1 The probability that a product of random numbers is less than a fixed constant

1.1 First proof (by Howard E. Haber)

Consider \( n \) random numbers, \( r_1, r_2, \ldots, r_n \) each uniformly distributed on the interval \([0, 1]\). Let \( P_n(a) \) be the probability that the product \( r_1 r_2 \cdots r_n < a \) for some number \( a \in [0, 1] \). To compute \( P_n(a) \), I shall first consider \( I_n(a) \equiv 1 - P_n(a) \) which is the probability that \( r_1 r_2 \cdots r_n > a \). Clearly, \( I_n(a) \) is equal to the hypervolume of the region bounded by the hypersurface \( x_1 x_2 \cdots x_n = a \) and the \( n \) hyperplanes \( x_1 = 1, x_2 = 1, \ldots, x_n = 1 \). Explicitly, this hypervolume is given by the \( n \)-fold integral:

\[
I_n(a) = \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_n .
\]

(1)

It is convenient to rewrite this in the form of a recursion relation. Start with

\[
I_{n+1}(a) = \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_n \int_a/(x_1 x_2 \cdots x_n) dx_{n+1} = I_n(a) - a J_n(a) ,
\]

(2)

where

\[
J_n(a) \equiv \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_n \frac{dx_n}{x_n} .
\]

(3)

after performing one integration over \( x_{n+1} \). Since \( P_n(a) = 1 - I_n(a) \), we have deduced the following recursion relation for the \( P_n(a) \):

\[
P_{n+1}(a) = P_n(a) + a J_n(a) .
\]

(4)

To evaluate \( J_n(a) \), make a change of variables from the \( x_n \) to \( y_k = -\ln x_n \). In addition, I define \( b \equiv -\ln a \) (note that \( b > 0 \) since \( a \in [0, 1] \)). Then,

\[
J_n(a) \equiv b^n \int_0^b dy_1 \int_0^{b-y_1} dy_2 \cdots \int_0^{b-y_1-y_2-\cdots-y_{n-1}} dy_n .
\]

(5)

Now observe the geometrical interpretation of \( J_n \). First, \( J_1 \) is the length of the line from 0 to \( b \). \( J_2 \) is the area within the unit square lying below the diagonal, which is equal to half the area of the square. \( J_3 \) is the volume within the cube lying below the plane \( y_1 + y_2 + y_3 = b \), which is equal to \( 1/3 \) or one sixth of the volume of the cube. Thus, \( J_n \) is the hypervolume lying below the hyperplane \( y_1 + y_2 + \cdots y_n = b \) which is equal to \( 1/n! \) times the volume of the hypercube (whose side has length \( b \)). That is,

\[
J_n(a) = \frac{b^n}{n!} = \left(\frac{-\ln a}{n!}\right)^n .
\]

(6)

Note that this result can also be derived by a straightforward change of variables. Namely, let \( z_1 = b - y_1, z_2 = b - y_1 - y_2, \ldots, z_n = b - y_1 - y_2 - \cdots - y_n \). The
Jacobian of the transformation is 1. Thus, we end up with:

\[ J_n(a) \equiv \int_0^a dz_1 \int_0^{z_1} dz_2 \cdots \int_0^{z_{n-1}} dz_n. \]  

(7)

All \( n \) integrals can now be performed sequentially, and one simply recovers the result for \( J_n(a) \) given by eq. (6). Thus, we have derived the recursion relation:

\[ P_{n+1}(a) = P_n(a) + \frac{a(-\ln a)^n}{n!}. \]  

(8)

To obtain a closed form expression for \( P_n(a) \), simply note that \( P_1(a) = a \). (In fact, we can define \( P_0(a) = 0 \).) Iterating eq. (8), one ends up with:

\[ P_n(a) = a \sum_{k=0}^{n-1} \frac{(-\ln a)^k}{k!}. \]  

(9)

It is interesting to consider the limit as \( n \to \infty \). For any fixed value of \( a \), we expect \( P_n \to 1 \) in this limit. But,

\[ \sum_{k=0}^{\infty} (-1)^k \frac{(-\ln a)^k}{k!} = e^{-\ln a} = \frac{1}{a} \]  

(10)

Inserting this into eq. (9), one indeed confirms the expectation that \( P_n \to 1 \) as \( n \to \infty \).

1.2 Second proof (following Harrison B. Prosper)

We start from the integral representation of \( P_n(a) \)

\[ P_n(a) = \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_n \Theta(a - x_1 x_2 \cdots x_n), \]  

(11)

where

\[ \Theta(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x < 0, \end{cases} \]  

(12)

is the Heaviside step function. It turns out that it is simpler to start from the equivalent representation

\[ P_n(a) = \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_n \Theta(\ln a - \ln x_1 - \ldots - \ln x_n), \]  

(13)

Taking the derivative with respect to \( \ln a \), and using

\[ \frac{d\Theta(x)}{dx} = \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{ixt}, \]  

(14)
It follows that

$$\frac{dP_n}{d\ln a} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \ e^{it \ln a} \ n! \ \prod_{i=1}^{n} \int_{0}^{1} dx_i \ e^{-it \ln x_i}. \quad (15)$$

Now,

$$\int_{0}^{1} dx \ e^{-it \ln x} = \int_{0}^{1} x^{-it} \ dx = \frac{1}{1-it}. \quad (16)$$

Thus,

$$\frac{dP_n}{d\ln a} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \ e^{it \ln a} \ (1-it)^n. \quad (17)$$

This integral is easily evaluated by the method of contour integration in the complex plane. One can close the contour in the lower half of the complex $t$-plane (since $0 < a < 1$ implies that $\ln a < 0$, the contribution from the semicircle at infinity vanishes); note that the contour of integration is clockwise. There is one $n$th order pole at $t = -i$ enclosed by the contour, so one obtains for the integral $-2\pi i$ multiplied by the residue of the integrand at $t = -i$. Recalling the formula:

$$\text{Res} p = \frac{1}{(n-1)!} \left( \frac{d}{dz} \right)^{n-1} \left[ (t - p)^n f(t) \right]_{t=p}, \quad (18)$$

for the residue of a function $f(t)$ that has an $n$th order pole at $t = p$, and carefully writing $(t+i)^n = (1-it)^n$ in the numerator, we end up with:

$$\frac{dP_n}{d\ln a} = \frac{a(-\ln a)^{n-1}}{(n-1)!}. \quad (19)$$

To complete the computation, write $d\ln a = da/a$ and integrate to get $P_n$, with the boundary condition that $P_n(a=0) = 0$.

$$P_n(a) = \frac{1}{(n-1)!} \int_{0}^{a} (-\ln x)^{n-1} \ dx$$

$$= \frac{1}{(n-1)!} \int_{-\ln a}^{\infty} e^{-y} \ y^{n-1} \ dy$$

$$= \frac{a(-\ln a)^{n-1}}{(n-1)!} + P_{n-1}(a). \quad (20)$$

Above, we have changed the integration variable to $y = -\ln x$ and integrated by parts once. Finally, letting $n-1 \to n$ yields precisely the recursion relation for $P_n(a)$ obtained in the first proof [see eq. (8)]. Thus the final expression for $P_n(a)$ given in eq. (9) follows.
1.3 Third proof (following Zoltan Ligeti)

It is easy to see that the $P_n$ satisfy the following recursion relation:

$$P_{n+1}(a) = P_n(a) + a \int_0^1 P'_n(x) \frac{dx}{x}, \quad (21)$$

where $P'_n(x) \equiv dP_n(x)/dx$. First, if $x_1 x_2 \cdots x_n < a$, then it follows that $x_1 x_2 \cdots x_{n+1} < a$, since $x_i \in [0,1]$. This accounts for the first term on the right hand side of eq. (21). Second, if $x_1 x_2 \cdots x_n \equiv x > a$, then $x_1 x_2 \cdots x_{n+1} < a$ only if $x_{n+1} < a/x$. But, $P'_n(x)dx$ is the probability that $x_1 x_2 \cdots x_n$ lies in the interval $[x, x + dx]$, while the probability that $x_{n+1}$ lies in the interval $[0, a/x]$ is simply $a/x$. This then accounts for the second term of eq. (21).

We next take the derivative of eq. (21) with respect to $a$. Then, two of the resulting three terms on the right hand side of eq. (21) cancel, which yields

$$P'_{n+1}(a) = \int_0^1 P'_n(x) \frac{dx}{x}, \quad (22)$$

with boundary condition of $P'_1(x) = 1$. We then compute the $P'_n$ successively for $n = 1, 2, \ldots, n$ in general,

$$P'_n(a) = \frac{1}{(n-2)!} \int_0^1 (-\ln x)^{n-2} \frac{dx}{x} = \frac{(-\ln a)^{n-1}}{(n-1)!}, \quad (23)$$

after changing the integration variable to $y = -\ln x$. Inserting this result into eq. (21), we reproduce eq. (8), and the final result of eq. (9) once again follows. [Alternatively, note that the result of eq. (23) is equivalent to eq. (19).]

Finally, we can make a connection to the first proof. Recall that eq. (4) states that $P_{n+1}(a) = P_n(a) + aJ_n(a)$. Using eq. (3), we can write:

$$J_n(a) = \int_0^1 \frac{dx_1}{x_1} \int_0^1 \frac{dx_2}{x_2} \cdots \int_0^1 \frac{dx_n}{x_n} \Theta(x_1 x_2 \cdots x_n - a) \quad (24)$$

$$= \int_a^1 \frac{dx}{x} \int_0^1 \frac{dx_1}{x_1} \int_0^1 \frac{dx_2}{x_2} \cdots \int_0^1 \frac{dx_n}{x_n} \delta(x - x_1 x_2 \cdots x_n) \quad (25)$$

$$= \int_a^1 P'_n(x) \frac{dx}{x}, \quad (26)$$

and eq. (21) then follows. Note that the last step above [eq. (26)] follows from eq. (11) [simply take the derivative to convert the $\Theta$-function into the $\delta$-function]. Moreover, if one carries out the integration over $x$ in eq. (25) by using the $\delta$-function, one obtains eq. (24) since the argument of the $\delta$-function vanishes inside the integration region $a < x < 1$ only if $x_1 x_2 \cdots x_n > a$.

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1. Clearly, $P_1(x) = x$ for $x \in [0,1]$, since this is the probability that a randomly chosen number from the interval $[0,1]$ is less than $x$. 