1 The probability that a product of random numbers is less than a fixed constant

1.1 First proof (by Howard E. Haber)

Consider n random numbers, r_1, r_2,...,r_n each uniformly distributed on the interval $[0,1]$. Let $P_n(a)$ be the probability that the product $r_1 r_2 \cdots r_n < a$ for some number $a \in [0,1]$. To compute $P_n(a)$, I shall first consider $I_n(a) \equiv 1 - P_n(a)$ which is the probability that $r_1 r_2 \cdots r_n > a$. Clearly, $I_n(a)$ is equal to the hypervolume of the region bounded by the hypersurface $x_1 x_2 \cdots x_n = a$ and the n hyperplanes $x_1 = 1, x_2 = 1, \ldots, x_n = 1$. Explicitly, this hypervolume is given by the n-fold integral:

$$I_n(a) = \int_a^1 dx_1 \int_{a/x_1}^1 dx_2 \cdots \int_{a/(x_1 x_2 \cdots x_{n-1})}^1 dx_n \,. \tag{1}$$

It is convenient to rewrite this in the form of a recursion relation. Start with

$$I_{n+1}(a) = \int_a^1 dx_1 \int_{a/x_1}^1 dx_2 \cdots \int_{a/(x_1 x_2 \cdots x_n)}^1 dx_{n+1} = I_n(a) - aJ_n(a)\,, \tag{2}$$

where

$$J_n(a) \equiv \int_a^1 \frac{dx_1}{x_1} \int_{a/x_1}^1 \frac{dx_2}{x_2} \cdots \int_{a/(x_1 x_2 \cdots x_{n-1})}^1 \frac{dx_n}{x_n} \,. \tag{3}$$

after performing one integration over x_{n+1}. Since $P_n(a) = 1 - I_n(a)$, we have deduced the following recursion relation for the $P_n(a)$:

$$P_{n+1}(a) = P_n(a) + aJ_n(a)\,. \tag{4}$$

To evaluate $J_n(a)$, make a change of variables from the x_n to $y_n = -\ln x_n$. In addition, I define $b \equiv -\ln a$ (note that $b > 0$ since $a \in [0,1]$). Then,

$$J_n(a) \equiv \int_0^b dy_1 \int_0^{b-y_1} dy_2 \cdots \int_0^{b-y_1-y_2-\ldots-y_{n-1}} dy_n \,. \tag{5}$$

Now observe the geometrical interpretation of J_n. First, J_1 is the length of the line from 0 to b. J_2 is the area within the unit square lying below the diagonal, which is equal to half the area of the square. J_3 is the volume within the cube lying below the plane $y_1 + y_2 + y_3 = b$, which is equal to $1/3!$ or one sixth of the volume of the cube. Thus, J_n is the hypervolume lying below the hyperplane $y_1 + y_2 + \cdots y_n = b$ which is equal to $1/n!$ times the volume of the hypercube (whose side has length b). That is,

$$J_n(a) = \frac{b^n}{n!} = \frac{(-\ln a)^n}{n!}\,. \tag{6}$$

Note that this result can also be derived by a straightforward change of variables. Namely, let $z_1 = b - y_1, z_2 = b - y_1 - y_2, \ldots, z_n = b - y_1 - y_2 - \ldots - y_n$. The

Jacobian of the transformation is 1. Thus, we end up with:

$$J_n(a) \equiv \int_0^b dz_1 \int_0^{z_1} dz_2 \cdots \int_0^{z_{n-1}} dz_n \,. \tag{7}$$

All n integrals can now be performed sequentially, and one simply recovers the result for $J_n(a)$ given by eq. (6). Thus, we have derived the recursion relation:

$$P_{n+1}(a) = P_n(a) + \frac{a(-\ln a)^n}{n!} \,. \tag{8}$$

To obtain a closed form expression for $P_n(a)$, simply note that $P_1(a) = a$. (In fact, we can define $P_0(a) = 0$.) Iterating eq. (8), one ends up with:

$$\boxed{P_n(a) = a \sum_{k=0}^{n-1} \frac{(-\ln a)^k}{k!} \,.} \tag{9}$$

It is interesting to consider the limit as $n \to \infty$. For any fixed value of a, we expect $P_n \to 1$ in this limit. But,

$$\sum_{k=0}^{\infty} (-1)^k \frac{(\ln a)^k}{k!} = e^{-\ln a} = \frac{1}{a} \tag{10}$$

Inserting this into eq. (9), one indeed confirms the expectation that $P_n \to 1$ as $n \to \infty$.

1.2 Second proof (following Harrison B. Prosper)

We start from the integral representation of P_n(a)

$$P_n(a) = \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_n \, \Theta(a - x_1 x_2 \cdots x_n) \,, \tag{11}$$

where

$$\Theta(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x < 0, \end{cases} \tag{12}$$

is the Heavyside step function. It turns out that it is simpler to start from the equivalent representation

$$P_n(a) = \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_n \, \Theta(\ln a - \ln x_1 - \ldots - \ln x_n) \,, \tag{13}$$

Taking the derivative with respect to $\ln a$, and using

$$\frac{d\Theta(x)}{dx} = \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, e^{ixt} \,, \tag{14}$$

it follows that

$$\frac{dP_n}{d\ln a} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{it\ln a} \prod_{i=1}^{n} \int_0^1 dx_i\, e^{-it\ln x_i}\,. \tag{15}$$

Now,

$$\int_0^1 dx\, e^{-it\ln x} = \int_0^1 x^{-it}\, dx = \frac{1}{1-it}\,. \tag{16}$$

Thus,

$$\frac{dP_n}{d\ln a} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, \frac{e^{it\ln a}}{(1-it)^n}\,. \tag{17}$$

This integral is easily evaluated by the method of contour integration in the complex plane. One can close the contour in the lower half of the complex t-plane (since $0 < a < 1$ implies that $\ln a < 0$, the contribution from the semicircle at infinity vanishes); note that the contour of integration is clockwise. There is one nth order pole at $t = -i$ enclosed by the contour, so one obtains for the integral $-2\pi i$ multiplied by the residue of the integrand at $t = -i$. Recalling the formula:

$$\operatorname{Res} p = \frac{1}{(n-1)!}\left(\frac{d}{dz}\right)^{n-1} \left[(t-p)^n f(t)\right]\Bigg|_{t=p}\,, \tag{18}$$

for the residue of a function $f(t)$ that has an nth order pole at $t = p$, and carefully writing $(t+i)^n = i^n(1-it)^n$ in the numerator, we end up with:

$$\frac{dP_n}{d\ln a} = \frac{a(-\ln a)^{n-1}}{(n-1)!}\,. \tag{19}$$

To complete the computation, write $d\ln a = da/a$ and integrate to get P_n, with the boundary condition that $P_n(a=0) = 0$.

$$\begin{aligned} P_n(a) &= \frac{1}{(n-1)!}\int_0^a (-\ln x)^{n-1}\, dx \\ &= \frac{1}{(n-1)!}\int_{-\ln a}^{\infty} e^{-y}\, y^{n-1}\, dy \\ &= \frac{a(-\ln a)^{n-1}}{(n-1)!} + P_{n-1}(a)\,. \end{aligned} \tag{20}$$

Above, we have changed the integration variable to $y = -\ln x$ and integrated by parts once. Finally, letting $n - 1 \to n$ yields precisely the recursion relation for $P_n(a)$ obtained in the first proof [see eq. (8)]. Thus the final expression for $P_n(a)$ given in eq. (9) follows.

1.3 Third proof (following Zoltan Ligeti)

It is easy to see that the P_n satisfy the following recursion relation:

$$P_{n+1}(a) = P_n(a) + a \int_a^1 P'_n(x) \frac{dx}{x} , \tag{21}$$

where $P'_n(x) \equiv dP_n(x)/dx$. First, if $x_1 x_2 \cdots x_n < a$, then it follows that $x_1 x_2 \cdots x_{n+1} < a$, since $x_i \in [0, 1]$. This accounts for the first term on the right hand side of eq. (21). Second, if $x_1 x_2 \cdots x_n \equiv x > a$, then $x_1 x_2 \cdots x_{n+1} < a$ only if $x_{n+1} < a/x$. But, $P'_n(x)dx$ is the probability that $x_1 x_2 \cdots x_n$ lies in the interval $[x, x + dx]$, while the probability that x_{n+1} lies in the interval $[0, a/x]$ is simply a/x. This then accounts for the second term of eq. (21).

We next take the derivative of eq. (21) with respect to a. Then, two of the resulting three terms on the right hand side of eq. (21) cancel, which yields

$$P'_{n+1}(a) = \int_a^1 P'_n(x) \frac{dx}{x} , \tag{22}$$

with boundary condition of $P'_1(x) = 1$.[1] We then compute the P'_n successively for $n = 1, 2 \ldots$; in general,

$$P'_n(a) = \frac{1}{(n-2)!} \int_a^1 (-\ln x)^{n-2} \frac{dx}{x} = \frac{(-\ln a)^{n-1}}{(n-1)!} , \tag{23}$$

after changing the integration variable to $y = -\ln x$. Inserting this result into eq. (21), we reproduce eq. (8), and the final result of eq. (9) once again follows. [Alternatively, note that the result of eq. (23) is equivalent to eq. (19).]

Finally, we can make a connection to the first proof. Recall that eq. (4) states that $P_{n+1}(a) = P_n(a) + aJ_n(a)$. Using eq. (3), we can write:

$$J_n(a) = \int_0^1 \frac{dx_1}{x_1} \int_0^1 \frac{dx_2}{x_2} \cdots \int_0^1 \frac{dx_n}{x_n} \Theta(x_1 x_2 \cdots x_n - a) \tag{24}$$

$$= \int_a^1 \frac{dx}{x} \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_n \, \delta(x - x_1 x_2 \cdots x_n) \tag{25}$$

$$= \int_a^1 P'_n(x) \frac{dx}{x} , \tag{26}$$

and eq. (21) then follows. Note that the last step above [eq. (26)] follows from eq. (11) [simply take the derivative to convert the Θ-function into the δ-function]. Moreover, if one carries out the integration over x in eq. (25) by using the δ-function, one obtains eq. (24) since the argument of the δ-function vanishes inside the integration region $a < x < 1$ only if $x_1 x_2 \cdots x_n > a$.

Howard E. Haber

December 6, 1998

[1] Clearly, $P_1(x) = x$ for $x \in [0, 1]$, since this is the probability that a randomly chosen number from the interval $[0, 1]$ is less than x.