Notes on the spontaneous breaking of SU($N$) and SO($N$) via a second-rank tensor multiplet

Howard E. Haber
Santa Cruz Institute for Particle Physics
University of California, Santa Cruz, CA 95064, USA

Abstract

The group theory of the spontaneous breaking of SU($N$) is explored. Two specific cases are analyzed in detail: (i) SU($N$) is broken to SO($N$) via a scalar field vacuum expectation value for a second-rank symmetric tensor multiplet, and (ii) SU($2N$) or SU($2N + 1$) is broken to Sp($2N$) via a scalar field vacuum expectation value for a second-rank antisymmetric tensor multiplet. The case of the spontaneous breaking of SO($2N$) or SO($2N + 1$) to U($N$) via a scalar field vacuum expectation value for a second-rank antisymmetric tensor multiplet is also treated.

1. Introduction

In these notes, I study the group theory of the spontaneous breaking of a global SU($N$)-symmetric field theory via a scalar field vacuum expectation value for second-rank tensor multiplet, $\langle \Sigma \rangle \equiv \Sigma_0$. The cases of a symmetric tensor and anti-symmetric tensor field are separately examined. I focus on one particular symmetry breaking pattern in each case corresponding to the maximal degeneracy of non-zero eigenvalues of $\Sigma_0^\dagger \Sigma_0$. The case of spontaneous breaking of SO($2N$) or SO($2N + 1$) to U($N$) via a scalar field vacuum expectation value for a second-rank antisymmetric tensor multiplet is very similar to the corresponding breaking of SU($2N$) or SU($2N + 1$). A previous analysis of these (and other) cases can be found in ref. [1].
2. Symmetry breaking via a second-rank symmetric tensor

Let $\Sigma^{ab}$ be a symmetric second-rank tensor which transforms under $\text{SU}(N)$ as:

$$\Sigma^{ab} \rightarrow U^a_c U^b_d \Sigma^{bd},$$

where $U$ is an $N \times N$ unitary matrix with unit determinant. Equivalently, in matrix form $\Sigma \rightarrow U \Sigma U^T$, where $U^T$ is the transpose of $U$. Suppose that $\Sigma$ is a multiplet of scalar fields whose Lagrangian is invariant under global $\text{SU}(N)$ transformations. If $\Sigma$ acquires a vacuum expectation value $\langle \Sigma \rangle \equiv \Sigma_0$, then the $\text{SU}(N)$ symmetry will be broken. If there exists a subgroup $H$ of $\text{SU}(N)$, such that $U \Sigma_0 U^T = \Sigma_0$ for all $U \in H$, then the global $\text{SU}(N)$ symmetry is spontaneously broken to $H$. Writing $U = \exp \{ i \theta_a T_a \}$ where the $T_a$ are the unbroken generators (which span the unbroken subgroup $H$), it follows that for infinitesimal $\theta_a$,

$$(1 + i \theta_a T_a) \Sigma_0 (1 + i \theta_a T_a^T) = \Sigma_0,$$

which implies that

$$T_a \Sigma_0 + \Sigma_0 T_a^T = 0.\quad (3)$$

Thus, it is possible to find a basis for the traceless hermitian $\text{SU}(N)$ generators given by $\{T_a, X_b\}$ such that the $T_a$ satisfy eq. (3). In this basis, the broken generators $X_b$ are orthogonal to the $T_a$, that is $\text{Tr} (T_a X_b) = 0$.

The identity of the unbroken subgroup $H$ depends on the choice of $\Sigma_0$ (which depends on the underlying dynamics responsible for the spontaneous symmetry breaking). Here, I shall consider the case of $H=\text{SO}(N)$, for which the most general form for $\Sigma_0$ is a complex symmetric $N \times N$ matrix that satisfies $\Sigma_0^T \Sigma_0 = \Sigma_0 \Sigma_0^T = |c|^2 I_N$, where $c \in \mathbb{C}$ and $I_N$ is the $N \times N$ unit matrix. This is summarized by the following theorem.

**Theorem:** Suppose that $\Sigma_0$ is an $N \times N$ complex symmetric matrix that satisfies $\Sigma_0^T \Sigma_0 = \Sigma_0 \Sigma_0^T = |c|^2 I_N$ for some complex number $c$. Then, if the generators of $\text{SU}(N)$ in the defining ($N$-dimensional) representation are given by $\{T_a, X_b\}$, where the $T_a$ and $X_b$ are traceless hermitian $N \times N$ matrices that satisfy:

$$T_a \Sigma_0 + \Sigma_0 T_a^T = 0,\quad (4)$$

$$X_b \Sigma_0 - \Sigma_0 X_b^T = 0,\quad (5)$$
then the $T_a$ span an unbroken SO($N$) Lie subalgebra, while the $X_b$ are the broken generators that span an SU($N$)/SO($N$) homogeneous space. Furthermore, $\text{Tr} (T_a X_b) = 0$.

**Proof:** First, I show that if $\Sigma_0^\dagger \Sigma_0 = \Sigma_0 \Sigma_0^\dagger = |c|^2 I_N$ and $T_a \Sigma_0 + \Sigma_0 T_a^T = 0$, then the $T_a$ span an SO($N$) Lie algebra. Note that these two conditions imply:

$$|c|^2 T_a^T = -\Sigma_0^\dagger T_a \Sigma_0 .$$  \hfill (6)

At this point, I note that the Takagi factorization [2, 3] for any complex symmetric matrix $M$ corresponds to the statement that there exists a unitary matrix $V$ such that $VMV^T$ is diagonal with non-negative entries given by the positive square roots of the eigenvalues of $MM^\dagger$ (or $M^\dagger M$). In our case, this result implies that there exists a unitary matrix $V$ such that

$$V \Sigma_0 V^T = c I_N .$$  \hfill (7)

The inverse of this result is $(V^T)^{-1} \Sigma_0^\dagger V^{-1} = c^* I_N$ (since $\Sigma_0^\dagger = |c|^2 \Sigma_0^{-1}$). I now define: $\tilde{T}_a \equiv VT_a V^{-1}$, where $V$ is the unitary matrix appearing in eq. (7). Then, inserting this result into eq. (6), it follows that:

$$\tilde{T}_a^T = -\frac{1}{|c|^2} (V^T)^{-1} \Sigma_0^\dagger V^{-1} \tilde{T}_a V \Sigma_0 V^T$$
$$= -\tilde{T}_a .$$  \hfill (8)

Likewise, one can use the same matrix $V$ to define $\tilde{X} \equiv VXV^{-1}$. By an analogous computation, $|c|^2 X^T = \Sigma_0^\dagger X \Sigma_0$, which implies that $\tilde{X}_b^T = \tilde{X}_b$. Moreover, since the generators of SU($N$) are traceless and hermitian, it follows that the $\tilde{X}_b$ are also traceless and real.

Thus, I have exhibited a similarity transformation that transforms the basis of the Lie algebra spanned by the $T_a$ to one that is spanned by the $\tilde{T}_a$. Since the $i\tilde{T}_a$ are real antisymmetric matrices, one immediately recognizes this Lie algebra as that of SO($N$). If an arbitrary element of the unbroken Lie algebra is exponentiated, it follows that $\exp i\theta_b T_a$ is related by a similarity transformation to $\exp i\theta_b \tilde{T}_a$. The latter consists of arbitrary $N \times N$ real orthogonal matrices, which implies that the $\exp i\theta_b T_a$ constitutes an $N$-dimensional representation of SO($N$).

---

1Strictly speaking, the Takagi factorization yields a diagonal matrix with non-negative diagonal elements. If $c = |c|e^{2i \xi}$, one can obtain $V \Sigma_0 V^T = |c| I_N$ by taking $V = e^{i \xi} V$. However, this step is not necessary for the present argument.
dimensional representation that is equivalent to the $N$-dimensional defining representation of $SO(N)$.

Finally, I note that from $|c|^2 T_a^T = -\Sigma_0^T T_a \Sigma_0$ and $|c|^2 X_b^T = \Sigma_0^T X_b \Sigma_0$ it follows that $|c|^2 T_a^T X_b^T = -\Sigma_0^T T_a X_b \Sigma_0$ (since $\Sigma_0^T \Sigma_0 = |c|^2 I_N$). Taking the trace yields $\text{Tr} T_a X_b = -\text{Tr} T_a X_b$, or equivalently $\text{Tr} T_a X_b = 0$. To show that the $\{T_a, X_b\}$ span the full $SU(N)$ Lie algebra, it is convenient to count the number of independent generators after applying the similarity transformation that converts the $\{T_a, X_b\}$ into $\{\tilde{T}_a, \tilde{X}_b\}$. I showed above that the $i\tilde{T}_a$ are real antisymmetric matrices whereas the $\tilde{X}_b$ are traceless real symmetric matrices. This implies that there are $\frac{1}{2}N(N-1)$ independent $\tilde{T}_a$ and $\frac{1}{2}N(N+1) - 1$ independent $\tilde{X}_b$ (the 1 is subtracted to account for the extra condition that the $\tilde{X}_b$ are traceless). The total number of $SU(N)$ generators is therefore $N^2 - 1$ as expected.

The above results are easily verified explicitly for $N = 3$. Consider for example a case in which

$$\Sigma_0 = c \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (9)$$

which clearly satisfies $\Sigma_0^T \Sigma_0 = \Sigma_0 \Sigma_0^T = |c|^2 I$. Using the Gell-Mann matrices $\frac{1}{2} \lambda_a$ as the generators of $SU(3)$, it is easy to check that $T_a \Sigma_0 + \Sigma_0 T_a^T = 0$ implies that $T_a = c_1 (\lambda_1 - \lambda_6) + c_2 (\lambda_2 - \lambda_7) + c_3 (\lambda_3 + \sqrt{3} \lambda_8)$, where $c_i \in \mathbb{R}$. That is, $\{\frac{1}{2} (\lambda_1 - \lambda_6), \frac{1}{2} (\lambda_2 - \lambda_7), \frac{1}{\sqrt{3}} (\lambda_3 + \sqrt{3} \lambda_8)\}$ spans an $SO(3)$ Lie subalgebra of the $SU(3)$ Lie algebra. This matrix representation corresponds to the adjoint representation of an $SU(2)$ subalgebra, which when exponentiated yields a representation equivalent to the defining representation of $SO(3)$.

3. Symmetry breaking of $SU(2N)$ via a second-rank antisymmetric tensor

The case of symmetry breaking via an antisymmetric tensor exhibits many similar features. First, I shall consider the case of a global $SU(2N)$ symmetry group. Let $\Sigma^{ab}$ be an antisymmetric second-rank tensor which transforms under $SU(2N)$ as $\Sigma \rightarrow U \Sigma U^T$, where $U$ is a $2N \times 2N$ unitary matrix with unit determinant. If $\Sigma$ acquires a vacuum expectation value
\[ \langle \Sigma \rangle \equiv \Sigma_0, \] then the SU(2N) symmetry will be spontaneously broken. Writing \( U = \exp \left( i \theta_a T_a \right) \) where the \( T_a \) are the unbroken generators, I again find

\[ T_a \Sigma_0 + \Sigma_0 T_a^T = 0. \tag{10} \]

Thus, it is possible to find a basis for the traceless hermitian SU(2N) generators given by \( \{ T_a, X_b \} \) such that the \( T_a \) satisfy eq. (10) and \( \text{Tr} \left( T_a X_b \right) = 0 \).

The unbroken subgroup \( H \) depends on the choice of \( \Sigma_0 \). Here, I shall consider the case of \( H = \text{Sp}(2N) \), for which the most general form for \( \Sigma_0 \) is a complex antisymmetric \( 2N \times 2N \) matrix that satisfies \( \Sigma_0^\dagger \Sigma_0 = \Sigma_0 \Sigma_0^\dagger = |c|^2 I_{2N} \), where \( c \in \mathbb{C} \) and \( I_{2N} \) is the \( 2N \times 2N \) unit matrix. This is summarized by the following theorem.

**Theorem:** Suppose that \( \Sigma_0 \) is a \( 2N \times 2N \) complex antisymmetric matrix that satisfies \( \Sigma_0^\dagger \Sigma_0 = \Sigma_0 \Sigma_0^\dagger = |c|^2 I_{2N} \) for some complex number \( c \). Then, if the generators of SU(2N) in the defining (2N-dimensional) representation are given by \( \{ T_a, X_b \} \), where the \( T_a \) and \( X_b \) are traceless hermitian \( 2N \times 2N \) matrices that satisfy:

\[ T_a \Sigma_0 + \Sigma_0 T_a^T = 0, \tag{11} \]
\[ X_b \Sigma_0 - \Sigma_0 X_b^T = 0, \tag{12} \]

then the \( T_a \) span an unbroken Sp(2N) Lie subalgebra, while the \( X_b \) are the broken generators that span an SU(2N)/Sp(2N) homogeneous space. Furthermore, \( \text{Tr} \left( T_a X_b \right) = 0 \).

**Proof:** First, I show that if \( \Sigma_0^\dagger \Sigma_0 = \Sigma_0 \Sigma_0^\dagger = |c|^2 I_{2N} \) and \( T_a \Sigma_0 + \Sigma_0 T_a^T = 0 \), then the \( T_a \) span an Sp(2N) Lie algebra. Note that these two conditions imply:

\[ |c|^2 T_a^T = -\Sigma_0^\dagger T_a \Sigma_0. \tag{13} \]

For any even-dimensional complex antisymmetric matrix \( M \), there exists a unitary matrix \( W \) such that \( WMW^T = \text{diag}(J_1, J_2, \ldots, J_n) \) is block diagonal, where each block is a \( 2 \times 2 \) matrix of the form \( J_n \equiv \begin{pmatrix} 0 & z_n \\ -z_n & 0 \end{pmatrix} \), where \( z_n \in \mathbb{C} \) and the \( |z_n|^2 \) are the eigenvalues of \( MM^\dagger \) (or \( M^\dagger M \)).\(^2\) Applying this result to \( \Sigma_0 \), I note that the eigenvalues of \( \Sigma_0 \Sigma_0^\dagger \) are all degenerate and equal

\(^2\)This result for complex antisymmetric matrices is the analog of the Takagi factorization for symmetric matrices [4]. Moreover, it is always possible to find a suitable choice for the unitary matrix \( W \) such that the \( z_i \) are real and non-negative. However, this step is not necessary for the present argument.
to $|c|^2$. Consider the matrix

$$J \equiv \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix},$$

(14)

where $I_N$ is the $N \times N$ identity matrix. Since $JJ^\dagger = I_{2N}$, it follows that one can find unitary matrices $W_1$ and $W_2$ such that

$$W_1 \Sigma_0 W_1^T = cW_2 J W_2^T = \text{diag}(cJ, cJ, \ldots, cJ),$$

where

$$J \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  

(15)

That is, the factorization of $\Sigma_0$ and $cJ$ both yield the same block diagonal matrix consisting of $N$ identical $2 \times 2$ blocks consisting of $cJ$. Thus, there exists a unitary matrix $V = W_2^{-1} W_1$ such that

$$V \Sigma_0 V^T = cJ.$$  

(16)

The inverse of this result is $(V^T)^{-1} \Sigma_0^T V^{-1} = -c^* J$ (since $\Sigma_0^T = |c|^2 \Sigma_0^{-1}$ and $J^{-1} = -J$). I now define $\tilde{T}_a \equiv V T_a V^{-1}$, where $V$ is the unitary matrix appearing in eq. (16). Then, inserting this result into eq. (13), it follows that:

$$\tilde{T}_a = \frac{1}{|c|^2} (V^T)^{-1} \Sigma_0^T V^{-1} V \Sigma_0 V^T$$

$$= J \tilde{T}_a J.$$  

(17)

Likewise, one can use the same matrix $V$ to define $\tilde{X}_b \equiv V X_b V^{-1}$. By an analogous computation, $|c|^2 X^T = \Sigma_0^T X \Sigma_0$, which implies that $\tilde{X}_b^T = -J \tilde{X}_b J$.

Thus, I have exhibited a similarity transformation that transforms the basis of the Lie algebra spanned by the $T_a$ to one that is spanned by the $\tilde{T}_a$. Since the $\tilde{T}_a$ are traceless hermitian matrices\(^3\) that satisfy $\tilde{T}_a^T = J \tilde{T}_a J$ [where $J$ is defined by eq. (14)], one immediately recognizes this Lie algebra as that of $\text{Sp}(2N)$.\(^4\)

\(^3\)Since $J^2 = -I_{2N}$, it follows from $\tilde{T}_a^T = J \tilde{T}_a J$ that $\text{Tr} T_a = 0$. This implies that the group theory for the breaking of $U(2N)$ to $\text{Sp}(2N)$ would work in almost precisely the same way with one difference—the unbroken generators $X_b$ would not be traceless.

\(^4\)This is actually the unitary symplectic Lie algebra, which is the compact real form of the complex symplectic Lie algebra. Some books use the notation $\text{Sp}(N)$ where I have used $\text{Sp}(2N)$. For more details, see ref. [5].
is exponentiated, it follows that $\exp i\theta_a T_a$ is related by a similarity transformation to $\exp i\theta_a \tilde{T}_a$. The latter consists of arbitrary $2N \times 2N$ unitary symplectic matrices, which implies that the $\exp i\theta_a T_a$ constitutes an $2N$-dimensional representation that is equivalent to the $2N$-dimensional defining representation of $\text{Sp}(2N)$.

Finally, I note that from $|c|^2 T_a^T = -\Sigma_0^\dagger T_a \Sigma_0$ and $|c|^2 X_b^T = \Sigma_0^\dagger X_b \Sigma_0$ it follows that $|c|^2 T_a^T X_b^T = \Sigma_0^\dagger T_a X_b \Sigma_0$ (since $\Sigma_0^\dagger \Sigma_0 = |c|^2 I_{2N}$). Taking the trace yields $\text{Tr} T_a X_b = -\text{Tr} T_a X_b$, or equivalently $\text{Tr} T_a X_b = 0$. To show that the $\{T_a, X_b\}$ span the full $\text{SU}(2N)$ Lie algebra, it is convenient to count the number of independent generators after applying the similarity transformation that converts the $\{T_a, X_b\}$ into $\{\tilde{T}_a, \tilde{X}_b\}$. Since these are all $\text{SU}(2N)$ generators, they are traceless hermitian matrices. Moreover, I showed above that the $\tilde{T}_a$ satisfy $\tilde{T}_a = J\tilde{T}_a J$, whereas the $\tilde{X}_b$ are traceless and satisfy $\tilde{X}_b^T = -J \tilde{X}_b J$. More explicitly,

$$
\tilde{T}_a = \begin{pmatrix} A & B \\ B^\dagger & -A^T \end{pmatrix}, \quad \tilde{X}_b = \begin{pmatrix} C & D \\ D^\dagger & C^T \end{pmatrix}, \quad (18)
$$

where $A$, $B$, $C$ and $D$ are $N \times N$ complex matrices such that $A$ and $C$ are hermitian, $B$ is symmetric, $D$ is antisymmetric and $\text{Tr} C = 0$. Thus, the number of independent real parameters that describe $\tilde{T}_a$ corresponds to the number of parameters needed to define the hermitian matrix $A$ and the complex symmetric matrix $B$, which is equal to $N^2 + N(N+1) = N(2N+1)$. Similarly, number of independent real parameters that describe $\tilde{X}_b$ corresponds to the number of parameters needed to define the traceless hermitian matrix $C$ and the complex antisymmetric matrix $D$, which is equal to $N^2 - 1 + N(N+1) = N(2N-1) - 1$. The total number of $\text{SU}(2N)$ generators is therefore $(2N)^2 - 1$ as expected.

### 4. Symmetry breaking of $\text{SU}(2N + 1)$ via a second-rank antisymmetric tensor

For the case of spontaneous breaking of an $\text{SU}(2N + 1)$ global symmetry by a second-rank antisymmetric tensor field, the analysis of the previous section requires some modification. In this case, $\Sigma$ is a $(2N + 1) \times (2N + 1)$ matrix, which acquires a vacuum expectation value $\Sigma_0$. Once again, the
unbroken generators $T_a$ satisfy:

$$T_a \Sigma_0 + \Sigma_0 T_a^T = 0.$$  \hfill (19)

In this case, one can find a basis for the traceless hermitian SU($2N+1$) generators given by $\{T_a, X_b, Y_c\}$ such that the $T_a$ satisfy eq. (19) and $\text{Tr} (T_a X_b) = \text{Tr} (T_a Y_c) = \text{Tr} (X_b Y_c) = 0$.

I shall identify the unbroken subgroup $H$ under the assumption that $\Sigma_0$ satisfies:

$$\Sigma_0^\dagger \Sigma_0 = \Sigma_0 \Sigma_0^\dagger = |c|^2 \begin{pmatrix} I_{2N} & 0 \\ 0 & 0 \end{pmatrix},$$  \hfill (20)

where $c \in \mathbb{C}$. In this case, $H = \text{Sp}(2N)$. This is summarized by the following theorem.

**Theorem:** Suppose that $\Sigma_0$ is a $(2N+1) \times (2N+1)$ complex antisymmetric matrix that satisfies eq. (20) for some complex number $c$. Then, if the generators of SU($2N+1$) in the defining $(2N+1)$-dimensional representation are given by $\{T_a, X_b, Y_c\}$, where the $T_a$, $X_b$ and $Y_c$ are traceless hermitian $(2N+1) \times (2N+1)$ matrices that satisfy:

$$T_a \Sigma_0 + \Sigma_0 T_a^T = 0,$$  \hfill (21)

$$X_b \Sigma_0 - \Sigma_0 X_b^T = 0,$$  \hfill (22)

$$\Sigma_0^\dagger Y_c \Sigma_0 = 0,$$  \hfill (23)

then the $T_a$ span an unbroken Sp($2N$) Lie algebra, while the $\{X_b, Y_c\}$ are the broken generators that span an SU($2N+1$)/Sp($2N$) homogeneous space. Furthermore, $\text{Tr} (T_a X_b) = \text{Tr} (T_a Y_c) = \text{Tr} (X_b Y_c) = 0$.

**Proof:** First, I show that if $\Sigma_0$ satisfies eq. (20) and $T_a \Sigma_0 + \Sigma_0 T_a^T = 0$, then the $T_a$ span an Sp($2N$) Lie algebra. Here, I note that for any odd-dimensional complex antisymmetric matrix $M$, there exists a unitary matrix $W$ such that $W M W^T = \text{diag}(\mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_N, 0)$ where $\mathcal{J}_n \equiv \begin{pmatrix} 0 & z_n \\ -z_n & 0 \end{pmatrix}$, with $z_n \in \mathbb{C}$ and the $|z_n|^2$ are the eigenvalues of $MM^\dagger$ (or $M^\dagger M$) [4]. Introduce the $(2N+1) \times (2N+1)$ matrix:

$$K \equiv \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix},$$  \hfill (24)

where $J$ is the $2N \times 2N$ matrix given by eq. (14). The zeros shown above fill out the last row and the last column of the matrix $K$. Then applying the
above factorization to the antisymmetric matrices $\Sigma_0$ and $K$, it follows that there exists a unitary matrix $V$ such that

$$V \Sigma_0 V^T = cK.$$  \hfill (25)

Next, I multiply eq. (19) on the left by $V$ and the right by $V^T$. Defining $\tilde{T}_a \equiv VT_a V^{-1}$ as before and using eq. (25), one easily derives:

$$K \tilde{T}_a^T = -\tilde{T}_a K.$$ \hfill (26)

Using the fact that $\tilde{T}_a$ is traceless and hermitian, eq. (26) has a unique solution:

$$\tilde{T}_a = \begin{pmatrix} t_a & 0 \\ 0 & 0 \end{pmatrix},$$ \hfill (27)

where $t_a$ is an $2N \times 2N$ hermitian matrix that satisfies $t_a^T = Jt_a J$. Thus, the $\tilde{T}_a$ span an $\text{Sp}(2N)$ Lie subalgebra of the $\text{SU}(2N+1)$ Lie algebra.

Consider next the unbroken generators $X_b$ and $Y_c$, which satisfy eqs. (22) and (23), and define $\tilde{X}_b \equiv VX_b V^{-1}$ and $\tilde{Y}_c \equiv VY_c V^{-1}$ Then eq. (25) implies that

$$K \tilde{X}_b^T = \tilde{X}_b K,$$ \hfill (28)

Using the fact that $\tilde{X}_b$ and $\tilde{Y}_c$ are traceless and hermitian, eq. (28) has a unique solution:

$$\tilde{X}_b = \begin{pmatrix} x_b & 0 \\ 0 & -\text{Tr} x_b \end{pmatrix}, \quad \tilde{Y}_c = \begin{pmatrix} 0 & y_c \\ y_c^\dagger & 0 \end{pmatrix},$$ \hfill (29)

where $x_b$ is an $2N \times 2N$ hermitian matrix that satisfies $x_b^T = -Jx_b J$, and $y_c$ is a complex $2N$-dimensional column vector. From the explicit forms above, it is easy to check that $\text{Tr} (\tilde{T}_a \tilde{X}_b) = \text{Tr} (\tilde{T}_a \tilde{Y}_c) = \text{Tr} (\tilde{X}_b \tilde{Y}_c) = 0$, which implies that $\text{Tr} (T_a X_b) = \text{Tr} (T_a Y_c) = \text{Tr} (X_b Y_c) = 0$.

To show that the $\{T_a, X_b, Y_c\}$ span the full $\text{SU}(2N+1)$ Lie algebra, it is convenient to count the number of independent generators after applying the similarity transformation that converts the $\{T_a, X_b, Y_c\}$ into $\{\tilde{T}_a, \tilde{X}_b, \tilde{Y}_c\}$. Since these are all $\text{SU}(2N+1)$ generators, they are traceless hermitian matrices. Moreover, I showed above that the $\tilde{T}_a$, $\tilde{X}_b$ and $\tilde{Y}_c$ are given by eqs. (27) and (29), where $t_a^T = Jt_a J$ and $x_b^T = -Jx_b J$ are $2N \times 2N$ hermitian matrices and $y_c$ is a complex $2N$-dimensional vector. However, in contrast to $\tilde{X}_b$ in the last section, $x_b$ is not traceless. Following the analysis at the end of the
previous section (but with $\text{Tr } x_b \neq 0$), it follows that the number of independent real parameters that describe $\tilde{T}_a$ and $\tilde{X}_b$ is given by $N(2N+1)$ and $N(2N-1)$, respectively. Adding this to the $4N$ parameters that describe $\tilde{Y}_c$ yields a total number of SU($2N+1$) generators equal to $(2N+1)^2 - 1$ as expected.

Finally, it is interesting to note that the generators of the type $Y_c$ do not appear in the breaking of SU($2N$) to Sp($2N$) described in the previous section. This is easy to see by noting that in the previous section, $\Sigma_0^T \Sigma_0 = |c|^2 I_{2N}$ implies that $\Sigma_0^{-1}$ exists. Thus, in this case $\Sigma_0^T Y_c \Sigma_0 = 0$ would imply that $Y_c = 0$. In the case of SU($2N+1$) breaking, since $\Sigma_0$ is an odd-dimensional antisymmetric matrix, it follows that $\text{det } \Sigma_0 = 0$. Thus, $\Sigma_0^{-1}$ does not exist and a non-trivial solution for $Y_c$ can arise, as we have explicitly shown above.

5. Symmetry breaking of SO($2N$) and SO($2N+1$) via a second-rank antisymmetric tensor

The case of spontaneous breaking of SO($2N$) or SO($2N+1$) to U($N$) via a scalar field vacuum expectation value for a second-rank antisymmetric tensor multiplet is very similar to the corresponding breaking of SU($2N$) or SU($2N+1$) considered in the previous two sections. Thus, we provide a few details here. In the case of SO($2N$) the relevant theorem is as follows.

**Theorem:** Suppose that $\Sigma_0$ is a $2N \times 2N$ real antisymmetric matrix that satisfies $\Sigma_0^T \Sigma_0 = \Sigma_0 \Sigma_0^T = c^2 I_{2N}$ for some real number $c$. Then, if the generators of SO($2N$) in the defining $(2N$-dimensional) representation are given by $\{T_a, X_b\}$, where the $iT_a$ and $iX_b$ are real antisymmetric $2N \times 2N$ matrices that satisfy:

$$T_a \Sigma_0 + \Sigma_0 T_a^T = 0, \quad (30)$$
$$X_b \Sigma_0 - \Sigma_0 X_b^T = 0, \quad (31)$$

then the $T_a$ span an unbroken U($N$) Lie subalgebra, while the $X_b$ are the broken generators that span an SO($2N$)/U($N$) homogeneous space. Furthermore, Tr $(T_a X_b) = 0$.

**Proof:** First, I show that if $\Sigma_0^T \Sigma_0 = \Sigma_0 \Sigma_0^T = c^2 I_{2N}$ and $T_a \Sigma_0 + \Sigma_0 T_a^T = 0$, then the $T_a$ span an U($N$) Lie subalgebra. Note that these two conditions
imply:
\[ c^2 T_a^T = -\Sigma_0^T T_a \Sigma_0. \]  
(32)

For any even-dimensional real antisymmetric matrix \( M \), there exists a real orthogonal matrix \( W \) such that \( W M W^T = \text{diag}(J_1, J_2, \ldots, J_n) \) is block diagonal, where each block is a \( 2 \times 2 \) matrix of the form \( J_n \equiv \begin{pmatrix} 0 & z_n \\ -z_n & 0 \end{pmatrix} \), where \( z_n \in \mathbb{R} \) and the \( z_n^2 \) are the eigenvalues of \( M M^T \) (or \( M^T M \)). Applying this result to \( \Sigma_0 \), note that the eigenvalues of \( \Sigma_0 \Sigma_0^T \) are all degenerate and equal to \( c^2 \). Moreover, since the matrix \( J \equiv \text{diag}(cJ, cJ, \ldots, cJ) \) satisfies \( JJ^T = I_2N \), it follows that one can find real orthogonal matrices \( W_1 \) and \( W_2 \) such that
\[ W_1 \Sigma_0 W_1^T = c W_2 J W_2^T = \text{diag}(cJ, \ldots, cJ), \]
where \( J \) is defined in eq. (15). That is, the factorization of \( \Sigma_0 \) and \( cJ \) both yield the same block diagonal matrix consisting of \( N \) identical \( 2 \times 2 \) blocks consisting of \( cJ \). Thus, there exists a real orthogonal matrix \( V = W_2^{-1} W_1 \) such that \( V \Sigma_0 V^T = cJ \). The inverse of this result is \( V \Sigma_0 V^T = -cJ \) (since \( J^T = -J \)). I now define \( \tilde{T}_a \equiv VT_a V^T \).

Then eq. (32) implies that
\[ \tilde{T}_a^T = \frac{-1}{c^2} V \Sigma_0^T V^T \tilde{T}_a V \Sigma_0 V^T = J \tilde{T}_a J. \]  
(33)

Likewise, one can use the same matrix \( V \) to define \( \tilde{X}_b \equiv VX_b V^T \). By an analogous computation, \( c^2 X_b^T = \Sigma_0^T X \Sigma_0 \), which implies that \( \tilde{X}_b^T = -J \tilde{X}_b J \).

Recall that that \( T_a \) and \( X_b \) are both antisymmetric \( 2N \times 2N \) matrices. Then, \( \tilde{T}_a \equiv VT_a V^T \) and \( \tilde{X}_a \equiv VX_a V^T \) are also antisymmetric. Hence, it follows that
\[ \tilde{T}_a = -J \tilde{T}_a J, \quad \tilde{X}_a = J \tilde{X}_a J. \]  
(34)

Using the explicit form for \( J \), eq. (34) implies that \( T_a \) and \( X_b \) take the following block form:
\[ i \tilde{T}_a = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \quad i \tilde{X}_b = \begin{pmatrix} C & D \\ D & -C \end{pmatrix}, \]  
(35)
where \( A, B, C \) and \( D \) are \( N \times N \) real matrices such that \( A, C \) and \( D \) are antisymmetric and \( B \) is symmetric. Thus, I have exhibited a similarity transformation (note that \( V^T = V^{-1} \)) that transforms the basis of the Lie algebra spanned by the \( T_a \) to one that is spanned by the \( \tilde{T}_a \). Moreover, consider the isomorphism that maps \( i \tilde{T}_a \) given in eq. (35) to the \( N \times N \)

\(^5\)This result for real antisymmetric matrices is the analog of the corresponding factorization of complex antisymmetric matrices quoted in Section 3.
matrix $A + iB$. Since $(A + iB)\dagger = (A - iB)^T = -(A + iB)$, we see that the $A + iB$ are anti-hermitian generators (which are not generally traceless) that span a $U(N)$ subalgebra of the $SO(2N)$. We can check the number of unbroken generators by counting the number of degrees of freedom of one real antisymmetric and one real symmetric matrix: $\frac{1}{2}N(N - 1) + \frac{1}{2}N(N + 1) = N^2$, as expected.

Finally, I note that from $c^2 T_a^T = -\Sigma_0^T T_a \Sigma_0$ and $c^2 X_b^T = \Sigma_0^T X_b \Sigma_0$ it follows that $c^2 T_a^T X_b = \Sigma_0^T T_a X_b \Sigma_0$ (since $\Sigma_0^T \Sigma_0 = c^2 I_{2N}$). Taking the trace yields $\text{Tr} T_a X_b = -\text{Tr} T_a X_b$, or equivalently $\text{Tr} T_a X_b = 0$. To show that the $\{T_a, X_b\}$ span the full $SO(2N)$ Lie algebra, we note that there are $N^2$ unbroken generators and $N(N - 1)$ broken generators (corresponding to the number of parameters describing two real antisymmetric matrices [see eq. (35)]). Thus, the total number of generators is $N(2N - 1)$ which matches the total number of $SO(2N)$ generators.

Finally, we turn to the case of $SO(2N + 1)$ breaking. In this case, we will make use of the fact that for any odd-dimensional real antisymmetric matrix $M$, there exists a real orthogonal matrix $W$ such that $WMW^T = \text{diag}(J_1, J_2, \ldots, J_N, 0)$ where $J_n \equiv \left(\begin{array}{cc} 0 & z_n \\ -z_n & 0 \end{array}\right)$, with $z_n \in \mathbb{R}$ and the $z_n^2$ are the eigenvalues of $MM^T$ (or $M^T M$). The relevant theorem for case of $SO(2N + 1)$ is as follows.

**Theorem:** Suppose that $\Sigma_0$ is a $(2N + 1) \times (2N + 1)$ real antisymmetric matrix that satisfies

\[ \Sigma_0^T \Sigma_0 = \Sigma_0 \Sigma_0^T = c^2 \left(\begin{array}{cc} I_{2N} & 0 \\ 0 & 0 \end{array}\right), \]  

where $c \in \mathbb{R}$. Then, if the generators of $SO(2N + 1)$ in the defining $[(2N + 1)$-dimensional] representation are given by $\{T_a, X_b, Y_c\}$, where the $iT_a$, $iX_b$ and $iY_c$ are real antisymmetric $(2N + 1) \times (2N + 1)$ matrices that satisfy:

\begin{align*}
T_a \Sigma_0 + \Sigma_0 T_a^T &= 0, \\
X_b \Sigma_0 - \Sigma_0 X_b^T &= 0, \\
\Sigma_0^T Y_c \Sigma_0 &= 0,
\end{align*}

then the $T_a$ span an unbroken $U(N)$ Lie subalgebra, while the $\{X_b, Y_c\}$ are the broken generators that span an $SO(2N + 1)/U(N)$ homogeneous space. Furthermore, $\text{Tr} (T_a X_b) = \text{Tr} (T_a Y_c) = \text{Tr} (X_b Y_c) = 0$.

**Proof:** Here, I shall only sketch the modifications to the proof given in Section 4. Again, we easily derive $KT_a^T = -\tilde{T}_a K$ [where $K$ is defined in
In the present case, we use the fact that $\tilde{T}_a$ is antisymmetric to conclude that $K \tilde{T}_a = \tilde{T}_a K$. That is, we may write the unbroken generators, $\tilde{T}_a$, in the form of eq. (27) where $t_a = -J t_a J$ and the $i t_a$ are $2N \times 2N$ real antisymmetric matrices. Using the results previously obtained, it follows that the $\tilde{T}_a$ span a $\text{U}(N)$ subalgebra. Likewise, the broken generators satisfy: $K \tilde{X}_a = -\tilde{X}_a K$ and $K^T \tilde{Y}_c K = 0$. The antisymmetry of $\tilde{X}_a$ and $\tilde{Y}_a$ implies

$$i \tilde{X}_b = \begin{pmatrix} x_b & 0 \\ 0 & 0 \end{pmatrix}, \quad i \tilde{Y}_c = \begin{pmatrix} 0 & y_c \\ -y_c^T & 0 \end{pmatrix},$$

where $x_b$ is an $2N \times 2N$ real antisymmetric matrix that satisfies $x_b = J x_b J$, and $y_c$ is a real $2N$-dimensional column vector. From the explicit forms above, it is easy to check that $\text{Tr} (\tilde{T}_a \tilde{X}_b) = \text{Tr} (\tilde{T}_a \tilde{Y}_c) = \text{Tr} (\tilde{X}_b \tilde{Y}_c) = 0$, which implies that $\text{Tr} (T_a X_b) = \text{Tr} (T_a Y_c) = \text{Tr} (X_b Y_c) = 0$.

Finally, we count the number of $\text{SO}(2N+1)$ generators $\{ \tilde{T}_a, \tilde{X}_b, \tilde{Y}_c \}$. There are $N^2$ unbroken generators and $N(N+1)$ degrees of freedom associated with $\tilde{X}_b$ as in the case of $\text{SO}(2N)$ breaking. Finally, adding in the $2N$ parameters that describes $\tilde{Y}_c$ yields a total number of $\text{SO}(2N+1)$ generators equal to $N(2N+1)$ as expected.

**References**


