Physics 110A Quiz #3 Solutions Winter '07

1) a) $$\mathcal{U} = \frac{1}{2}\int \rho V d\tau = \frac{\varepsilon_0}{2}\int (\nabla\cdot\vec{E}) V d\tau$$

where I used Gauss' law $\rho = \varepsilon_0 \nabla\cdot\vec{E}$

Now, integrate by parts using

$$(\nabla\cdot\vec{E})V = \nabla\cdot(V\vec{E}) - \vec{E}\cdot\nabla V$$
$$= \nabla\cdot(V\vec{E}) + |\vec{E}|^2$$

since $\vec{E} = -\nabla V$

$$\mathcal{U} = \frac{\varepsilon_0}{2}\int_V \nabla\cdot(V\vec{E}) d\tau + \frac{\varepsilon_0}{2}\int_V |\vec{E}|^2 d\tau$$

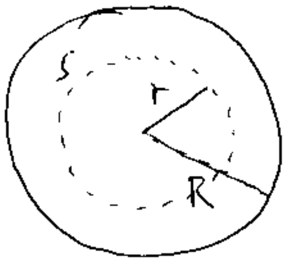
$$= \frac{\varepsilon_0}{2}\int_S V\vec{E}\cdot d\vec{a} + \frac{\varepsilon_0}{2}\int_V |\vec{E}|^2 d\tau$$ using the divergence theorem

The volume can be any volume enclosing all the charge, which is obvious from the original integral.

b) as $R\to\infty$ $V \sim \frac{1}{R}$ and $\vec{E} \sim \frac{1}{R^2}$ (or they fall off even faster if the net charge is zero).

The surface area increases as R^2 so the surface integral $\sim \frac{1}{R}\frac{1}{R^2}R^2 \sim \frac{1}{R} \to 0$ as $r\to\infty$

2.



$\rho = kr$

For the spherical gaussian surface S at radius r

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0}\int_V \rho\, d\tau$$

V = volume enclosed by S

From symmetry, $\vec{E}(\vec{r}) = E_r(r)\,\hat{r}$ so E_r comes out of the surface integral

$$E_r \int_S da = E_r 4\pi r^2$$

$$E_r 4\pi r^2 = \frac{1}{\varepsilon_0}\int d\Omega \int_0^r k r' r'^2 dr'$$

$$E_r 4\pi r^2 = \frac{1}{\varepsilon_0} 4\pi\, k \frac{1}{4} r^4$$

$$\boxed{\vec{E} = \frac{k}{4\varepsilon_0} r^2\,\hat{r}}$$