Homework Assignment #3
Due Friday, October 16 by 4:00 pm.

1. Show that if a driven oscillator is only lightly damped (so $\omega_d \approx \omega_0$) and is driven near resonance, then the $Q$ of the system is approximately

$$Q \approx 2\pi \times \left( \frac{\text{Total Energy}}{\text{Energy loss during one period}} \right).$$

This expression gives a good physical intuition for the meaning of the $Q$.

2. Problem 3-25, plotting solutions for a driven oscillator. You can use whatever computer program you prefer for making these plots (e.g. MathCad or Mathematica). If you don’t have anything better at hand, Excel or Open-Office Calc can certainly do it. In that case, make a column of time $t$, a column for $x_s$, a column for $x_p$, and a column for $x_s + x_p$. Put the parameters such as $\beta$ in cells somewhere off to the right, for reference. Enter the formulas into the columns in such a way that they can be dragged downward to fill many rows (remember the $\$ signs for the parameter references, as in Problem 1 of HW-2). Then use “Insert Chart” to plot the columns versus time. Be sure to answer all of the questions posed in the problem statement.

3. Numerically integrate the nonlinear differential equation for a simple pendulum to find $x(t)$ and $v(t)$ over one period. Use Excel or Open-Office Calc and the classic Euler algorithm (the simplest method, but not very efficient). Here is a recipe that illustrates the very basics of the numerical technique that we will use to solve nonlinear mechanics problems. The 2$^{nd}$-order ODE is $\ddot{x} = -\sin x$, where $x$ is the angle, and I’ve chosen the unit of time to make $\omega_0 = \sqrt{gL}$ equal to unity. Define $y = \dot{x}$ to be the angular velocity, in order to rewrite the problem as two coupled 1$^{st}$-order differential equations:

$$\frac{dx}{dt} = y,$$

$$\frac{dy}{dt} = -\sin x.$$

Then, make 3 columns in your spread sheet for $t$, $x$, and $y$ and enter initial values 0, 1, and 0 respectively (i.e. the initial conditions are 1 radian amplitude and zero initial velocity). Enter a time step $\Delta t = 0.002$ into a cell somewhere, say $G2$, and use the following algorithm to update the angle and velocity at each step (i.e. enter the appropriate formulas into the spread sheet row just below the initial values):

$$t_{n+1} = t_n + \Delta t,$$

$$x_{n+1} = x_n + \frac{dx}{dt} \bigg|_{t_n} \cdot \Delta t,$$

$$y_{n+1} = y_n + \frac{dy}{dt} \bigg|_{t_n} \cdot \Delta t,$$

where of course the derivatives are evaluated according to the 1$^{st}$-order differential equations above. Copy the formulas by dragging down about 3500 rows. Then make two plots, one of $x$ versus $t$ and another of $y$ versus $x$. The latter is a phase-space diagram. Estimate the period of the motion from your plot. How does it compare to $2\pi/\omega_0$? Note that the step size is large enough that some error in the integration will be easily visible after one period (the amplitude doesn’t quite stay constant). A 4$^{th}$-order Runge-Kutta algorithm would improve this result tremendously even with a much larger time step. Finally, take a look at the MathCad or Mathematica solution to this problem, both of which are posted on the course web page just under this assignment. How well does your result agree? In the remainder of this course you will do such integrations and plots with MathCad or Mathematica.
4. Derive the Fourier representation of a triangle function, which is defined by

\[ F(t) = \begin{cases} 
\pi + t & -\pi < t \leq 0 \\
\pi - t & 0 < t < \pi 
\end{cases} \]

Plot the sum of the first 5 non-zero terms of the expansion and compare with the exact function (using the MathCad template referred to below). Use your result to write down the steady-state solution for the oscillator system described by

\[ \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{1}{m} F(t). \]

Plot the solution for a system with \( \beta = 0.4 \) and \( m = 1 \), for 3 values of the oscillator frequency \( \omega_0 \): 0.5, 3.5, and 6.5. Also plot the last case for higher damping: \( \beta = 2 \). Explain why the very different responses result. The plots are all best done using MathCad or Mathematica. A template for each is available on the course web site, into which you need only to enter the correct formula for the expansion coefficients and change the values of the oscillator frequency and damping. Also posted is a complete example for a different driving force, in both MathCad and Mathematica.

5. A particle at rest is attracted toward a center of force according to the relation \( F = -mk^2/x^3 \). Show that the time required for the particle to reach the force center from a distance \( d \) is \( d^2/k \).

6. Calculate the gravitational potential due to a thin rod of length \( \ell \) and mass \( M \) at a distance \( R \) from the center of the rod and in a direction perpendicular to the rod.