Simple Plane Pendulum

\[ m := 1 \quad g := 9.8 \quad L := 9.8 \quad \text{Pendulum parameters} \]

\[ \omega_0 := \sqrt{\frac{g}{L}} = 1 \quad \tau_0 := \frac{2\pi}{\omega_0} = 6.283 \quad \text{Frequency and period of small oscillation} \]

\[ \theta_0 := 25 \cdot \frac{\pi}{180} \quad \text{Initial angle, assuming release from rest} \]

\[ U(\theta) := m \cdot g \cdot L \cdot (1 - \cos(\theta)) \quad \text{Potential energy function} \]

\[ E := U(\theta_0) \quad E = 8.998 \quad \text{Total conserved energy (release from rest at angle } \theta_0) \]

\[ \tau := 2 \int_{\theta_0}^{\theta} \frac{L}{\sqrt{\frac{2}{m}(E - U(\theta))}} d\theta \quad \text{Oscillation Period} \]

\[ (\text{See equation 2.98}) \]

Repeat the calculation using Eqn. 4.28 in the text (i.e. elliptic integral)

\[ F(x, k) := \int_{0}^{x} \frac{1}{\sqrt{(1-z^2)(1-k^2z^2)}} \, dz \quad \text{Elliptic integral of the 1st kind.} \]

\[ k := \sin \left( \frac{\theta_0}{2} \right) \quad \tau := 4 \cdot \frac{L}{\sqrt{g}} \cdot F(1, k) = 6.359 \quad \text{Same result} \]
Integrate numerically to plot the motion

\[
x_0 := \begin{pmatrix} \theta_0 \\ 0 \end{pmatrix}
\]

Initial conditions on theta and theta-dot

\[
D(t, X) := \begin{pmatrix} X_2 \\ -\frac{g}{L} \sin(X_1) \end{pmatrix}
\]

Derivatives from Newton's second law. \(X_1\) is theta while \(X_2\) is theta-dot. The first row is the derivative of theta w.r.t. time, while the second is the derivative of theta-dot w.r.t. time. Note that the array origin is set to 1 (the default is 0 in MathCad).

\[
t_0 := 0 \quad t_1 := 4 \cdot \tau \quad N := 400
\]

Initial and final times, and number of time steps

\[
S := \text{rkfixed}(x_0, t_0, t_1, N, D)
\]

Fourth order Runge-Kutta integration algorithm

\[
T := S^{(1)}
\]

List of times

\[
\Theta := S^{(2)}
\]

List of angles

\[
\Theta_{\text{dot}} := S^{(3)}
\]

List of angular velocities
Phase-Space Plot

Setup for making an animation of the pendulum motion. Go to the Tools menu and selection animation-record. Enter the range for FRAME (1 to 400), select the plot below with the mouse rubber-band cursor, and then click the animate button.

\[ i \quad \text{:= FRAME} \]
Simple Euler method numerical integration of the equation of motion of a simple plane pendulum

<table>
<thead>
<tr>
<th>t (s)</th>
<th>theta</th>
<th>thetadot</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.002</td>
<td>1</td>
<td>-0.000168</td>
</tr>
<tr>
<td>0.004</td>
<td>0.99997</td>
<td>-0.00337</td>
</tr>
<tr>
<td>0.006</td>
<td>0.9999</td>
<td>-0.00505</td>
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<tr>
<td>0.008</td>
<td>0.9998</td>
<td>-0.00673</td>
</tr>
<tr>
<td>0.01</td>
<td>0.99966</td>
<td>-0.00841</td>
</tr>
<tr>
<td>0.012</td>
<td>0.9995</td>
<td>-0.0101</td>
</tr>
<tr>
<td>0.014</td>
<td>0.99929</td>
<td>-0.01178</td>
</tr>
<tr>
<td>0.016</td>
<td>0.99906</td>
<td>-0.01346</td>
</tr>
<tr>
<td>0.018</td>
<td>0.99879</td>
<td>-0.01515</td>
</tr>
<tr>
<td>0.02</td>
<td>0.99849</td>
<td>-0.01683</td>
</tr>
<tr>
<td>0.022</td>
<td>0.99815</td>
<td>-0.01851</td>
</tr>
<tr>
<td>0.024</td>
<td>0.99778</td>
<td>-0.02019</td>
</tr>
<tr>
<td>0.026</td>
<td>0.99737</td>
<td>-0.02188</td>
</tr>
<tr>
<td>0.028</td>
<td>0.99694</td>
<td>-0.02356</td>
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<td>0.99647</td>
<td>-0.02524</td>
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<td>0.99596</td>
<td>-0.02693</td>
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<td>0.034</td>
<td>0.99542</td>
<td>-0.02861</td>
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<tr>
<td>0.036</td>
<td>0.99485</td>
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<tr>
<td>0.038</td>
<td>0.99424</td>
<td>-0.03197</td>
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<tr>
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<td>-0.03365</td>
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<tr>
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<tr>
<td>0.044</td>
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<td>0.99148</td>
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<tr>
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<tr>
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<tr>
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<td>0.98728</td>
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<tr>
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<td>0.98536</td>
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</tr>
<tr>
<td>0.062</td>
<td>0.98435</td>
<td>-0.05215</td>
</tr>
<tr>
<td>0.064</td>
<td>0.98331</td>
<td>-0.05384</td>
</tr>
</tbody>
</table>

theta vs time

![Theta vs Time Graph](attachment:image.png)
thetadot vs theta

(0.066, 0.998223, -0.05552)
(0.068, 0.998112, -0.0572)
(0.07, 0.997998, -0.05888)
(0.072, 0.99788, -0.06056)
(0.074, 0.997759, -0.06224)
(0.076, 0.997634, -0.06392)
(0.078, 0.997506, -0.0656)
(0.08, 0.997375, -0.06728)
(0.082, 0.997241, -0.06896)
(0.084, 0.997103, -0.07064)
(0.086, 0.996962, -0.07232)
(0.088, 0.996817, -0.074)
(0.09, 0.996669, -0.07568)
(0.092, 0.996517, -0.07736)
(0.094, 0.996363, -0.07904)
(0.096, 0.996205, -0.08072)
(0.098, 0.996043, -0.0824)
(0.1, 0.995878, -0.08408)
(0.102, 0.99571, -0.08575)
(0.104, 0.995539, -0.08743)
(0.106, 0.995364, -0.08911)
(0.108, 0.995186, -0.09079)
(0.11, 0.995004, -0.09247)
(0.112, 0.994819, -0.09414)
(0.114, 0.994631, -0.09582)
(0.116, 0.994439, -0.0975)
(0.118, 0.994244, -0.09918)
(0.12, 0.994046, -0.10085)
(0.122, 0.993844, -0.10253)
(0.124, 0.993639, -0.1042)
(0.126, 0.993431, -0.10588)
(0.128, 0.993219, -0.10756)
(0.13, 0.993004, -0.10923)
(0.132, 0.992785, -0.11091)
(0.134, 0.992564, -0.11258)
Pendulum Phase Diagram

\[ E_0 = mg \cdot 2\ell \]
Integration of the van der Pol equation.

\[
\begin{align*}
    x_0 &= \begin{pmatrix} 0.1 \\ 0 \end{pmatrix} & \text{Initial conditions} & \mu &= .5 & \text{Parameter} \\
    D(t,X) &= \begin{bmatrix} X_2 \\ -\mu[(X_1)^2 - 1]X_2 - X_1 \end{bmatrix} & \text{Vector of derivatives describing the 2nd-order nonlinear ODE. } X_1 \text{ is the displacement while } x_2 \text{ is the velocity. The first row is the derivative of } X_1 \text{ w.r.t. time, while the second row is the derivative of } X_2 \text{ w.r.t. time.} \\
    t_0 &= 0 & t_1 &= 100 & N &= 1000 & \text{Initial and final times, and the number of time steps.} \\
    S &:= \text{rkfixed}(x_0,t_0,t_1,N,D) & \text{Fourth order Runge-Kutta integration algorithm with fixed step size.} \\
    T &:= S^{(1)} & \text{List of } N \text{ equally spaced times} \\
    x &:= S^{(2)} & \text{List of } N \text{ displacements, one for each time.} \\
    x_{\text{dot}} &:= S^{(3)} & \text{List of } N \text{ velocities} \\
\end{align*}
\]
Driven Pendulum
Poincaré Sections

- A useful way to project a 3-D phase space onto a 2-D plot.
- Plot the values of $x$ and $y$ only when $z$ is a multiple of $2\pi$. 

![Diagram of Poincaré Sections](image)
Driven Nonlinear Pendulum (Note: units are such that the natural angular frequency for small oscillation is unity)

\[ \omega := 0.7 \quad F := 0.5 \quad \text{Driving frequency and amplitude} \quad \frac{\omega}{2\pi} = 0.111 \]

\[ c := 0.05 \quad \text{Damping coefficient} \]

\[ x_0 := \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{Initial condition for the position and velocity (or angle and angular velocity)} \]

\[ D(t, X) := \begin{pmatrix} X_2 \\ -c \cdot X_2 - \sin(X_1) + F \cdot \cos(\omega \cdot t) \end{pmatrix} \quad \text{First derivatives of the unknown function.} \]

\[ t_0 := 0 \quad t_1 := 640 \cdot \pi \quad m := 18 \quad N := 2^m - 1 \quad \text{Initial and final times, and number of time steps} \]

\[ S := \text{rkfixed}(x_0, t_0, t_1, N, D) \quad \text{Fourth order Runge-Kutta numerical integration algorithm} \]

\[ T := S^{(1)} \quad \text{Times at which the solution is evaluated} \]

\[ X := S^{(2)} \quad \text{Position at each time step} \]

\[ V := S^{(3)} \quad \text{Velocity at each time step} \]

Angular Velocity vs Time

Time (units of 2pi)
\( F := \cos(\omega \cdot T) \)

![Angle vs Time and Driving Function](image)

Define arrays containing just the 2nd half of the times, to remove startup transients:

\[
N_2 := \frac{N + 1}{2} \quad k \ := \ 1, 2 \ldots N_2 \quad T_{s_k} := T_{N_2 + k} \quad V_{s_k} := V_{N_2 + k} \quad X_{s_k} := X_{N_2 + k}
\]

![Phase Plot](image)
$U := \text{mod}(X_s + \pi, 2\pi)$

Folded Phase Plot

Angular Velocity

$\frac{U}{\pi}$

Angle (units of $\pi$)
Calculate numerically the Fourier power spectrum, to see what frequencies exist in the motion.

\[ M := 2^{m-2} \quad M = 65536 \quad i := 1..M \]

\[ C := \text{fft}(V_s) \]  
Fast Fourier Transform (FFT) of the velocity versus time

\[ f_s := \frac{N + 1}{t_1} \]  
Sampling frequency of the time

\[ f_i := \frac{2i}{N + 1} \cdot f_s \]  
Frequency values in the FFT

\[ f_d := \frac{\omega}{2\pi} \quad f_d = 0.111 \]  
Driving frequency
Program to select values for the Poincare section plot.

\[
\omega = 0.7
\]

\[
\Delta t := T_2 - T_1 = 7.67 \times 10^{-3}
\]

\[
p := \begin{cases} 
  i & \quad i < 0 \\
  \text{for } n \in 2, 3 \ldots N_2 \\
  z_2 & \quad \text{mod}(\omega T_n, 2\pi) \\
  z_1 & \quad \text{mod}(\omega T_{n-1}, 2\pi) \\
  \text{if } z_2 < z_1 \\
  i & \quad i + 1 \\
  p_i, 1 & \quad X_{s_n} \\
  p_i, 2 & \quad V_{s_n} \\
\end{cases}
\]

This program simply loops over the times and looks for the point where \( \omega \times \text{time} \) changes from being just below a multiple of 2\( \pi \) to being just above 2\( \pi \).

\[
X_p := p^{(1)} \\
V_p := p^{(2)} \\
U_p := \text{mod}(X_p, 2\pi)
\]

Poincare Section (Folded phase space plot)

Angle (units of \( \pi \))

Angular Velocity

\[
\begin{array}{c}
\text{Angular Velocity} \\
V_p \\
\end{array}
\]

\[
\begin{array}{c}
\text{Up} \\
\frac{\pi}{\text{Up}} \\
\end{array}
\]

\[
\begin{array}{c}
\text{Poincare Section (Folded phase space plot)} \\
\end{array}
\]

\[
\begin{array}{c}
\text{Up} \\
\frac{\pi}{\text{Up}} \\
\end{array}
\]