - m^2 + i\varepsilon)^r} = \quad i(-1)^r (p^2 + m^2)^{2-\epsilon-r}(4\pi)^{\epsilon-2}\, \frac{\Gamma(\epsilon + r - 2)}{\Gamma(r)}\,, \tag{3}$$

where $\epsilon \equiv 2 - \frac{1}{2}n$, it follows that

$$\begin{aligned} B_0(p^2; m_1^2, m_2^2) &= (4\pi\mu^2)^\epsilon\, \Gamma(\epsilon) \int_0^1 \big[p^2x^2 - (p^2 + m_1^2 - m_2^2)x + m_1^2 - i\varepsilon\big]^{-\epsilon} dx \\ &= \left(\frac{1}{\epsilon} - \gamma + \ln(4\pi)\right)\left[1 - \epsilon \int_0^1 \ln\left(\frac{p^2x^2 - (p^2 + m_1^2 - m_2^2)x + m_1^2 - i\varepsilon}{\mu^2}\right) dx + \mathcal{O}(\epsilon^2)\right] \\ &= \Delta - \int_0^1 \ln\left(\frac{p^2x^2 - (p^2 + m_1^2 - m_2^2)x + m_1^2 - i\varepsilon}{\mu^2}\right) dx + \mathcal{O}(\epsilon)\,, \end{aligned} \tag{4}$$

after expanding in ϵ and defining $\Delta \equiv \epsilon^{-1} - \gamma + \ln(4\pi)$, where γ is Euler's constant.

[1] The definition of $B_0(p^2; m_1^2, m_2^2)$ given in eq. (1) differs slightly from the original definition appearing in Ref. [1]. Here, we have modified the multiplicative prefactor, and we have employed the mostly minus metric, $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, following the notation of eq. (2.5.64) of Ref. [2].

2. The evaluation of a logarithmic integral

Eq. (4) motivates the computation of the following integral,

$$\mathcal{I} \equiv \int_0^1 \ln(Ax^2 + Bx + C - i\varepsilon)\, dx\,, \tag{5}$$

where A, B and C are real numbers and ε is a positive infinitesimal constant. As a warmup, we first consider the case of $A = 0$ and $B \neq 0$. Then,

$$\mathcal{I}_0 \equiv \int_0^1 \ln(Bx + C - i\varepsilon)\, dx\,. \tag{6}$$

Integrating by parts by taking $u = \ln(Bx + C + i\varepsilon)$ and $dv = dx$, it follows that

$$\begin{aligned}
\mathcal{I}_0 &= \ln(B + C - i\varepsilon) - \int_0^1 \frac{Bx\, dx}{Bx + C - i\varepsilon} \\
&= \ln(B + C - i\varepsilon) - \int_0^1 \frac{Bx + C - i\varepsilon - (C - i\varepsilon)\, dx}{Bx + C - i\varepsilon} \\
&= \ln(B + C - i\varepsilon) - 1 + C \int_0^1 \frac{dx}{Bx + C - i\varepsilon}\,,
\end{aligned} \tag{7}$$

where it is safe to drop the $i\varepsilon$ factor in the numerator. Defining $C = rB$,

$$\begin{aligned}
\mathcal{I}_0 = &= \ln(B + C - i\varepsilon) - 1 + r\big[\ln(1 + r - i\varepsilon \operatorname{sgn} B) - \ln(r - i\varepsilon \operatorname{sgn} B)\big] \\
&= \ln(B + C - i\varepsilon) - 1 + \frac{C}{B}\left[\ln\left(\frac{B + C - i\varepsilon}{B}\right) - \ln\left(\frac{C - i\varepsilon}{B}\right)\right], \qquad \text{for } B \neq 0.
\end{aligned} \tag{8}$$

Two equivalent forms of eq. (8) are of some interest. By combining the two logarithms, we end up with,

$$\mathcal{I}_0 = \ln(B + C - i\varepsilon) - 1 + \frac{C}{B} \ln\left(\frac{B + C}{C} + i\varepsilon \operatorname{sgn} B\right), \qquad \text{for } B \neq 0. \tag{9}$$

Alternatively, one can rewrite eq. (8) as

$$\mathcal{I}_0 = L - 1 + \left(1 + \frac{C}{B}\right) \ln\left(\frac{B + C - i\varepsilon}{B}\right) - \frac{C}{B} \ln\left(\frac{C - i\varepsilon}{B}\right), \tag{10}$$

where

$$L \equiv \ln(B + C - i\varepsilon) - \ln\left(\frac{B + C - i\varepsilon}{B}\right). \tag{11}$$

If $B > 0$, then $L = \ln B$. If $B < 0$ and $B + C > 0$,

$$L = \ln(B + C) - i\pi - \ln(B + C) + \ln(-B) = \ln(-B) - i\pi\,. \tag{12}$$

If $B<0$ and $B+C<0$,

$$L=\ln(-B-C)-i\pi-\ln(-B-C)+\ln(-B)=\ln(-B)-i\pi\,. \tag{13}$$

Thus, we conclude that in all three cases considered above,

$$L=\ln(B-i\varepsilon)\,. \tag{14}$$

Hence, another alternative form for eq. (8) is given by,[2]

$$\mathcal{I}_0=\ln(B-i\varepsilon)-1+\left(1+\frac{C}{B}\right)\ln\left(\frac{B+C-i\varepsilon}{B}\right)-\frac{C}{B}\ln\left(\frac{C-i\varepsilon}{B}\right)\,,\qquad\text{for } B\neq 0. \tag{15}$$

Perhaps you are concerned with some of the manipulations used to go from eq. (7) to eq. (8). As a check, we can use eq. (7) compute $\operatorname{Im}\mathcal{I}_0$. If $Bx+C$ does not vanish in the integration region of $0<x<1$, then $Bx+C>0$ over the full integration range if $C>0$ and $B+C>0$ whereas $Bx+C<0$ over the full integration range if $C<0$ and $B+C<0$. In contrast, if $Bx+C$ vanishes within the integration region of $0<x<1$, then $0<-C/B<1$. It then follows that the argument of the logarithm is negative over the region $0<x<-C/B$ if $B>0$, $C<0$ and $B+C>0$, whereas the argument of the logarithm is negative over the region $-C/B<x<1$ if $B<0$, $C>0$ and $B+C<0$. It then follows that,

$$\operatorname{Im}\mathcal{I}_0=\begin{cases} 0\,, & \text{if } C>0 \text{ and } B+C>0,\\[1ex] -\pi\int_0^1=-\pi\,, & \text{if } C<0 \text{ and } B+C<0,\\[1ex] -\pi\int_0^{-C/B}dx=\pi C/B\,, & \text{if } B>0,\ C<0 \text{ and } B+C>0,\\[1ex] -\pi\int_{-C/B}^1 dx=-\pi(B+C)/B\,, & \text{if } B<0,\ C>0 \text{ and } B+C<0.\end{cases} \tag{16}$$

It is straightforward to check that eqs. (8), (9) and (15) all yield the result quoted in eq. (16).

We now return to the evaluation of $\mathcal{I}$, under the assumption that $A\neq 0$. We again integrate by parts with $u=\ln(A^2x+Bx+C-i\varepsilon)$ and $dv=dx$ to obtain

$$\begin{aligned}\mathcal{I}&=\ln(A+B+C-i\varepsilon)-\int_0^1\frac{(2Ax^2+Bx)\,dx}{Ax^2+Bx+C-i\varepsilon}\\ &=\ln(A+B+C-i\varepsilon)-\int_0^1\frac{2(Ax^2+Bx+C-i\varepsilon)-(Bx+2C-2i\varepsilon)\,dx}{Ax^2+Bx+C-i\varepsilon}\\ &=\ln(A+B+C-i\varepsilon)-2+\int_0^1\frac{(Bx+2C)\,dx}{Ax^2+Bx+C-i\varepsilon}\,,\end{aligned} \tag{17}$$

where it is safe to drop the $i\varepsilon$ factor in the numerator. Defining $B=r_1A$ and $C=r_2A$,

$$\mathcal{I}=\ln(A+B+C-i\varepsilon)-2+\int_0^1\frac{(r_1x+2r_2)\,dx}{x^2+r_1x+r_2-i\varepsilon\operatorname{sgn}A}\,, \tag{18}$$

[2] Eq. (15) corrects the incorrect version of this formula that appears in eq. (D.14) in Appendix D of Ref. [3]. Indeed, one can correct eq. (D.14) of the cited paper by replacing ε with $-\varepsilon\operatorname{sgn}b$.

We consider three cases.

Case 1: $r_1^2 < 4r_2$

In this case, the roots of polynomial equation $x^2+r_1x+r_2=0$ are complex. Consequently, the denominator of the integrand above never vanishes, and we are free to set $\varepsilon=0$. After factoring the denominator,

$$x^2+r_1x+r_2=(x-x_+)(x-x_-)\,, \tag{19}$$

where

$$x_\pm \equiv \tfrac{1}{2}\big[-r_1 \pm i\sqrt{4r_2-r_1^2}\big]\,, \tag{20}$$

we perform a partial fractioning,

$$\frac{r_1x+2r_2}{(x-x_+)(x-x_-)} = -\left(\frac{x_+}{x-x_+}+\frac{x_-}{x-x_-}\right)\,, \tag{21}$$

after noting that

$$x_+ + x_- = -r_1\,, \qquad\qquad x_+x_- = r_2\,. \tag{22}$$

Hence, it follows that

$$\begin{aligned}
\mathcal{I} &= \ln(A+B+C-i\varepsilon) - 2 - \int_0^1 \left(\frac{x_+}{x-x_+}+\frac{x_-}{x-x_-}\right)dx \\
&= \ln(A+B+C-i\varepsilon) - 2 - x_+\big[\ln(1-x_+)-\ln(-x_+)\big] - x_-\big[\ln(1-x_-)-\ln(-x_-)\big] \\
&= \ln(A+B+C-i\varepsilon) - 2 - 2\,\mathrm{Re}\left\{x_+\big[\ln(1-x_+)-\ln(-x_+)\big]\right\} \\
&= \ln(A+B+C-i\varepsilon) - 2 + \mathrm{Re}\left\{\left(r_1 - i\sqrt{4r_2-r_1^2}\right)\big[\ln(1-x_+)-\ln(-x_+)\big]\right\},
\end{aligned} \tag{23}$$

after using $(x_+)^* = x_1$ in the penultimate step above. The logarithms above are the principal values of the corresponding complex logarithms defined on the cut complex plane, where the branch cut runs along the real axis from $-\infty$ to the origin.

Evaluating the real part of the expression above is straightforward.

$$\begin{aligned}
&\mathrm{Re}\left\{\left(r_1 - i\sqrt{4r_2-r_1^2}\right)\big[\ln(1-x_+)-\ln(-x_+)\big]\right\} \\
&\quad = r_1\big\{\ln|1-x_+| - \ln|x_+|\big\} + \sqrt{4r_2-r_1^2}\,\big[\arg(1-x_+)-\arg(-x_+)\big] \\
&\quad = \tfrac{1}{2}r_1 \ln\left(\frac{r_1+r_2+1}{r_2}\right) + \sqrt{4r_2-r_1^2}\,\Bigg[\arg\left(1+\tfrac{1}{2}r_1-\tfrac{1}{2}i\sqrt{4r_2-r_1^2}\right) \\
&\qquad\qquad - \arg\left(\tfrac{1}{2}r_1-\tfrac{1}{2}i\sqrt{4r_2-r_1^2}\right)\Bigg],
\end{aligned} \tag{24}$$

where the principal value of the argument lies in the range $-\pi < \arg z \leq \pi$ for any complex number z. Since the roots of polynomial equation $x^2 + r_1 x + r_2 = 0$ are complex, it follows that $x^2 + r_1 x + r_2 > 0$ for all values of x. Setting $x = 0$ and $x = 1$, respectively, in the inequality, one can conclude that $r_2 > 0$ and $r_1 + r_2 + 1 > 0$.

In order to evaluate the argument functions in eq. (24), we make use of the following result for the principal value of the argument function,

$$\arg(x - iy) = \begin{cases} \arctan(y/x)\,, & \text{for } x > 0 \text{ and } y > 0, \\ \pi + \arctan(y/x)\,, & \text{for } x < 0 \text{ and } y > 0, \end{cases} \tag{25}$$

where we have employed the principal value of the real arctangent function, which satisfies $|\arctan(y/x)| \leq \frac{1}{2}\pi$. Referring to p. 119 of Ref. [4],

$$\arctan(y/x) = \begin{cases} \frac{1}{2}\pi - \arctan(x/y)\,, & \text{for } y/x > 0, \\ -\frac{1}{2}\pi - \arctan(x/y)\,, & \text{for } y/x < 0. \end{cases} \tag{26}$$

Noting that $\arctan(x/y) = -\arctan(-x/y)$, it follows that

$$\arg(x - iy) = \tfrac{1}{2}\pi + \arctan(-x/y)\,, \qquad \text{for } y > 0, \tag{27}$$

which holds for both signs of x.

Hence,

$$\arg\left(1 + \tfrac{1}{2}r_1 - \tfrac{1}{2}i\sqrt{4r_2 - r_1^2}\right) - \arg\left(\tfrac{1}{2}r_1 - \tfrac{1}{2}i\sqrt{4r_2 - r_1^2}\right)$$
$$= \arctan\left(\frac{2 + r_1}{\sqrt{4r_2 - r_1^2}}\right) - \arctan\left(\frac{r_1}{\sqrt{4r_2 - r_1^2}}\right)\,. \tag{28}$$

Combining the results of eqs. (23), (24) and (28),

$$\mathcal{I} = \ln(A + B + C - i\varepsilon) + \tfrac{1}{2}r_1 \ln\left(\frac{r_1 + r_2 + 1}{r_2}\right) - 2$$
$$+ \sqrt{4r_2 - r_1^2}\left[\arctan\left(\frac{2 + r_1}{\sqrt{4r_2 - r_1^2}}\right) - \arctan\left(\frac{r_1}{\sqrt{4r_2 - r_1^2}}\right)\right]\,. \tag{29}$$

Plugging in $r_1 = B/A$ and $r_2 = C/A$, we obtain our final result,

$$\mathcal{I} = \ln(A + B + C - i\varepsilon) + \frac{B}{2A}\ln\left(\frac{A + B + C}{C}\right) - 2$$
$$+ \frac{\sqrt{4AC - B^2}}{A}\left[\arctan\left(\frac{2A + B}{\sqrt{4AC - B^2}}\right) - \arctan\left(\frac{B}{\sqrt{4AC - B^2}}\right)\right],$$
$$\text{for } A \neq 0 \text{ and } B^2 - 4AC < 0, \tag{30}$$

in agreement with the result quoted in Appendix D of Ref. [3]. In particular,

$$\operatorname{Im}\mathcal{I} = -\pi\Theta(-A - B - C)\,, \qquad \text{for } A \neq 0 \text{ and } B^2 - 4AC < 0. \tag{31}$$

Case 2: $r_1^2 > 4r_2$

In this case, the roots of polynomial equation $x^2 + r_1x + r_2 = 0$ are real, and

$$x^2 + r_1x + r_2 = (x - x_+)(x - x_-)\,, \tag{32}$$

where

$$x_\pm \equiv \tfrac{1}{2}\big[-r_1 \pm \sqrt{r_1^2 - 4r_2}\big]\,. \tag{33}$$

The real roots satisfy,

$$x_+ + x_- = -r_1\,, \qquad x_+ - x_- = \sqrt{r_1^2 - 4r_2}\,, \qquad x_+x_- = r_2\,, \tag{34}$$

If either (or both) x_+ or x_- lie in the integration region of $0 < x < 1$, then the denominator of the integrand of eq. (18) would vanish if we set $\varepsilon = 0$. Hence, we keep the $i\varepsilon$ term present and note that

$$x^2 + rx_1 + r_2 - i\varepsilon\,\mathrm{sgn}\,A = (x - x_+ + i\varepsilon\,\mathrm{sgn}\,A)(x - x_- - i\varepsilon\,\mathrm{sgn}\,A)\,. \tag{35}$$

Hence, performing a partial fractioning of the integrand of eq. (18) yields

$$\frac{(r_1x + 2r_2)}{x^2 + r_1x + r_2 - i\varepsilon\,\mathrm{sgn}\,A} = \frac{x_+}{x - x_+ - i\varepsilon\,\mathrm{sgn}\,A} + \frac{x_-}{x - x_- + i\varepsilon\,\mathrm{sgn}\,A}\,, \tag{36}$$

after omitting the term proportional to $i\varepsilon\,\mathrm{sgn}\,A$ in the numerator, which can be safely dropped in the limit of $\varepsilon \to 0$. Hence, it follows that

$$\begin{aligned}\mathcal{I} &= \ln(A + B + C - i\varepsilon) - 2 - \int_0^1 \left(\frac{x_+}{x - x_+ - i\varepsilon\,\mathrm{sgn}\,A} + \frac{x_-}{x - x_- + i\varepsilon\,\mathrm{sgn}\,A}\right) dx \\ &= \ln(A + B + C - i\varepsilon) - 2 - x_+\big[\ln(1 - x_+ - i\varepsilon\,\mathrm{sgn}\,A) - \ln(-x_+ - i\varepsilon\,\mathrm{sgn}\,A)\big] \\ &\qquad - x_-\big[\ln(1 - x_- + i\varepsilon\,\mathrm{sgn}\,A) - \ln(-x_- + i\varepsilon\,\mathrm{sgn}\,A)\big]\,. \end{aligned} \tag{37}$$

Note that

$$x_\pm = \frac{-B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2|A|}\,. \tag{38}$$

In the literature, it is more typical to define,

$$y_\pm = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}\,, \tag{39}$$

under the assumption that $B^2 - 4AC > 0$. Note that

$$y_+ + y_- = -\frac{B}{A}\,, \qquad y_+y_- = \frac{C}{A}\,, \qquad y_+ - y_- = \frac{\sqrt{B^2 - 4AC}}{A}\,. \tag{40}$$

and it follows that $y_+ > y_-$ if $\mathrm{sgn}\,A > 0$, but $y_+ < y_-$ if $\mathrm{sgn}\,A < 0$. In this notation, eq. (37) can be rewritten as,

$$\begin{aligned}\mathcal{I} &= \ln(A + B + C - i\varepsilon) - 2 - y_+\big[\ln(1 - y_+ - i\varepsilon) - \ln(-y_+ - i\varepsilon)\big] \\ &\qquad - y_-\big[\ln(1 - y_- + i\varepsilon) - \ln(-y_- + i\varepsilon)\big]\,, \qquad \text{for } A \neq 0 \text{ and } B^2 - 4AC > 0. \end{aligned} \tag{41}$$

It is convenient to provide an alternative expression for $\mathcal{I}$ as follows. Note that

$$\begin{aligned}
\ln(1-y_-+i\varepsilon)+\ln(1-y_+-i\varepsilon) &= \ln\bigl[1-y_+-y_-+y_+y_--i\varepsilon(y_+-y_-)\bigr] \\
&= \ln\left(1+\frac{B}{A}+\frac{C}{A}-i\varepsilon\,\mathrm{sgn}\,A\right) \\
&= \ln\left(\frac{A+B+C-i\varepsilon}{A}\right), \qquad (42)
\end{aligned}$$

where we have used the fact that $y_+-y_-<0$ when $A<0$. Hence, it follows that

$$\begin{aligned}
\mathcal{I} = L-2+(1-y_-)\ln\left(1-y_-+i\varepsilon\right)+y_-\ln\left(-y_-+i\varepsilon\right) \\
+(1-y_+)\ln\left(1-y_+-i\varepsilon\right)+y_+\ln\left(-y_+-i\varepsilon\right), \qquad (43)
\end{aligned}$$

where

$$L=\ln(A+B+C-i\varepsilon)-\ln\left(\frac{A+B+C-i\varepsilon}{A}\right). \qquad (44)$$

Consider first the case of $A>0$. Then,

$$\ln\left(\frac{A+B+C-i\varepsilon}{A}\right)=\ln(A+B+C-i\varepsilon)-\ln A\,, \qquad (45)$$

and it follows that $L=\ln A$. Next, consider the case of $A<0$ and $A+B+C>0$. Then,

$$\ln\left(\frac{A+B+C-i\varepsilon}{A}\right)=\ln\left(\frac{A+B+C}{A}+i\varepsilon\right)=\ln(A+B+C)-\ln(-A)+i\pi\,, \qquad (46)$$

and it follows that

$$L=\ln(-A)-i\pi\,. \qquad (47)$$

Finally, consider the case of $A<0$ and $A+B+C<0$. In this case,

$$\begin{aligned}
\ln(A+B+C-i\varepsilon) &= \ln(-A-B-C)-i\pi\,, \\
\ln\left(\frac{A+B+C-i\varepsilon}{A}\right) &= \ln\left(\frac{A+B+C}{A}+i\varepsilon\right)=\ln(-A-B-C)-\ln(-A)\,, \qquad (48)
\end{aligned}$$

and we again recover eq. (47).

Thus, we have demonstrated that

$$L=\begin{cases}\ln A\,, & \text{for } A>0,\\ \ln(-A)-i\pi\,, & \text{for } A<0.\end{cases} \qquad (49)$$

Equivalently, one can write

$$L=\ln(A-i\varepsilon)\,, \qquad (50)$$

which yields eq. (49) for both $A>0$ and $A<0$ in the limit of $\varepsilon\to 0^+$. Hence, an equivalent form for eq. (41) is

$$\begin{aligned}
\mathcal{I} = \ln(A-i\varepsilon)-2+(1-y_-)\ln\left(1-y_-+i\varepsilon\right)+y_-\ln\left(-y_-+i\varepsilon\right) \\
+(1-y_+)\ln\left(1-y_+-i\varepsilon\right)+y_+\ln\left(-y_+-i\varepsilon\right), \qquad \text{for } A\neq 0 \text{ and } B^2-4AC>0, \quad (51)
\end{aligned}$$

in agreement with the result quoted in Appendix D of Ref. [3].

Case 3: $r_1^2 = 4r_2$

In this limit, $B^2 - 4AC = 0$ and eq. (17) yields,

$$\mathcal{I} = \ln\left(A + B + \frac{B^2}{4A} - i\varepsilon\right) - 2 + \frac{B}{A}\int_0^1 \frac{\left(x + \frac{B}{2A}\right)\,dx}{\left(x + \frac{B}{2A}\right)^2 - i\varepsilon\,\mathrm{sgn}\,A}\,. \tag{52}$$

Employing the Sokhotski-Plemelj formula, with $x_0 \equiv B/(2A)$,

$$\lim_{\varepsilon\to 0^+}\frac{(x+x_0)}{(x+x_0)^2 - i\varepsilon\,\mathrm{sgn}\,A} = \mathrm{P}\frac{1}{x+x_0} + i\pi(x+x_0)\delta\big((x+x_0)^2\big) = \mathrm{P}\frac{1}{x+x_0}\,, \tag{53}$$

where we have used the well-known property of δ-functions that $f(x)\delta(x) = f(0)\delta(x)$. Hence,

$$\begin{aligned}
\mathcal{I} &= \ln\left(A + B + \frac{B^2}{4A} - i\varepsilon\right) - 2 + \frac{B}{A}\,\mathrm{P}\int_0^1 \frac{dx}{x+x_0} \\
&= \ln\left[A\left(1 + \frac{B}{2A}\right)^2 - i\varepsilon\right] - 2 + \frac{B}{A}\lim_{\delta\to 0^+}\left\{\int_0^{-x_0-\delta}\frac{dx}{x+x_0} + \int_{-x_0+\delta}^1 \frac{dx}{x+x_0}\right\} \\
&= \ln\left[(A - i\varepsilon)\left(1 + \frac{B}{2A}\right)^2\right] - 2 + \frac{B}{A}\lim_{\delta\to 0^+}\left\{\ln\left|\frac{\delta}{x_0}\right| + \ln\left|\frac{1+x_0}{\delta}\right|\right\} \\
&= \ln(A - i\varepsilon) + 2\ln\left|1 + \frac{B}{2A}\right| - 2 + \frac{B}{A}\ln\left|\frac{1+x_0}{x_0}\right| \\
&= \ln(A - i\varepsilon) - 2 + \left(\frac{B+2A}{A}\right)\ln\left|1 + \frac{B}{2A}\right| - \frac{B}{A}\ln\left|\frac{B}{2A}\right|\,.
\end{aligned} \tag{54}$$

One further simplification yields our final result,

$$\mathcal{I} = \ln(A - i\varepsilon) - 2 + 2\ln\left|1 + \frac{B}{2A}\right| + \frac{B}{A}\ln\left|1 + \frac{2A}{B}\right|\,, \qquad \text{for } A \neq 0 \text{ and } B^2 - 4AC = 0. \tag{55}$$

It is straightforward to check that in the limit of $B^2 = 4AC$, both eqs. (30) and (51) yield the result quoted in eq. (55). For example, in this limit, $y_+ = y_- = -B/(2A)$. Hence, we can take the limit of $y_+ = y_-$ of eq. (51) to obtain,

$$\mathcal{I} = \ln(A - i\varepsilon) - 2 + 2\left(1 + \frac{B}{2A}\right)\mathrm{Re}\ln\left(1 + \frac{B}{2A} + i\varepsilon\right) - \frac{B}{A}\,\mathrm{Re}\ln\left(\frac{B}{2A} + i\varepsilon\right)\,, \tag{56}$$

which is equivalent to eq. (54).

3. The finite part of $B_0(p^2, m_1^2, m_2^2)$

We shall now apply the results of Section 2 to $B_0(p^2; m_1^2, m_2^2)$. Comparing eqs. (4) and (5), we identify,

$$A = \frac{p^2}{\mu^2}\,, \qquad B = -\frac{p^2 + m_1^2 - m_2^2}{\mu^2}\,, \qquad C = \frac{m_1^2}{\mu^2}\,. \tag{57}$$

It follows that

$$r_1 = -1 + \frac{m_2^2 - m_1^2}{p^2}\,, \qquad r_2 = \frac{m_1^2}{p^2}\,, \qquad r_1^2 - 4r_2 = \frac{\lambda(p^2, m_1^2, m_2^2)}{p^4}\,, \tag{58}$$

where $\lambda(a,b,c)$ is the well-known kinematical triangle function [5],

$$\lambda(a,b,c) \equiv a^2 + b^2 + c^2 - 2ab - 2ac - 2bc = (a+b-c)^2 - 4ab\,. \tag{59}$$

Thus,

$$\boxed{B_0(p^2; m_1^2, m_2^2) = \Delta - G(p^2; m_1^2, m_2^2)\,,} \tag{60}$$

where Δ is defined below eq. (4) and,

$$\begin{aligned} G(p^2; m_1^2, m_2^2) &= \ln\left(\frac{m_2^2}{\mu^2}\right) - \left(\frac{p^2 + m_1^2 - m_2^2}{2p^2}\right)\ln\left(\frac{m_2^2}{m_1^2}\right) - 2 \\ &\quad + \frac{(-\lambda)^{1/2}}{p^2}\left[\arctan\left(\frac{p^2 - m_1^2 + m_2^2}{(-\lambda)^{1/2}}\right) + \arctan\left(\frac{p^2 + m_1^2 - m_2^2}{(-\lambda)^{1/2}}\right)\right], \\ &\qquad \text{for } p^2 \neq 0 \text{ and } \lambda \equiv \lambda(p^2, m_1^2, m_2^2) < 0\,, \end{aligned} \tag{61}$$

$$\begin{aligned} G(p^2; m_1^2, m_2^2) &= \ln\left(\frac{m_2^2}{\mu^2}\right) - 2 \\ &\quad - \left(\frac{p^2 + m_1^2 - m_2^2 + \lambda^{1/2}}{2p^2}\right)\left[\ln\left(\frac{p^2 - m_1^2 + m_2^2 - \lambda^{1/2}}{2p^2} - i\varepsilon\right)\right. \\ &\qquad\qquad \left. - \ln\left(\frac{-p^2 - m_1^2 + m_2^2 - \lambda^{1/2}}{2p^2} - i\varepsilon\right)\right] \\ &\quad - \left(\frac{p^2 + m_1^2 - m_2^2 - \lambda^{1/2}}{2p^2}\right)\left[\ln\left(\frac{p^2 - m_1^2 + m_2^2 + \lambda^{1/2}}{2p^2} - i\varepsilon\right)\right. \\ &\qquad\qquad \left. - \ln\left(\frac{-p^2 - m_1^2 + m_2^2 + \lambda^{1/2}}{2p^2} - i\varepsilon\right)\right] \\ &\qquad \text{for } p^2 \neq 0 \text{ and } \lambda \equiv \lambda(p^2, m_1^2, m_2^2) > 0\,, \end{aligned} \tag{62}$$

$$\begin{aligned} G(p^2; m_1^2, m_2^2) &= \ln\left(\frac{p^2}{\mu^2}\right) - 2 + \left(\frac{p^2 - m_1^2 + m_2^2}{p^2}\right)\ln\left|\frac{p^2 - m_1^2 + m_2^2}{2p^2}\right| \\ &\quad + \left(\frac{p^2 + m_1^2 - m_2^2}{p^2}\right)\ln\left|\frac{p^2 + m_1^2 - m_2^2}{2p^2}\right|\,, \\ &\qquad \text{for } p^2 \neq 0 \text{ and } \lambda(p^2, m_1^2, m_2^2) = 0, \end{aligned} \tag{63}$$

$$G(0; m_1^2, m_2^2) = \frac{1}{m_1^2 - m_2^2}\left[m_1^2 \ln\left(\frac{m_1^2}{\mu^2}\right) - m_2^2 \ln\left(\frac{m_2^2}{\mu^2}\right)\right] - 1\,, \qquad \text{for } m_1^2 \neq m_2^2, \tag{64}$$

$$G(0; m^2, m^2) = \ln\left(\frac{m^2}{\mu^2}\right)\,. \tag{65}$$

Note that an alternate expression for λ is given by [5],

$$\lambda(p^2, m_1^2, m_2^2) = \left[p^2 - (m_1 + m_2)^2\right]\left[p^2 - (m_1 - m_2)^2\right] . \tag{66}$$

It follows that,

$$\begin{aligned} \lambda(p^2, m_1^2, m_2^2) < 0 \quad &\Longrightarrow \quad (m_1 - m_2)^2 < p^2 < (m_1 + m_2)^2 , \\ \lambda(p^2, m_1^2, m_2^2) > 0 \quad &\Longrightarrow \quad p^2 < (m_1 - m_2)^2 \quad \text{or} \quad p^2 > (m_1 + m_2)^2 . \end{aligned} \tag{67}$$

In light of Cutkosky's cutting rules [6,7], $\operatorname{Im} G(p^2; m_1^2, m_2^2) \neq 0$ if and only if $p^2 > (m_1 + m_2)^2$, in which case the internal lines of the one-loop self-energy graph can go on-shell.

Thus, we can simplify the expression given by eq. (62) as follows. Using the definition of the principal value of the complex logarithm,

$$\ln(x - i\varepsilon) = \ln|x| - i\pi\Theta(-x) , \quad \text{for } x \in \mathbb{R},\ x \neq 0 \text{ and positive infinitesimal } \varepsilon, \tag{68}$$

it follows that $\operatorname{Re} \ln x = \ln|x|$. Hence, after combining logarithms, eq. (62) yields,

$$\begin{aligned} \operatorname{Re} G(p^2; m_1^2, m_2^2) = \ln\left(\frac{m_2^2}{\mu^2}\right) - 2 - \left(\frac{p^2 + m_1^2 - m_2^2}{2p^2}\right) \ln\left(\frac{m_2^2}{m_1^2}\right) \\ + \frac{\lambda^{1/2}(p^2, m_1^2, m_2^2)}{2p^2} \ln\left(\frac{p^2 - m_1^2 - m_2^2 + \lambda^{1/2}(p^2, m_1^2, m_2^2)}{p^2 - m_1^2 - m_2^2 - \lambda^{1/2}(p^2, m_1^2, m_2^2)}\right) , \end{aligned} \tag{69}$$

In the case of $p^2 > (m_1 + m_2)^2$, one can check that

$$p^2 - m_1^2 + m_2^2 \pm \lambda^{1/2} > 0 \quad \text{and} \quad -p^2 - m_1^2 + m_2^2 \pm \lambda^{1/2} < 0.$$

Hence,

$$\operatorname{Im} G(p^2; m_1^2, m_2^2) = -\frac{\pi\lambda^{1/2}(p^2, m_1^2, m_2^2)}{p^2}\, \Theta\big(p^2 - (m_1 + m_2)^2\big) . \tag{70}$$

It then follows that an alternate expression for eq. (62) is

$$\begin{aligned} G(p^2; m_1^2, m_2^2) = \ln\left(\frac{m_2^2}{\mu^2}\right) - 2 - \left(\frac{p^2 + m_1^2 - m_2^2}{2p^2}\right) \ln\left(\frac{m_2^2}{m_1^2}\right) \\ + \frac{\lambda^{1/2}(p^2, m_1^2, m_2^2)}{2p^2} \left[\ln\left(\frac{p^2 - m_1^2 - m_2^2 + \lambda^{1/2}(p^2, m_1^2, m_2^2)}{p^2 - m_1^2 - m_2^2 - \lambda^{1/2}(p^2, m_1^2, m_2^2)}\right) - 2i\pi\, \Theta\big(p^2 - (m_1 + m_2)^2\big)\right] , \\ \text{for } p^2 \neq 0 \text{ and } \lambda \equiv \lambda(p^2, m_1^2, m_2^2) > 0 . \end{aligned} \tag{71}$$

One can perform one further simplification by noting that

$$\frac{p^2 - m_1^2 - m_2^2 + \lambda^{1/2}(p^2, m_1^2, m_2^2)}{p^2 - m_1^2 - m_2^2 - \lambda^{1/2}(p^2, m_1^2, m_2^2)} = \left(\frac{\left[p^2 - (m_1 - m_2)^2\right]^{1/2} + \left[p^2 - (m_1 + m_2)^2\right]^{1/2}}{\left[p^2 - (m_1 - m_2)^2\right]^{1/2} - \left[p^2 - (m_1 + m_2)^2\right]^{1/2}}\right)^2 , \tag{72}$$

which is useful in the case of $p^2 > (m_1+m_2)^2$. Hence,

$$\boxed{\begin{aligned} G(p^2;m_1^2,m_2^2) &= \ln\left(\frac{m_2^2}{\mu^2}\right) - 2 - \left(\frac{p^2+m_1^2-m_2^2}{2p^2}\right)\ln\left(\frac{m_2^2}{m_1^2}\right) \\ &\quad + \frac{\lambda^{1/2}(p^2,m_1^2,m_2^2)}{p^2}\left[\ln\left(\frac{\left[p^2-(m_1-m_2)^2\right]^{1/2}+\left[p^2-(m_1+m_2)^2\right]^{1/2}}{\left[p^2-(m_1-m_2)^2\right]^{1/2}-\left[p^2-(m_1+m_2)^2\right]^{1/2}}\right) - i\pi\right], \\ &\qquad \text{for } p^2 > (m_1+m_2)^2. \end{aligned}} \tag{73}$$

Likewise,

$$\frac{p^2-m_1^2-m_2^2+\lambda^{1/2}(p^2,m_1^2,m_2^2)}{p^2-m_1^2-m_2^2-\lambda^{1/2}(p^2,m_1^2,m_2^2)} = \left(\frac{\left[(m_1+m_2)^2-p^2\right]^{1/2}+\left[(m_1-m_2)^2-p^2\right]^{1/2}}{\left[(m_1+m_2)^2-p^2\right]^{1/2}-\left[(m_1-m_2)^2-p^2\right]^{1/2}}\right)^2, \tag{74}$$

which is useful in the case of $p^2 < (m_1-m_2)^2$. Hence,

$$\boxed{\begin{aligned} G(p^2;m_1^2,m_2^2) &= \ln\left(\frac{m_2^2}{\mu^2}\right) - 2 - \left(\frac{p^2+m_1^2-m_2^2}{2p^2}\right)\ln\left(\frac{m_2^2}{m_1^2}\right) \\ &\quad + \frac{\lambda^{1/2}(p^2,m_1^2,m_2^2)}{p^2}\ln\left(\frac{\left[(m_1+m_2)^2-p^2\right]^{1/2}+\left[(m_1-m_2)^2-p^2\right]^{1/2}}{\left[(m_1+m_2)^2-p^2\right]^{1/2}-\left[(m_1-m_2)^2-p^2\right]^{1/2}}\right), \\ &\qquad \text{for } p^2 < (m_1-m_2)^2 \text{ and } p^2 \neq 0. \end{aligned}} \tag{75}$$

One can also obtain expressions that are valid for $p^2 = (m_1 \pm m_2)^2$ by taking the appropriate limits in eqs. (73) and (75). Since $\lambda(p^2,m_1^2,m_2^2) \to 0$ in both limiting cases, we find

$$G(p^2;m_1^2,m_2^2) = \begin{cases} \dfrac{1}{m_1+m_2}\left[m_1\ln\left(\dfrac{m_1^2}{\mu^2}\right) + m_2\ln\left(\dfrac{m_2^2}{\mu^2}\right)\right] - 2\,, & \text{for } p^2 = (m_1+m_2)^2\,, \\ \dfrac{1}{m_1-m_2}\left[m_1\ln\left(\dfrac{m_1^2}{\mu^2}\right) - m_2\ln\left(\dfrac{m_2^2}{\mu^2}\right)\right] - 2\,, & \text{for } p^2 = (m_1-m_2)^2\,. \end{cases} \tag{76}$$

It is straightforward to check that these expressions match the expected result of eq. (63) for $p^2 = (m_1 \pm m_2)^2$, respectively.

It is instructive to verify the results of eqs. (73) and (75) in the limit of $m_1 = m_2 = m$,

$$G(p^2;m^2,m^2) = \begin{cases} \ln\left(\dfrac{m^2}{\mu^2}\right) - 2 + \sqrt{1-\dfrac{4m^2}{p^2}}\ln\left(\dfrac{\sqrt{1-\frac{4m^2}{p^2}}+1}{\sqrt{1-\frac{4m^2}{p^2}}-1}\right), & \text{for } p^2 < 0 \\ \ln\left(\dfrac{m^2}{\mu^2}\right) - 2 + \sqrt{1-\dfrac{4m^2}{p^2}}\left[\ln\left(\dfrac{1+\sqrt{1-\frac{4m^2}{p^2}}}{1-\sqrt{1-\frac{4m^2}{p^2}}}\right) - i\pi\right], & \text{for } p^2 > 4m^2, \end{cases} \tag{77}$$

which is in agreement with the results of eqs. (95) and (109).

The limit of $p^2 \to 0$ of eq. (75) exists, and can be evaluated by expanding out the argument of the logarithm in powers of p^2. We leave this as an exercise for the reader. The end result of this computation is

$$G(0;m_1^2,m_2^2) = \frac{1}{m_1^2-m_2^2}\left[m_1^2\ln\left(\frac{m_1^2}{\mu^2}\right) - m_2^2\ln\left(\frac{m_2^2}{\mu^2}\right)\right] - 1\,, \quad \text{for } m_1 \neq m_2,$$
$$G(0;m^2,m^2) = \ln\left(\frac{m^2}{\mu^2}\right)\,, \tag{78}$$

in agreement with eqs. (64) and (65).

Finally, we can simplify the expression given by eq. (61) by employing the following relation given on p. 58 of Ref. [8],

$$\arctan x + \arctan y = \arctan\left(\frac{x+y}{1-xy}\right) + \pi\,\text{sgn}(x)\Theta(xy-1)\,, \qquad \text{for } x,y\in\mathbb{R}. \tag{79}$$

Using this identity, eq. (61) yields,

$$G(p^2;m_1^2,m_2^2) = \ln\left(\frac{m_2^2}{\mu^2}\right) - \left(\frac{p^2+m_1^2-m_2^2}{2p^2}\right)\ln\left(\frac{m_2^2}{m_1^2}\right) - 2$$
$$+\frac{\bigl[-\lambda(p^2,m_1^2,m_2^2)\bigr]^{1/2}}{p^2}\left[\arctan\left(\frac{\bigl[-\lambda(p^2,m_1^2,m_2^2)\bigr]^{1/2}}{m_1^2+m_2^2-p^2}\right) + \pi\,\Theta(p^2-m_1^2-m_2^2)\right]\,,$$
$$\text{for } (m_1-m_2)^2 < p^2 < (m_1+m_2)^2. \tag{80}$$

One further simplification is possible by employing the following relation given on p. 59 of Ref. [8],

$$2\arctan x = \arctan\left(\frac{2x}{1-x^2}\right) + \pi\,\text{sgn}(x)\Theta(|x|-1)\,, \qquad \text{for } x\in\mathbb{R}. \tag{81}$$

Using this identity, it follows that

$$2\arctan\left(\frac{\sqrt{p^2-(m_1-m_2)^2}}{\sqrt{(m_1+m_2)^2-p^2}}\right) = \arctan\left(\frac{\bigl[-\lambda(p^2,m_1^2,m_2^2)\bigr]^{1/2}}{m_1^2+m_2^2-p^2}\right) + \pi\,\Theta(p^2-m_1^2-m_2^2)\,, \tag{82}$$

after making use of eq. (66). Hence, we end up with

$$\boxed{\begin{aligned} G(p^2;m_1^2,m_2^2) &= \ln\left(\frac{m_2^2}{\mu^2}\right) - \left(\frac{p^2+m_1^2-m_2^2}{2p^2}\right)\ln\left(\frac{m_2^2}{m_1^2}\right) - 2 \\ &\quad + \frac{2\bigl[-\lambda(p^2,m_1^2,m_2^2)\bigr]^{1/2}}{p^2}\arctan\left(\frac{\sqrt{p^2-(m_1-m_2)^2}}{\sqrt{(m_1+m_2)^2-p^2}}\right)\,, \\ &\qquad \text{for } (m_1-m_2)^2 < p^2 < (m_1+m_2)^2. \end{aligned}} \tag{83}$$

The boxed formulae exhibited in eqs. (73), (75) and (83) reproduce the explicit results given in Appendix B of Ref. [9].

It is straightforward to check that the limiting behavior of eq. (83) as $p^2 \to (m_1 \pm m_2)^2$ reproduces the results of eq. (76). As an aside, one could choose to present an equivalent expression for eq. (83) by replacing the arctangent function with an arcsine function by using the identity,

$$\arctan\left(\frac{\sqrt{p^2-(m_1-m_2)^2}}{\sqrt{(m_1+m_2)^2-p^2}}\right) = \arcsin\left[\left(\frac{p^2-(m_1-m_2)^2}{4m_1m_2}\right)^{1/2}\right], \tag{84}$$

which is valid for $(m_1-m_2)^2 < p^2 < (m_1+m_2)^2$ (cf. formula 2. on p. 57 of Ref. [8]),

It is instructive to verify the result of eq. (83) in the limit of $m_1 = m_2 = m$,

$$G(p^2; m^2, m^2) = \ln\left(\frac{m_2^2}{\mu^2}\right) - 2 + 2\left(\frac{4m^2}{p^2} - 1\right)^{1/2} \arctan\left(\frac{1}{\sqrt{\frac{4m^2}{p^2}-1}}\right), \quad \text{for } 0 < p^2 < 4m^2, \tag{85}$$

which is in agreement with the result of eq. (100).

4. A special case: $A = -B = z$ and $C = 1$

In this section, we examine a special case of eq. (5) with $A = z$, $B = -z$ and $C = 1$. We can perform an independent computation, and use the results as a check of the more general formulae obtained in Section 2.

Consider the function of a *real* parameter z

$$F(z) \equiv \int_0^1 dx\, \ln\bigl[1 - zx(1-x) - i\varepsilon\bigr], \tag{86}$$

where ε is a positive infinitesimal quantity.

First, we evaluate $\operatorname{Im} F$. Let us denote the argument of the logarithm in eq. (86) by the function,

$$f(x) \equiv zx^2 - zx + 1 \geq 0,$$

which satisfies the conditions $f(0) = f(1) = 1$. Next, we compute the first and second derivatives,

$$f'(x) = z(2x-1), \qquad f''(x) = 2z,$$

Thus, $f(x)$ has an extremum at $x = \frac{1}{2}$. Since $f''(\frac{1}{2}) = 2z$, it follows that $x = \frac{1}{2}$ is a maximum if $z < 0$ and $x = \frac{1}{2}$ is a minimum if $z > 0$. At $z = 0$, we have $f(x) = 1$ for all x. Moreover, for $z > 0$, the minimum value of $f(x)$ is equal to $f(\frac{1}{2}) = 1 - \frac{1}{4}z$. That is, for values of $0 \leq z \leq 4$, the minimum value of $f(x)$ is nonnegative for all $0 \leq x \leq 1$. Moreover, for values of $z \leq 0$, we have $f(x) \geq 1$ in the region where $0 \leq x \leq 1$.

Observe that $\operatorname{Im} F(z) = 0$ if $f(x) > 0$ for $0 \leq x \leq 1$, which implies that $\operatorname{Im} F(z) = 0$ if $z < 4$. When $z > 4$, the minimum value of $f(x)$ at $x = \frac{1}{2}$ is negative. Since $f(0) = f(1) = 1$, it follows that $f(x) < 0$ for values of $x_- < x < x_+$, where $x_\pm$ are the roots of $f(x)$,

$$x_\pm = \tfrac{1}{2}\left[1 \pm \sqrt{1 - \frac{4}{z}}\right]. \tag{87}$$

Thus,

$$\operatorname{Im} F(z) = \Theta(z-4)\int_{x_-}^{x_+} dx\; \operatorname{Im}\ln\bigl[1 - zx(1-x) - i\varepsilon\bigr]\,, \tag{88}$$

where we have explicitly included the step function to enforce the condition that $\operatorname{Im} F(z) = 0$ if $z < 4$. To evaluate the imaginary part of the logarithm, we employ the principal value of the complex-valued logarithm, with the branch cut taken along the negative real axis. In particular, assuming that x is a non-zero real number and ε is a *positive* infinitesimal,

$$\ln(x - i\varepsilon) = \ln|x| - i\pi\Theta(-x)\,. \tag{89}$$

It follows that $\operatorname{Im}\ln(x - i\varepsilon) = -\pi\Theta(-x)$. Employing this result in eq. (88),

$$\operatorname{Im} F(z) = -\Theta(z-4)\pi\int_{x_-}^{x_+} dx = -\Theta(z-4)\pi\bigl(x_+ - x_-\bigr) = -\Theta(z-4)\pi\sqrt{1-\frac{4}{z}}\,, \tag{90}$$

after using the explicit form for $x_\pm$ given in eq. (87). Note that $\operatorname{Im} F(z = 4) = 0$ at the boundary between the regions where $\operatorname{Im} F(z)$ is nonzero and where it vanishes.

We now turn to the computation of $\operatorname{Re} F(z)$. We shall examine separately the cases of $z < 4$ and $z > 4$. In the case of $z < 4$, the argument of the logarithm in eq. (86) is positive, in which case we can drop the $-i\varepsilon$ term and write

$$F(z) \equiv \int_0^1 dx\; \ln\bigl[1 - zx(1-x)\bigr]\,, \qquad \text{for } z < 4. \tag{91}$$

Let us set $x = \frac{1}{2}(1-y)$. Then, $1 - x = \frac{1}{2}(1+y)$ and $x(1-x) = \frac{1}{4}(1-y^2)$. Thus,

$$F(z) = \tfrac{1}{2}\int_{-1}^{1} \ln\bigl[1 - \tfrac{1}{4}z(1-y^2)\bigr]\,dy = \int_0^1 \ln\bigl[1 - \tfrac{1}{4}z(1-y^2)\bigr]\,dy\,, \qquad \text{for } 0 \le z < 4, \tag{92}$$

after noting that the integrand is an even function of y. Integrating by parts, we take $u = \ln\bigl[1 - \frac{1}{4}z(1-y^2)\bigr]$ and $dv = dy$, which yields,

$$\begin{aligned} F(z) &= y\ln\bigl[1 - \tfrac{1}{4}z(1-y^2)\bigr]\Big|_0^1 - \int_0^1 \frac{\frac{1}{2}zy^2\,dy}{1 - \frac{1}{4}z + \frac{1}{4}zy^2} \\ &= -2\int_0^1 \frac{y^2\,dy}{\frac{4}{z} - 1 + y^2} = -2\left[1 - \left(\frac{4}{z} - 1\right)\int_0^1 \frac{dy}{\frac{4}{z} - 1 + y^2}\right]. \end{aligned} \tag{93}$$

Two subcases will now be treated.

First, if $z < 0$, then it is convenient to denote $a^2 \equiv 1 - 4/z$ with $a > 1$. Hence,

$$F(z) = -2 - 2a^2\int_0^1 \frac{dy}{y^2 - a^2} = -2 + a\int_0^1\left(\frac{1}{y+a} - \frac{1}{y-a}\right)dy = -2 + a\ln\left(\frac{a+1}{a-1}\right). \tag{94}$$

Hence, we end up with

$$F(z) = -2 + \sqrt{1-\frac{4}{z}}\,\ln\left(\frac{\sqrt{1-\frac{4}{z}}+1}{\sqrt{1-\frac{4}{z}}-1}\right), \qquad \text{for } z < 0. \tag{95}$$

This result agrees with eq. (41) after setting $A = -B = z$, $C = 1$ and noting that $B^2 - 4AC = z(z-4) > 0$, In particular, $y_\pm = \frac{1}{2}\bigl[1 \mp \sqrt{1-4/z}\bigr]$ (note the order of the signs in the case of $z < 0$), in which case it follows that

$$F(z) = -2 + \sqrt{1 - \frac{4}{z}}\,\ln\left(-\frac{y_-}{y_+}\right)\,, \qquad \text{for } z < 0, \tag{96}$$

in agreement with eq. (95). One can check that $\lim_{z\to 0} F(z) = 0$ as expected from setting $z = 0$ in eq. (91).

One can also rewrite eq. (95) in a slightly different form. Employing the relation,

$$\operatorname{arcsinh} x = \ln\bigl(x + \sqrt{1+x^2}\,\bigr) = \tfrac{1}{2}\ln\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right) = \tfrac{1}{2}\ln\left(\frac{\sqrt{1+\frac{1}{x^2}}+1}{\sqrt{1+\frac{1}{x^2}}-1}\right)\,, \qquad \text{for } x \geq 0, \tag{97}$$

it follows that

$$F(z) = 2\left\{\left(1 - \frac{4}{z}\right)^{1/2} \operatorname{arcsinh}\bigl(\tfrac{1}{2}\sqrt{-z}\bigr) - 1\right\}\,, \qquad \text{for } z < 0. \tag{98}$$

Second, if $0 \leq z < 4$, then it is convenient to denote $a^2 = 4/z - 1$ with $a > 0$. Hence,

$$F(z) = -2 + 2a^2 \int_0^1 \frac{dy}{y^2 + a^2} = -2 + 2a \arctan\left(\frac{1}{a}\right)\,. \tag{99}$$

Hence, we end up with

$$F(z) = -2 + 2\left(\frac{4}{z} - 1\right)^{1/2} \arctan\left(\frac{1}{\sqrt{\frac{4}{z}-1}}\right)\,, \qquad \text{for } 0 \leq z \leq 4. \tag{100}$$

This result agrees with eq. (30) after setting $A = -B = z$, $C = 1$ and noting that $4AC - B^2 = z(4-z) > 0$.

One can rewrite eq. (100) in a slightly different form. In light of the relation,

$$\arcsin x = \arctan\left(\frac{x}{\sqrt{1-x^2}}\right)\,, \qquad \text{for } x^2 < 1, \tag{101}$$

it follows that

$$F(z) = 2\left\{\left(\frac{4}{z} - 1\right)^{1/2} \arcsin\bigl(\tfrac{1}{2}\sqrt{z}\bigr) - 1\right\}\,, \qquad \text{for } 0 \leq z \leq 4. \tag{102}$$

Note that eqs. (98) and (102) are analytic continuations of each other, in light of the relation, $\arcsin(ix) = i \operatorname{arcsinh} x$.

As indicated in eqs. (100) and (102), the above results are also applicable at $z = 4$, since eq. (92) yields

$$F(z = 4) = \int_0^1 \ln(y^2)\, dy = -2\,, \tag{103}$$

in agreement with eqs. (100) and (102) in the limit of $z \to 4$.

Finally, we assume that $z > 4$. In this case, $\mathrm{Im}\, F(z) \neq 0$ and is given explicitly in eq. (90). In light of eq. (89) it follows that

$$\mathrm{Re}\, F(z) = \int_0^1 dx\, \ln\bigl|1 - zx(1-x)\bigr|\,. \tag{104}$$

We can again employ eq. (92), which yields

$$\mathrm{Re}\, F(z) = \int_0^1 \ln\bigl|1 - \tfrac{1}{4}z(1-y^2)\bigr|\, dy\,, \qquad \text{for } z > 4\,. \tag{105}$$

Integrating by parts, we take $u = \ln\bigl|1 - \frac{1}{4}z(1-y^2)\bigr|$ and $dv = dy$. In the computation of du, we shall use the relation derived on pp. 25–26 of Ref. [10],

$$\frac{d}{dx}\, \ln|x| = \mathrm{P}\,\frac{1}{x}\,, \tag{106}$$

where P indicates the principal value prescription. Hence, the integration by parts yields

$$\begin{aligned} \mathrm{Re}\, F(z) &= y\ln\bigl|1 - \tfrac{1}{4}z(1-y^2)\bigr|\Bigr|_0^1 - \mathrm{P}\int_0^1 \frac{\frac{1}{2}zy^2\, dy}{1 - \frac{1}{4}z + \frac{1}{4}zy^2} = -2\,\mathrm{P}\int_0^1 \frac{y^2\, dy}{\frac{4}{z} - 1 + y^2} \\ &= -2\left[1 - \left(\frac{4}{z} - 1\right)\mathrm{P}\int_0^1 \frac{dy}{\frac{4}{z} - 1 + y^2}\right]\,, \end{aligned} \tag{107}$$

Using the definition of the principal value prescription, it follows that for $0 < a < 1$,

$$\begin{aligned} \mathrm{P}\int_0^1 \frac{dy}{y^2 - a^2} &= -\frac{1}{2a}\,\mathrm{P}\int_0^1 \left(\frac{1}{y+a} - \frac{1}{y-a}\right) dy = -\frac{1}{2a}\left[\ln\left(\frac{1+a}{a}\right) - \mathrm{P}\int_0^1 \frac{dy}{y-a}\right] \\ &= -\frac{1}{2a}\left[\ln\left(\frac{1+a}{a}\right) - \lim_{\delta\to 0^+}\left\{\int_0^{a-\delta} \frac{dy}{y-a} + \int_{a+\delta}^1 \frac{dy}{y-a}\right\}\right] \\ &= -\frac{1}{2a}\left[\ln\left(\frac{1+a}{a}\right) - \lim_{\delta\to 0^+}\left\{\ln\left(\frac{\delta}{a}\right) + \ln\left(\frac{1-a}{\delta}\right)\right\}\right] \\ &= -\frac{1}{2a}\ln\left(\frac{1+a}{1-a}\right)\,. \end{aligned} \tag{108}$$

Hence, after setting $a = (1 - 4/z)^{1/2}$, eqs. (90), (107) and (108) yield,

$$F(z) = -2 + \sqrt{1 - \frac{4}{z}}\left[\ln\left(\frac{1 + \sqrt{1 - \frac{4}{z}}}{1 - \sqrt{1 - \frac{4}{z}}}\right) - i\pi\right]\,, \qquad \text{for } z \geq 4. \tag{109}$$

This result agrees with eq. (41) after setting $A = -B = z$, $C = 1$ and noting that $B^2 - 4AC = z(z-4) > 0$, In particular, $y_\pm = \frac{1}{2}\bigl[1 \pm \sqrt{1 - 4/z}\bigr]$ (note the order of the signs in the case of $z > 4$), in which case it follows that

$$F(z) = -2 + \sqrt{1 - \frac{4}{z}}\,\ln\left(\frac{y_+}{y_-}\right) - i\pi(y_+ - y_-)\,, \qquad \text{for } z > 4, \tag{110}$$

in agreement with eq. (109).

In light of eq. (103), we see that eq. (109) is also valid at the boundary where $z = 4$. One can also rewrite eq. (109) in a slightly different form. Employing the relation,

$$\operatorname{arccosh} x = \ln\bigl(x + \sqrt{x^2-1}\,\bigr) = \tfrac{1}{2}\ln\left(\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}\right) = \tfrac{1}{2}\ln\left(\frac{1+\sqrt{1-\frac{1}{x^2}}}{1-\sqrt{1-\frac{1}{x^2}}}\right)\,, \qquad \text{for } x \geq 1, \tag{111}$$

it follows that

$$F(z) = 2\left\{\left(1-\frac{4}{z}\right)^{1/2}\left[\operatorname{arccosh}\bigl(\tfrac{1}{2}\sqrt{z}\bigr) - \tfrac{1}{2}i\pi\right] - 1\right\}\,, \qquad \text{for } z \geq 4. \tag{112}$$

An alternative derivation of eqs. (98), (102) and (112) is given in the Appendix to these notes.

Of course, the results of this section are immediately applicable to the evaluation of $B_0(p^2; m^2, m^2)$. In particular,

$$B_0(p^2; m^2, m^2) = \Delta - \ln\left(\frac{m^2}{\mu^2}\right) - F(p^2/m^2) + \mathcal{O}(\epsilon)\,, \tag{113}$$

where Δ is defined below eq. (4) and $F(z)$ is explicitly given in eqs. (95), (102) and (109).

REMARKS:

One can easily verify that eqs. (102) and (112) are analytic continuations of each other by keeping track of the $i\varepsilon$ factors. In particular, note that $F(z) \equiv \lim_{\varepsilon\to 0^+} F(z + i\varepsilon)$. Thus, for real values of $z > 4$,

$$\lim_{\varepsilon\to 0^+}\sqrt{\frac{4}{z+i\varepsilon}-1} = \lim_{\varepsilon\to 0^+}\sqrt{\frac{4}{z}-1-i\varepsilon} = -i\sqrt{1-\frac{4}{z}}\,, \tag{114}$$

since we are evaluating the square root of a number that lies just *below* the branch cut of the complex square root function that runs along the negative real axis. Hence, if we analytically continue the expression given by eq. (102) into the parameter regime where $z > 4$,[3] we recover the result previously obtained in eq. (112),

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} F(z+i\varepsilon) &= \lim_{\varepsilon\to 0^+} -2 + 2\left(\frac{4}{z+i\varepsilon}-1\right)^{1/2}\arcsin\bigl(\tfrac{1}{2}\sqrt{z}+i\varepsilon\bigr) \\
&= -2 - 2i\left(1-\frac{4}{z}\right)^{1/2}\lim_{\varepsilon\to 0^+}\left[\tfrac{1}{2}\pi - \arccos\bigl(\tfrac{1}{2}\sqrt{z}+i\varepsilon\bigr)\right] \\
&= -2 + 2\left(1-\frac{4}{z}\right)^{1/2}\left[-\tfrac{1}{2}i\pi + \operatorname{arccosh}\bigl(\tfrac{1}{2}\sqrt{z}\bigr)\right]\,.
\end{aligned} \tag{115}$$

[3]The principal value of the complex arccosine function, $\arccos(x+iy)$, is defined in the cut complex plane, where the cuts comprise the real intervals $(-\infty,-1]\cup[1,\infty)$. For example, for values of $x+i\varepsilon$ where $x\geq 1$ and ε is a positive infinitesimal, $\lim_{\varepsilon\to 0^+}\arccos(x+i\varepsilon) = -i\operatorname{arccosh} x$, which has been employed in obtaining the final result of eq. (115) [cf. eqs. (4.23.24) and (4.37.19) of Ref. [4]

APPENDIX: An alternative treatment of eq. (86)

One can also evaluate the integral defined in eq. (86) by employing a convenient change of variables. Although the following method turns out to be slightly more involved than the one presented in Section 3, it turns out to be especially useful in evaluating a related integral in which dx is replaced by dx/x. Thus, for completeness, the details of this alternative approach are provided in this Appendix.

Consider first the case of $z < 0$, where $F(z)$ is given by eq. (91). In this case, it is convenient to define,

$$z = -4\sinh^2 w\,, \qquad \text{for } 0 < w < \infty. \tag{116}$$

Next, we take the derivative of F with respect to w,

$$\begin{aligned}
\frac{dF}{dw} &= \frac{d}{dw}\int_0^1 \ln\bigl[1+4x(1-x)\sinh^2 w\bigr]\,dx = \sinh 2w\int_0^1 \frac{4x(1-x)dx}{1+4x(1-x)\sinh^2 w} \\
&= \frac{\sinh 2w}{\sinh^2 w}\int_0^1 \frac{1+4x(1-x)\sinh^2 w - 1}{1+4x(1-x)\sinh^2 w}\,dx \\
&= 2\coth w - 2\coth w\int_0^1 \frac{1}{1+4x(1-x)\sinh^2 w}\,dx\,.
\end{aligned} \tag{117}$$

To evaluate the above integral, we first factor the denominator of the integrand and then apply a partial fractioning. That is,

$$1+4x(1-x)\sinh^2 w = -4\sinh^2 w(x-x_+)(x-x_-)\,, \qquad \text{where } x_\pm = \pm\frac{e^{\pm w}}{2\sinh w}\,. \tag{118}$$

Hence, it follows that

$$\begin{aligned}
\frac{dF}{dw} &= 2\coth w + \frac{\cosh w}{\sinh^3 w}\int_0^1 \frac{dx}{(x-x_+)(x-x_-)} \\
&= 2\tanh w + \frac{\cosh w}{\sinh^3 w(x_+ - x_-)}\int_0^1 \left(\frac{1}{x-x_+} - \frac{1}{x-x_-}\right) dx\,.
\end{aligned} \tag{119}$$

Note that $x_+ \in (1,\infty)$ and $x_- \in (-\infty, 0)$. Hence, the integrands above are not singular for $0 \leq x \leq 1$, and the corresponding integrals are well defined.

Using eq. (118), it follows that

$$x_+ - x_- = \frac{\cosh w}{\sinh w}\,, \qquad x_+ + x_- = 1\,. \tag{120}$$

It follows that

$$\begin{aligned}
\frac{dF}{dw} &= 2\coth w + \frac{1}{2\sinh^2 w}\left[\ln\left(\frac{x_+ - 1}{x_+}\right) - \ln\left(\frac{x_- - 1}{x_-}\right)\right] \\
&= 2\coth w + \frac{1}{\sinh^2 w}\ln\left(\frac{-x_-}{x_+}\right) = 2\coth w + \frac{1}{\sinh^2 w}\ln(e^{-2w}) \\
&= 2\left[\coth w - \frac{w}{\sinh^2 w}\right]\,,
\end{aligned} \tag{121}$$

after employing eqs. (118) and (120). Integrating both sides of eq. (133) and making use of the indefinite integrals,

$$\int \coth w \, dw = \ln|\sinh w| \,, \qquad\qquad \int \frac{w\, dw}{\sinh^2 w} = -w\coth w + \ln|\sinh w| \,, \tag{122}$$

it follows that

$$F(w) = 2\left[w\coth w - 1\right] \,, \tag{123}$$

where we employed the boundary condition, $F(0) = 0$. In light of eq. (116), $w = \text{arcsinh}\,\frac{1}{2}\sqrt{-z}$ and

$$\coth w = \frac{\cosh w}{\sinh w} = \frac{\sqrt{1+\sinh^2 w}}{\sinh w} = \sqrt{1-\frac{4}{z}} \,. \tag{124}$$

Hence,

$$F(z) = 2\left\{\left(1-\frac{4}{z}\right)^{1/2} \text{arcsinh}\left(\tfrac{1}{2}\sqrt{-z}\right) - 1\right\} \,, \qquad \text{for } z < 0. \tag{125}$$

in agreement with eq. (98).

Next, consider the case of $0 < z < 4$, where $F(z)$ is given by eq. (91). It is convenient to define,

$$z = 4\sin^2\theta \,, \qquad \text{for } 0 < \theta \leq \tfrac{1}{2}\pi. \tag{126}$$

Taking the derivative of F with respect to θ,

$$\begin{aligned} \frac{dF}{d\theta} &= \frac{d}{d\theta}\int_0^1 dx\, \ln\bigl[1-4x(1-x)\sin^2\theta\bigr] = -4\sin 2\theta \int_0^1 \frac{x(1-x)dx}{1-4x(1-x)\sin^2\theta} \\ &= \frac{\sin 2\theta}{\sin^2\theta}\int_0^1 \frac{1-4x(1-x)\cos^2\theta - 1}{1-4x(1-x)\sin^2\theta}\, dx \\ &= 2\cot\theta - 2\cot\theta \int_0^1 \frac{dx}{1-4x(1-x)\sin^2\theta} \,. \end{aligned} \tag{127}$$

To evaluate the above integral, we first factor the denominator of the integrand and then apply a partial fractioning. That is,

$$1-4x(1-x)\sin^2\theta = 4\sin^2\theta(x-x_+)(x-x_-) \,, \qquad \text{where } x_\pm \equiv \pm\frac{ie^{\mp i\theta}}{2\sin\theta}. \tag{128}$$

Hence, it follows that

$$\begin{aligned} \frac{dF}{d\theta} &= 2\cot\theta - \frac{\cot\theta}{2\sin^2\theta}\int_0^1 \frac{dx}{(x-x_+)(x-x_-)} \\ &= 2\cot\theta - \frac{\cot\theta}{2\sin^2\theta(x_+ - x_-)}\int_0^1 \left(\frac{1}{x-x_+} - \frac{1}{x-x_-}\right) dx \,. \end{aligned} \tag{129}$$

Using eq. (128), it follows that

$$x_+ - x_- = i\cot\theta \,, \qquad\qquad x_+ + x_- = 1 \,. \tag{130}$$

Moreover,

$$\int_0^1 \frac{dx}{x-x_-} = \int_0^1 \frac{dx}{x-1+x_+} = -\int_0^1 \frac{dx}{x-x_+}\,, \tag{131}$$

after changing the integration variable $x \to 1-x$ in the final step above. In light of these last two results, eq. (129) yields,

$$\begin{aligned}\frac{dF}{d\theta} &= 2\cot\theta + \frac{i}{\sin^2\theta}\int_0^1 \frac{dx}{x-x_+} = 2\cot\theta + \frac{i}{\sin^2\theta}\ln\left(\frac{1-x_+}{-x_+}\right) \\ &= 2\cot\theta + \frac{i}{\sin^2\theta}\ln\left(\frac{x_-}{-x_+}\right) = 2\cot\theta + \frac{i}{\sin^2\theta}\ln\left(e^{2i\theta}\right)\,. \end{aligned} \tag{132}$$

after using eq. (128) to obtain the final result.

To complete our analysis, recall that the principal value of the complex logarithm is given by eq. (89). Since $0 < \theta \leq \frac{1}{2}\pi$ [cf. eq. (126)], it follows that $\ln(e^{2i\theta}) = 2i\theta$. Hence, eq. (132) yields,

$$\frac{dF}{d\theta} = 2\cot\theta - \frac{2\theta}{\sin^2\theta}\,. \tag{133}$$

Integrating both sides of eq. (133) and making use of the indefinite integrals,

$$\int \cot\theta\, d\theta = \ln|\sin\theta|\,, \qquad \int \frac{\theta\, d\theta}{\sin^2\theta} = -\theta\cot\theta + \ln|\sin\theta|\,, \tag{134}$$

we end up with

$$F(\theta) = 2(\theta\cot\theta - 1)\,, \tag{135}$$

where we employed the boundary condition, $F(0) = 0$. In light of eq. (126), $\theta = \arcsin\left(\frac{1}{2}\sqrt{z}\right)$, and

$$\cot\theta = \frac{\sqrt{1-\sin^2\theta}}{\sin\theta} = \sqrt{\frac{4}{z} - 1}\,. \tag{136}$$

Hence,

$$F(z) = 2\left\{\left(\frac{4}{z} - 1\right)^{1/2} \arcsin\left(\tfrac{1}{2}\sqrt{z}\right) - 1\right\}\,, \qquad \text{for } 0 \leq z \leq 4, \tag{137}$$

in agreement with eq. (102).

Finally, we consider the case of $z > 4$,where $\operatorname{Re} F(z)$ is given by eq. (104). It is convenient to define,

$$z = 4\cosh^2 w\,, \qquad \text{for } 0 < w < \infty\,. \tag{138}$$

After employing eq. (138), we take the derivative of $\operatorname{Re} F$ with respect to w,

$$\begin{aligned}\frac{d}{dw}\operatorname{Re} F &= \frac{d}{dw}\int_0^1 dx\, \ln\bigl|1 - 4x(1-x)\cosh^2 w\bigr| = -4\sinh 2w\, \mathrm{P}\int_0^1 \frac{x(1-x)dx}{1-4x(1-x)\cosh^2 w} \\ &= -\frac{\sinh 2w}{\cosh^2 w}\mathrm{P}\int_0^1 \frac{-1+4x(1-x)\cosh^2 w + 1}{1-4x(1-x)\cosh^2 w}\, dw \\ &= 2\tanh w - 2\tanh w\, \mathrm{P}\int_0^1 \frac{dx}{1-4x(1-x)\cosh^2 w}\,, \end{aligned} \tag{139}$$

To evaluate the above integral, we first factor the denominator of the integrand and then apply a partial fractioning. That is,

$$1-4x(1-x)\cosh^2 w = 4\cosh^2 w(x-x_+)(x-x_-)\,, \qquad \text{where } x_\pm \equiv \frac{e^{\pm w}}{2\cosh w}. \tag{140}$$

Hence, it follows that

$$\begin{aligned}\frac{d}{dw}\,\text{Re}\,F &= 2\tanh w - \frac{\tanh w}{2\cosh^2 w}\,\text{P}\int_0^1 \frac{dx}{(x-x_+)(x-x_-)} \\ &= 2\tanh w - \frac{\tanh w}{2\cosh^2 w(x_+-x_-)}\,\text{P}\int_0^1\left(\frac{1}{x-x_+}-\frac{1}{x-x_-}\right)dx\,. \end{aligned}\tag{141}$$

Using eq. (140), it follows that

$$x_+-x_- = \tanh w\,, \qquad x_++x_-=1\,. \tag{142}$$

Moreover,

$$\text{P}\int_0^1\frac{dx}{x-x_-} = \text{P}\int_0^1\frac{dx}{x-1+x_+} = -\text{P}\int_0^1\frac{dx}{x-x_+}\,, \tag{143}$$

after changing the integration variable $x\to 1-x$ in the final step above. Using the definition of the principal value prescription,

$$\begin{aligned}\text{P}\int_0^1\frac{dx}{x-x_+} &= \lim_{\delta\to 0^+}\left\{\int_0^{x_+-\delta}\frac{dx}{x-x_+}+\int_{x_++\delta}^1\frac{dx}{x-x_+}\right\} \\ &= \lim_{\delta\to 0^+}\left\{\ln(x_+-x)\Big|_0^{x_+-\delta}+\ln(x-x_+)\Big|_{x_++\delta}^1\right\} \\ &= \lim_{\delta\to 0^+}\left\{\ln\delta-\ln x_++\ln(1-x_+)-\ln\delta\right\} \\ &= \ln\left(\frac{1-x_+}{x_+}\right) = \ln\left(\frac{x_-}{x_+}\right) = -2w\,, \end{aligned}\tag{144}$$

after making use of eqs. (140) and (142).

Using the above results, eq. (141) yields,

$$\frac{d}{dw}\,\text{Re}\,F = 2\tanh w - \frac{1}{\cosh^2 w}\,\text{P}\int_0^1\frac{dx}{x-x_+} = 2\tanh w+\frac{2w}{\cosh^2 w}\,. \tag{145}$$

Integrating both sides of eq. (145) and making use of the indefinite integrals,

$$\int \tanh w\, dw = \ln\cosh w\,, \qquad \int\frac{d\theta}{\cosh^2 w} = \tanh w \tag{146}$$

and employing an integration by parts to obtain

$$\int\frac{w\,dw}{\cosh^2 w} = w\tanh w - \int\tanh w\,dw = w\tanh w - \ln\cosh w\,, \tag{147}$$

we end up with

$$\operatorname{Re} F(w) = 2w \tanh w + C\,. \tag{148}$$

The integration constant C can be obtained by noting that $F(z=4) = F(w=0) = -2$ in light of eq. (103). Hence, $C = -2$. Employing eq. (138),

$$\tanh^2 w = \frac{\cosh^2 w - 1}{\cosh^2 w} = 1 - \frac{4}{z}\,, \qquad\qquad \cosh w = \tfrac{1}{2}\sqrt{z}\,. \tag{149}$$

Hence, eq. (148) with $C = -2$ yields,

$$\operatorname{Re} F(z) = 2\left[\left(1 - \frac{4}{z}\right)^{1/2} \operatorname{arccosh}\bigl(\tfrac{1}{2}\sqrt{z}\bigr) - 1\right], \qquad \text{for } z \geq 4. \tag{150}$$

Combining with the result for $\operatorname{Im} F(z)$ obtained in eq. (90), we end up with

$$F(z) = 2\left\{\left(1 - \frac{4}{z}\right)^{1/2} \left[\operatorname{arccosh}\bigl(\tfrac{1}{2}\sqrt{z}\bigr) - \tfrac{1}{2} i\pi\right] - 1\right\}, \qquad \text{for } z \geq 4, \tag{151}$$

in agreement with eq. (112).

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