MEASURING THE MASS OF THE EARTH
The Cavendish Experiment and the Acceleration Due to Gravity

1 Introduction

In 1687 Newton published his law of Gravity asserting (i) that the force between any two point masses was given by

$$F_g = \frac{G m_1 m_2}{r^2} \tag{1}$$

and (ii) that the same force that attracts an apple to the ground governed the motion of the planets and stars. That is, this is a Universal law of gravity.

However, while this law may be applicable for point masses or between astronomical objects separated by distances which are great compared to their diameter (that is, we can approximate the masses to point objects), it is not obvious how we can apply this formula to work out the attraction of the earth to the apple except by using calculus to sum the attraction of each elemental volume of the earth to the apple. Although Newton was not the first person to propose an inverse square law, he was the first person to derive a proof that, for such a law, and for a mass outside or on the surface of a spherical mass M, the mass acts as though all its mass were concentrated in its center. This is not intuitive and it took Newton ten years to derive the theorem. His proof was geometric and extremely elegant.

Once it is established that you can treat the earth as a point mass at the center of the earth, the apple falling from Newtons tree experiences a force equal to

$$F_g = \frac{G M m_a}{r_e^2} \tag{2}$$

where M is the mass of the earth, r_e is the radius of the earth, and m_a is the mass of the apple. Using Newton's other law, the law of motion, $F = ma$, he could write the acceleration of gravity on the earth's surface as

$$g = \frac{GM}{r_e^2} \tag{3}$$

Where g is the acceleration of the mass as it falls to earth in the absence of air resistance.

Some decades before Newtons work, Galileo had proposed that in the absence of any other force, a body would travel in a straight line at a constant velocity. If a mass is constrained to move in a circle, say by swinging a ball on the end of a string, a force is required to change its direction, which is called centrifugal force which acts along the string. Applying Newtons law of motion to this problem, it is also valid to assume that the rotating body experiences an acceleration towards the center of rotation which can be shown to be given by

$$\text{Centripetal acceleration} \equiv a_c = \frac{4\pi^2 r}{P^2}. \tag{4}$$

Newton assumed that the moon is kept in a circular orbit by the gravitational attraction between the earth and moon, so that the acceleration due to gravity of the earth at the distance of the moon would equal the centripetal acceleration required to keep the moon in a circular orbit. Since the gravitational acceleration falls as the inverse square law with the distance from the center of the earth, we can write

$$g(r_e/r)^2 = \frac{4\pi^2 r}{P^2} \tag{5}$$

[Note that we only have to use the acceleration due to gravity at the earths surface; also note that this equation gives Keplers third law (How?)]

This equation can therefore be used either to prove that the gravitational force does indeed obey an inverse square law (at least out to the distance of the moon) if the distance of the moon is known, or, alternatively, if Newtons theory is considered true, to determine the distance of the moon itself. If we are interested in proving Newtons law of gravity it is necessary to have an independent way of measuring the distance to the moon. This can be carried out using parallax [How?].

2 Measuring the Mass of the Earth

The mass of the earth can be derived directly from equation (2) provided we can measure the Gravitational constant, G, and the acceleration of a body on the earths surface due to gravity. The determination of G is of fundamental importance in physics and astronomy and its measurement is, the principle is straightforward, all that is necessary is to measure the force of attraction between a sphere of and a test particle of known masses. However, for laboratory size objects the gravitational force is very small (see below) and it is difficult to measure accurately. The first person the measure G was Henry Cavendish in 1798, over 100 years after Newtons theory had been published. Cavendish attached two small masses to the ends of a beam and supported the beam from its center of gravity from a long thin Copper wire to form a very sensitive torsion balance. Two large sphere of lead were supported on a massive beam and could be moved from one side of the test balls to the other. The

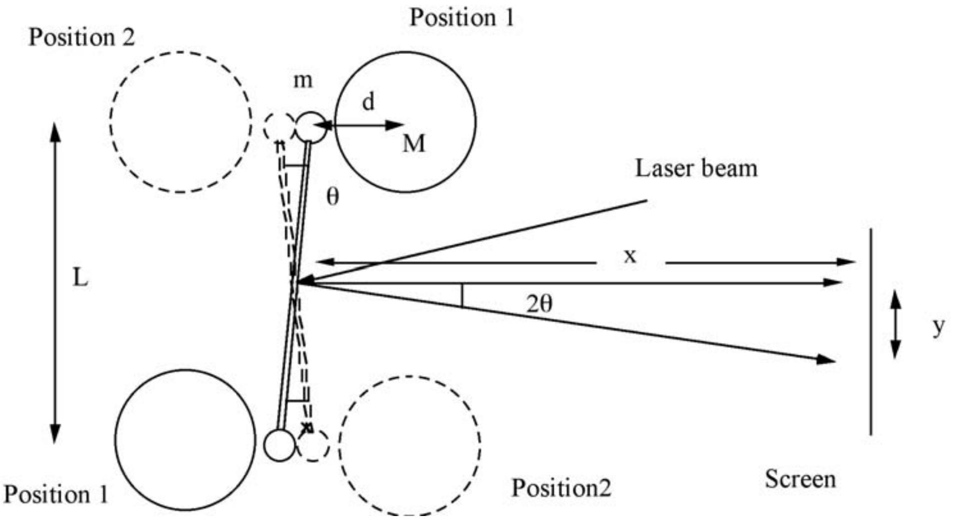

Figure 1: Schematic of Cavendish setup

attractive force of gravity applied a torque to the copper wire and the beam rotated by a small angle, which could be measured using a telescope. The apparatus was very large (Cavendish was very rich and had a building constructed just to house the experiment! The current experimental apparatus uses a very thin wire made of tungsten in place of the Copper wire. The properties of the wire enable the size of the experiment to be much reduced). Cavendish was able to obtain a value for G within 5% of the accepted value determined today.

3 The Cavendish Experiment Calculations

This experiment depends on a small bar-bell shaped mass suspended from an extremely thin wire which gets twisted by the force of gravity between the mass and a pair of external lead masses. When the external masses are moved from one side of the bar-bell to the other, the equilibrium angle of the suspended bar-bell changes. This change in angle is measured by reflecting laser light from a small mirror mounted on the bar-bell and watching the small motion of the reflected beam.

The torque exerted by gravitational attraction is given by

$$T = \frac{GMmL}{d^2} \qquad \text{[dyne cm]} \tag{6}$$

And the wire generates an opposing torque given by an unknown torque constant, κ, and the angle of rotation, θ, hence:

$$G = \frac{\kappa y d^2}{4xMmL} \qquad \text{[what are the units?]} \tag{7}$$

Where y is the displacement of the laser beam on a screen a distance x from the experimental apparatus.

We require to measure the torque constant, κ, of the wire. This is carried out by observing the dynamical motion of the beam. Even if the beam was at rest at the start of the experiment, when the masses are moved from one side of the beam to the other the beam does not move directly to its new position but overshoots and oscillates with an amplitude which slowly decays with time. The period of the oscillation is longer the weaker the torque constant and is given by

$$P = 2\pi(I/\kappa)^{1/2} \qquad \text{[sec]} \tag{8}$$

Where I is the Moment of Inertia of the beam ($I = mL^2/2$). It is therefore necessary to measure the period of oscillation of the beam P and once this is measured we can write

$$G = \frac{2\pi L y d^2}{xMP^2} \tag{9}$$

To get some idea of the difficulty of the experiment, calculate the ratio between the force exerted by the attracting masses and the gravitational force exerted by the earth (the weight of the balls attached to the beam). The attractive force between the spheres is given by $F_s = GM_e m/r_e^2$ and the gravitational force of the earth by $F_e = GM_e m/r_e^2$. You can write the mass of the sphere and earth in terms of their radius and density. If you assume that the mean density of the earth and spheres are similar what is the ratio between the two forces? What does this say about the difficulty of the experiment?

NOTE: Due to the delicate nature of the torsion balance in the Cavindish experiment, it is very important that the instrument not get jarred bumped or hit in any way. Moving the attracting masses too quickly or bumping them in the apparatus will set the bar-bell mass swinging wildly. It will not settle down for several hours and thus will make the instrument unusable for the rest of the lab session.

To get any measurement at all the wire supporting the beam is stretched almost to breaking point and the beam must be very nearly at rest before the experiment can be attempted. It is therefore MOST important not to hit or jar in any way the apparatus either by bumping it or (the most likely accident) moving the attracting masses too quickly and bumping the balls against the apparatus. If this does happen, call the TA or Professor who may be able recover the situation. [If you really bang it the wire holding the small beam will break and it takes days to fix!]

The position of the laser beam on the screen should be measured about once every 20 or 30 seconds. You should record at least one cycle and preferably two. When you are happy with the data, one member of the team should be CAREFULLY move to masses on the turntable so as to nearly BUT NOT ACTUALLY touch the other side of the apparatus. Repeat the measurements as before. The mean position of the beam will be at the mid-point between the maximum and minimum position of the laser spot. The Period can be estimate by measuring between peaks of the oscillation. When the experiment is complete, measure the distance between the positions of the balls on the turntable (distance $2d$ and distance L in Figure 1).

Procedure for the Cavendish Experiment

The instruments will be setup when you arrive in lab. There will be a light spot on the chalk board which is moving horizontally back and forth slowly with a several minute period. This motion is due to the slow torsion oscillation of the bar-bell in the experiment.

Measure the extrema of the oscillatory motion for several full periods. You can draw directly on the chalk board. You will notice that the amplitude of the motion gets smaller with time. You should find a way to measure the center of the motion. Also during this time, use a stop watch to measure the period of the oscillatory motion. Use several full cycles and measure each one so you can estimate the accuracy of your period measurement.

After you have found the center of the motion and the period, ask the TA to change the position of the external lead masses. This will shift the average position of the motion but should not change the period (check this). Now again find the center of the motion and mark it on the caulk board.

4 The Measurement of g

To measure the mass of the earth it is necessary to measure the acceleration of gravity, g, This can be done either by measuring the acceleration of a mass as it falls to earth or, more accurately, by measuring the period of oscillations of a pendulum. By similar arguments used to derive the period of the Cavendish beam we can show that the period of a pendulum is given by

$$P = 2\pi \left(\frac{L}{g}\right)^{1/2} \tag{10}$$

From which we see that

$$g = \frac{4\pi^2 L}{P^2} \tag{11}$$

It is convenient to time the pendulum over a number of oscillations to improve the accuracy of the measurement. The periods should be measured for a number of different lengths of pendulum and the value of P^2 plotted against L. From this data we can obtain the mass of the earth.

The second way of measuring g is to use the apparatus that measures the position of a falling body as a function of time. A high voltage pulse is generated by the pulse generator which creates a mark on the paper. Measure the positions of the marks and hence derive g using the formula you remember from high school:

$$x = \frac{at^2}{2} \tag{12}$$

Once you have measured both G and g, calculate the mass and mean density of the earth. Using the value of g and the distance of the moon derived either from the lunar eclipse or (if the event was clouded out) from a book, determine the period of the moon and compare this value with the value of the lunar month. Discuss the result.

5 Procedure for the Measurement of g

In the apparatus you will use for the measurement of g, a weight is initially held up by an electromagnet. When you are ready, you can turn off the electromagnet and the weight will fall toward the floor because of the force of gravity acting on it. A spark marks the a strip of paper at regular intervals permitting you to accurately determine the position of

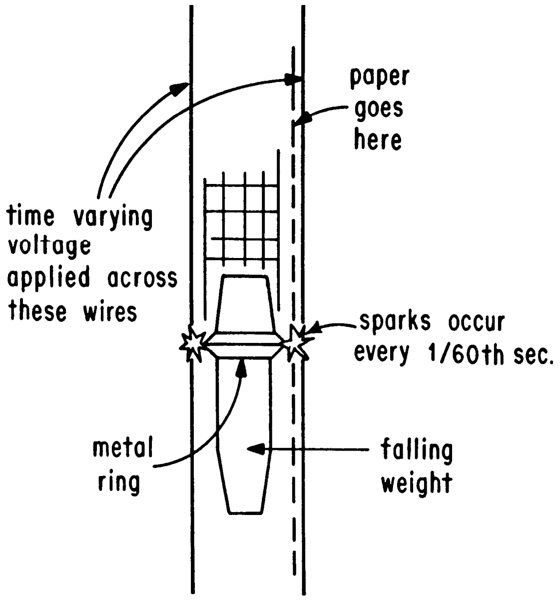

Figure 2: Schematic of Cavendish setup

the weight as it falls. This is done by applying a rapidly varying voltage across the pair of wires between which the weight falls (see Figure 2). As the voltage varies, it causes sparks to jump from one wire to the weight through spark sensitive paper at the position of the weight. The marks indicate where the weight was at times $\Delta T, 2\Delta T, 3\Delta T, \ldots$, where ΔT is the time interval between sparks.

Warning: It is easy to get an unpleasant, though not dangerous, shock in the experiment. When the sparker is on, you should not touch any metallic parts of any of the apparatus.

We need to know the time interval if we want to get a numerical answer for the acceleration. The spark rate is too fast to follow with your eye. The apparatus uses the variation in the line voltage us make the sparks which goes at 60 Hz. This means that the interval between sparks is 1/60 sec.

Perform the experiment and obtain your paper tape. Then measure the distance of the mark from the *starting point.* You should leave out the first few marks because they are too close together and this may make your measurement inaccurate. Also beware of “missing” points which occasionally occur because a spark may not have left a mark.

1. Label the distances $d_1, d_2, d_3, \ldots$ and tabulate them. Plot these points on a graph with the distance increasing upwards and time increasing to the right. You now have a graph of the *distance* of the weight from its starting point versus *time.*
2. Compute the average velocity that the weight had during each time interval:

$$v_2 = \frac{d_2 - d_1}{\Delta T}, v_3 = \frac{d_3 - d_2}{\Delta T}, \ldots$$

Note that an object traveling a constant velocity would make marks that were equally spaced. Tabulate the velocities and plot them with time along the horizontal axis.

3. You now have a graph of the velocity of the weight versus time. Notice that the velocity is increasing as the weight falls. The velocity points fall along a straight line meaning that the velocity is increasing at a constant rate, the acceleration. Determine the acceleration by taking the slope of the line that best fits your data. Express the value of the acceleration in cm s^{-2}.

You have now measured g. Use this value to determine M, the mass of the earth using the formulas from the first par to the experiment together with your value for G from the Cavendish experiment.