1 Complex representations of scalar fields

Let $\Phi_i(x)$ be a set of n complex scalar fields. The scalar Lagrangian

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \Phi_i)^\dagger(\partial^\mu \Phi_i) - V(\Phi_i, \Phi_i^\dagger) \tag{1}$$

is assumed to be invariant under a compact symmetry group G, under which the scalar fields transform as:

$$\Phi_i \to \mathcal{U}_i{}^j \Phi_j\,, \qquad\qquad \Phi^{\dagger\, i} \to \Phi^{\dagger\, j}(\mathcal{U}^\dagger)_j{}^i\,, \tag{2}$$

where $\mathcal{U}$ is a complex representation of G. Using a well-known theorem, all complex representations of a compact group are equivalent (via a similarity transformation) to a unitary representation. Thus, without loss of generality, we may take $\mathcal{U}$ to be a unitary $n \times n$ matrix. Explicitly,

$$\mathcal{U} = \exp[-ig_a \Lambda^a \mathcal{T}^a]\,, \tag{3}$$

where the generators $\mathcal{T}^a$ are $n \times n$ hermitian matrices. The corresponding infinitesimal transformation law is

$$\delta\Phi_i(x) = -ig_a \Lambda^a (\mathcal{T}^a)_i{}^j \Phi_j(x)\,, \tag{4}$$

$$\delta\Phi^{\dagger\, i}(x) = +ig_a \Phi^{\dagger\, j}(x) \Lambda^a (\mathcal{T}^a)_j{}^i\,, \tag{5}$$

where the g_a and Λ^a are real. One can check that the scalar kinetic energy term is invaraint under U(n) transformations. The scalar potential, which is not invariant in general under the full U(n) group, is invariant under G [which is a subgroup of U(n)] if

$$(\mathcal{T}^a)_i{}^j \Phi_j \frac{\partial V}{\partial \Phi_i} - (\mathcal{T}^a)_j{}^i \Phi^{\dagger\, j} \frac{\partial V}{\partial \Phi^{\dagger\, i}} = 0 \tag{6}$$

is satisfied.

There are $2n$ independent scalar degrees of freedom, corresponding to the fields Φ_i and $\Phi^{\dagger\, i}$. We can also express these degrees of freedom in terms of $2n$ hermitian scalar fields consisting of ϕ_{Aj} and ϕ_{Bj} ($j = 1, 2, \ldots, n$) defined by:

$$\Phi_j = \frac{1}{\sqrt{2}}(\phi_{Aj} + i\phi_{Bj})\,, \qquad \Phi^{\dagger\, j} = \frac{1}{\sqrt{2}}(\phi_{Aj} - i\phi_{Bj})\,. \tag{7}$$

It is straightforward to compute the group transformation laws for the hermitian fields ϕ_{Aj} and ϕ_{Bj}. These are conveniently expressed by introducing a $2n$-dimensional scalar multiplet:

$$\phi(x) = \begin{pmatrix} \phi_A(x) \\ \phi_B(x) \end{pmatrix}\,. \tag{8}$$

That is, $\phi_{Aj}(x) = \phi_j(x)$ and $\phi_{Bj}(x) = \phi_{j+n}(x)$. Then the inifintesimal form of the group transformation law for $\phi(x)$ is given by $\phi_k(x) \to \phi_k(x) + \delta\phi_k(x)$ for $k = 1, 2, \ldots, 2n$, where

$$\delta\phi_k(x) = -ig\Lambda^a (T^a)_k{}^\ell \phi_\ell(x)\,, \tag{9}$$

and

$$iT^a = \begin{pmatrix} -\text{Im } \mathcal{T}^a & -\text{Re } \mathcal{T}^a \\ \text{Re } \mathcal{T}^a & -\text{Im } \mathcal{T}^a \end{pmatrix} . \tag{10}$$

Note that Re $\mathcal{T}^a$ is symmetric and Im $\mathcal{T}^a$ is antisymmetric (which follow from the hermiticity of $\mathcal{T}^a$). Thus, iT^a is a real antisymmetric $2n \times 2n$ matrix, which when exponentiated yields a real orthogonal $2n$-dimensional representation of G.

2 The embedding of U(n) in SO(2n)

Consider a scalar field theory consisting of n identical complex fields Φ_i, with a Lagrangian

$$\mathscr{L} = \tfrac{1}{2}(\partial_\mu \Phi_i)^\dagger (\partial^\mu \Phi_i) - V(\Phi^\dagger \Phi) , \tag{11}$$

where the potential function V is a function of $\Phi^{\dagger\, i}\Phi_i$. Such a theory is invariant under the U(n) transformation $\Phi \to U\Phi$, where U is an $n \times n$ unitary matrix.

Rewrite the Lagrangian in terms of hermitian fields ϕ_{Ai} and ϕ_{Bi} defined by:

$$\Phi_j = \frac{1}{\sqrt{2}}(\phi_{Aj} + i\phi_{Bj}) , \qquad \Phi^{\dagger\, j} = \frac{1}{\sqrt{2}}(\phi_{Aj} - i\phi_{Bj}) , \tag{12}$$

and introduce the $2n$-dimensional hermitian scalar field:

$$\phi(x) = \begin{pmatrix} \phi_A(x) \\ \phi_B(x) \end{pmatrix} . \tag{13}$$

One can show that the Lagrangian is actually invariant under a larger symmetry group O($2n$), corresponding to the transformation $\phi \to \mathcal{O}\phi$ where $\mathcal{O}$ is a $2n \times 2n$ orthogonal matrix.

Working in the complex basis, one can show that the Lagrangian [eq. (11)] is invariant under the transformation:

$$\Phi_i \to U_i{}^j \Phi_j + \Phi^{\dagger\, j} (V^\dagger)_j{}^i , \tag{14}$$

where U and V are complex $n \times n$ matrices, provided that the following two conditions are satisfied:

$$(i) \quad (U^\dagger U + V^\dagger V)_i{}^j = \delta_i{}^j , \tag{15}$$

$$(ii) \quad V^T U \text{ is an antisymmetric matrix} . \tag{16}$$

In particular, the $2n \times 2n$ matrix

$$\mathcal{Q} = \begin{pmatrix} \text{Re } (U+V) & -\text{Im } (U+V) \\ \text{Im } (U-V) & \text{Re } (U-V) \end{pmatrix} \tag{17}$$

is an orthogonal matrix if U and V satisfy eqs. (15) and (16). One can prove that any $2n \times 2n$ orthogonal matrix can be written in the form of eq. (17) by verifying that $\mathcal{Q}$ is determined by $n(2n-1)$ independent parameters. This is most easily done with an infinitesimal analysis.

Using the above results, it follows that if U is a unitary $n \times n$ matrix, then the $2n \times 2n$ matrix

$$\mathcal{Q}_U = \begin{pmatrix} \mathrm{Re}\,U & -\mathrm{Im}\,U \\ \mathrm{Im}\,U & \mathrm{Re}\,U \end{pmatrix} \tag{18}$$

provides an explicit embedding of the subgroup U(n) inside O($2n$). By writing $\mathcal{Q}_U = \exp[-ig\Lambda^a T^a]$ and $U = \exp[-ig\Lambda^a \mathcal{T}^a]$, one can show that T^a is given by eq. (10) in terms of the $\mathcal{T}^a$.

Moreover, using the well-known formula for the determinant of a block-partitioned matrix:

$$\det \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \det P \, \det\,(S - RP^{-1}Q)\,, \tag{19}$$

and writing $U_R \equiv \mathrm{Re}\,U$ and $U_I \equiv \mathrm{Im}\,U$, it follows that

$$\det\,\mathcal{Q}_U = \det\,U^{\mathsf{T}} \det\,[U_R + U_I U_R^{-1} U_I]\,, \tag{20}$$

after using $\det\,U = \det\,U^{\mathsf{T}}$. Since U is unitary by assumption (since we have chosen $V=0$ in defining $\mathcal{Q}_U$), $U^\dagger U = I$ implies that

$$U_R^{\mathsf{T}} U_R + U_I^{\mathsf{T}} U_I = I\,, \qquad\qquad U_R^{\mathsf{T}} U_I = U_I^{\mathsf{T}} U_R\,, \tag{21}$$

after separating out the real and imaginary parts. Inserting these results into eq. (20) and using eq. (21), we find:

$$\det\,\mathcal{Q}_U = \det\,[U_R^{\mathsf{T}} U_R + U_R^{\mathsf{T}} U_I U_R^{-1} U_I] = \det\,[I - U_I^{\mathsf{T}} U_I + U_I^{\mathsf{T}} U_I] = \det\,I = 1\,. \tag{22}$$

That is, $\mathcal{Q}_U$ is an element of SO($2n$).

Likewise, define $\mathcal{Q}_V$ by taking $U=0$ in $\mathcal{Q}$ [eq. (17)]. That is, take V to be a unitary $n \times n$ matrix, and define the $2n \times 2n$ matrix

$$\mathcal{Q}_V = \begin{pmatrix} \mathrm{Re}\,V & -\mathrm{Im}\,V \\ -\mathrm{Im}\,V & -\mathrm{Re}\,V \end{pmatrix}\,. \tag{23}$$

Following the same computation as above, we also find that $\mathcal{Q}_V$ provides an embedding of the subgroup U(n) inside O($2n$). Moreover,

$$\det\,\mathcal{Q}_V = \det\,(-I) = (-1)^n\,. \tag{24}$$

For n even, $\mathcal{Q}_V$ provides another embedding of the subgroup U(n) inside SO($2n$).

Define the unitary matrix

$$A = \begin{pmatrix} I_n & -iI_n \\ iI_n & I_n \end{pmatrix}\,, \tag{25}$$

where I_n is the $n \times n$ identity matrix. Consider a real orthogonal $2n \times 2n$ matrix R that satisfies:

$$R^T A R = A\,. \tag{26}$$

Using an infinitesimal analysis (where $R \simeq I + Z$ where Z is an infinitesimal real antisymmetric $2n \times 2n$ matrix), it is easy to prove that R provides an $\boldsymbol{n} \oplus \boldsymbol{n}^*$ reducible representation of U(n). One can easily verify that both $\mathcal{Q}_U$ and $\mathcal{Q}_V$ satisfy the constraint given by eq. (26)

Finally, consider the matrix

$$\mathcal{U} = \begin{pmatrix} U & V^* \\ V & U^* \end{pmatrix}\,, \tag{27}$$

where U and V satisfy eqs. (15) and (16). One can easily verify that $\mathcal{U}^\dagger \mathcal{U} = I$. That is, $\mathcal{U}$ is a $2n$-dimensional unitary matrix. But, it must also be true that $\mathcal{U}\mathcal{U}^\dagger = I$, which yields:

$$(iii) \quad (UU^\dagger + V^* V^{\mathsf{T}})_i{}^j = \delta_i{}^j\,, \tag{28}$$

$$(iv) \quad UV^\dagger \text{ is an antisymmetric matrix}\,. \tag{29}$$