Final Exam

Closed book, except for the front and back inside covers of Griffiths. Math tables are allowed. However, all necessary integrals are very elementary, so don’t try to work the problems the hard way!

Electrostatic boundary conditions: \( \vec{E}_{\text{above}} - \vec{E}_{\text{below}} = \frac{\sigma}{\varepsilon_0} \hat{n} \), \( V_{\text{above}} = V_{\text{below}} \)

or with a linear dielectric: \( \varepsilon_{\text{above}} \vec{E}_{\text{above}} - \varepsilon_{\text{below}} \vec{E}_{\text{below}} = \sigma_f \)

Magnetostatic boundary conditions: \( \vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \mu_0 (\vec{K} \times \hat{n}) \), \( \vec{A}_{\text{above}} = \vec{A}_{\text{below}} \)

Solutions of Laplace’s equations:

- spherical, \( \varphi \) symmetric: \( V(r, \vartheta) = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \vartheta) \)
- cylindrical, \( z \) symmetric:

\[
V(r, \varphi) = A + B \ln r + \sum_{n=1}^{\infty} \left( C_n r^n \cos n\varphi + D_n r^{-n} \cos n\varphi \right) + \sum_{n=1}^{\infty} \left( E_n r^n \sin n\varphi + F_n r^{-n} \sin n\varphi \right)
\]

Legendre polynomials:

\[
P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = (3x^2 - 1)/2, \quad P_3(x) = (5x^3 - 3x)/2
\]

\[
\int_{-1}^{1} P_l(x) P_{l'}(x) dx = \frac{2}{2l + 1} \delta_{l,l'}
\]

Trig functions:

\[
\int_0^a \sin \left( \frac{n\pi}{a} x \right) \sin \left( \frac{m\pi}{a} x \right) dx = \frac{a}{2} \delta_{n,m}
\]

\[
\int_0^a \cos \left( \frac{n\pi}{a} x \right) \cos \left( \frac{m\pi}{a} x \right) dx = \frac{a}{2} \delta_{n,m} \quad \text{(for } m, n \neq 0 \text{)}
\]

Electric dipole:

\[
\vec{V}(\vec{r}) = \frac{\vec{p} \cdot \hat{r}}{4\pi \varepsilon_0 r^2}, \quad \vec{E} = \frac{1}{4\pi \varepsilon_0} \int \left[ \delta(\vec{p} \cdot \hat{r}) \vec{r} - \vec{p} \right] \cdot d\vec{r} \quad \vec{p} = \vec{\rho} \times \vec{E}, \quad \vec{F} = (\vec{p} \cdot \nabla) \vec{E}
\]

Magnetic dipole:

\[
\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}, \quad \vec{B} = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[ \delta(\vec{m} \cdot \hat{r}) \vec{r} - \vec{m} \right], \quad \vec{N} = \vec{m} \times \vec{B}, \quad \vec{F} = \nabla(\vec{m} \cdot \vec{B})
\]

\[
\vec{m} = \frac{1}{2} \int (\vec{r} \times d\vec{l}) = \frac{1}{2} \int (\vec{r} \times \vec{J}) d\tau \quad (\vec{m} = I \vec{a} \text{ for a planar loop})
\]

Energy stored in a magnetic field:

\[
W = \frac{1}{2} C I^2 = \frac{1}{2} \int_{\text{volume}} \left( \vec{A} \cdot \vec{J} \right) d\tau = \frac{1}{2} \mu_0 \int_{\text{all space}} \left| \vec{B} \right|^2 d\tau
\]

Energy stored in an electric field:

\[
W = \frac{1}{2} CV^2 = \frac{1}{2} \int_{\text{volume}} \rho V d\tau = \frac{\varepsilon_0}{2} \int_{\text{all space}} \left| \vec{E} \right|^2 d\tau
\]

Biot-Savart law:

\[
\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \left( \vec{J}(\vec{r}') \times \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \right) d\tau', \quad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \left( \vec{J}(\vec{r}') \right) d\tau'
\]

Derivatives:

\[
\nabla \frac{1}{|\vec{r} - \vec{r}'|} = -\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \quad \nabla^2 \frac{1}{|\vec{r} - \vec{r}'|} = -4\pi \delta^3(\vec{r} - \vec{r}')
\]
Short Answer Questions

Note that any calculation needed here is short enough to be done quickly in your head.

1. (2 pnts) A permanent bar magnet has uniform magnetization \( \vec{M} \) inside. There are no free currents present anywhere.
   a) From Ampère’s law for the \( \vec{H} \) field, we cannot conclude that \( \vec{H} = 0 \) everywhere, and therefore \( \vec{B} = \mu_0 \vec{M} \) everywhere.
      i) True
      ii) False
   b) \( \vec{B} = \mu \vec{H} \) everywhere, where \( \mu \) is a constant that is characteristic of the type of material from which the magnet is composed.
      i) True
      ii) False

2. (2 pnts) A bar electret has a frozen-in polarization \( \vec{P} \). The work done to establish this polarization and associated electric field can be calculated from this integral: \[ W = \frac{1}{2} \int_{\text{all space}} \vec{D} \cdot \vec{E} \ d\tau \]
   a) True
   b) False

3. (4 pnts) A charge \( q \) sits exactly at the center of an imaginary cubical surface. What is the flux \( \int \vec{E} \cdot d\vec{a} \) through one of the six sides of the cube, assuming this charge to be the only source of the field \( \vec{E} \)?

4. (4 pnts) A hollow, empty spherical shell is divided into two conducting hemispheres separated by a thin insulator. The potential of the upper hemisphere is 11 volts, while the potential of the lower hemisphere is 3 volts. What is the potential at the center of the sphere?

5. (2 pnts) Two charges of equal magnitude but opposite sign are separated by a distance \( d \), with the negative charge located at the origin. Circle all of the moments of this charge distribution that are zero.
   a) Monopole.
   b) Dipole.
   c) Quadrupole.

6. (2 pnts) Three charges of equal magnitude, two positive and one negative, are arranged at the points of an equilateral triangle. The dipole moment of this charge distribution depends on the location of the origin of the coordinate system.
   a) True
   b) False

7. (2 pnts) Two parallel wires carry current in the same direction. The magnetic force (choose one answer)
   a) repels the wires from each other.
   b) attracts the wires toward each other.

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8. (2 pnts) Consider the dipole field shown in this figure. Drawn in the plane are two paths, $P$ and $Q$, where $P$ closes on itself and $Q$ goes from point $a$ to point $b$, both of which lie on the perpendicular bisector of the line connecting the two charges. Compare the integral $\int \vec{E} \cdot d\vec{l}$ for the two paths.
   
   a) The integral is equal and zero for the two paths.
   b) The integral for path $Q$ is greater than for path $P$.
   c) The integral for path $P$ is greater than for path $Q$.
   d) The integral for path $Q$ is of the opposite sign than the integral for path $P$.

9. (4 pnts) Two conducting loops are located nearby each other, as shown below (not to scale).
   
   a) The current in loop 1 is flowing counterclockwise, viewed from above, as indicated by the arrow, and it is decreasing. Indicate with arrows the direction of the induced current in loop 2.
   
   b) Suppose that when a 1 amp DC current passes through loop number 1 the magnetic flux through loop number two is $0.02 \text{ N} \cdot \text{m/A}$. What then is the magnetic flux through loop number 1 when a 10 amp DC current passes through loop number 2 (with no current in loop 1)?

\[ \Phi_B = \]

10. (2 pnts) Diamagnetism is (choose one answer)
   
   a) due to alignment of electron spin magnetic moments and results in magnetization antiparallel to an applied field.
   b) due to alignment of electron spin magnetic moments and results in magnetization parallel to an applied magnetic field.
   c) due to modification of electron atomic orbital motion and results in magnetization antiparallel to an applied field.
   d) due to modification of electron atomic orbital motion and results in magnetization parallel to an applied field.

11. (4 pnts) The figure shows a cross-section of a solenoid on the left and capacitor on the right. The current in the solenoid and the voltage on the capacitor are increasing with time.
   
   a) Roughly sketch the induced $E$ field lines both inside and outside the solenoid, indicating clearly the field direction.
   
   b) Similarly, sketch the induced $B$ field lines both inside and outside the capacitor.
Problems
(Show all of your work neatly on separate sheets of paper.)

12. (14 pnts) Two infinite conducting planes parallel to the $x,z$ plane are grounded ($V=0$) and separated by a distance $a$. A flat strip of uniform charge density, infinite in the $z$ direction and of width $a$ in the $y$ direction, divides the volume between the conducting planes into two halves ($x>0$ and $x<0$). The strip is insulated from the grounded planes. Find a solution for the electric potential everywhere between the planes, in the form of Fourier series. You must do the integrals and give a formula for all of the coefficients in the series.

13. (12 pnts) A triangle wire loop of base $a$ and height $b$ lies in the $x,y$ plane along with a circular wire loop of radius $r$. The distance between the loops, $d$, is much larger than the dimensions of the loops. Calculate the approximate mutual inductance of the two loops. (Hint: this is a short calculation using approximations that we studied but nearly impossible to do exactly by hand!)

14. (10 pnts) A charge $Q$ is distributed uniformly along a thin rod of length $a$. Find the $y$ component $E_y$ of the electric field $\vec{E}$ at a point $P$ a distance $x$ from one end of the rod, as indicated in the figure.
15. (12 pnts) An infinitely long straight conducting cylinder of inner radius $a$ and outer radius $b$ is oriented parallel to the $z$ axis and carries a current density given by $\vec{J}_f = \frac{k}{r} \hat{z}$. It is composed of a linear medium of magnetic susceptibility $\chi_m$.

a) Calculate $\vec{H}$ and $\vec{M}$ at all points in space.

b) Find both the surface and volume bound currents.

16. (10 pnts) Consider a magnetic field $\vec{B}$ produced by a current flowing through some volume with current density $\vec{J}$. The energy stored in such a configuration is given by

$$W = \frac{1}{2} \int_{\text{volume}} (\vec{A} \cdot \vec{J}) \, d\tau$$

where $\vec{B} = \nabla \times \vec{A}$. Starting with this expression, derive the equivalent expression

$$W = \frac{1}{2\mu_0} \int_{\text{all space}} |\vec{B}|^2 \, d\tau.$$

Show or explain all steps rigorously, using correct vector notation.

17. (12 pnts) The current in a long, straight wire varies with time as $I(t) = I_0 e^{-t/\tau}$. A rectangular loop of resistance $R$ is near to the wire, as shown in the figure. Find the induced current in the loop, including the direction that it is flowing.