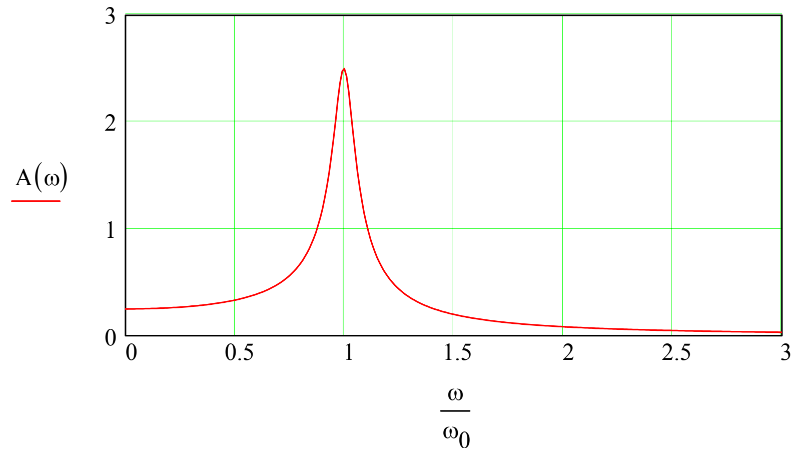
Analysis of a Forced Damped Harmonic Oscillator

$\omega_0 := 2$ Natural Frequency of the Oscillator with no damping

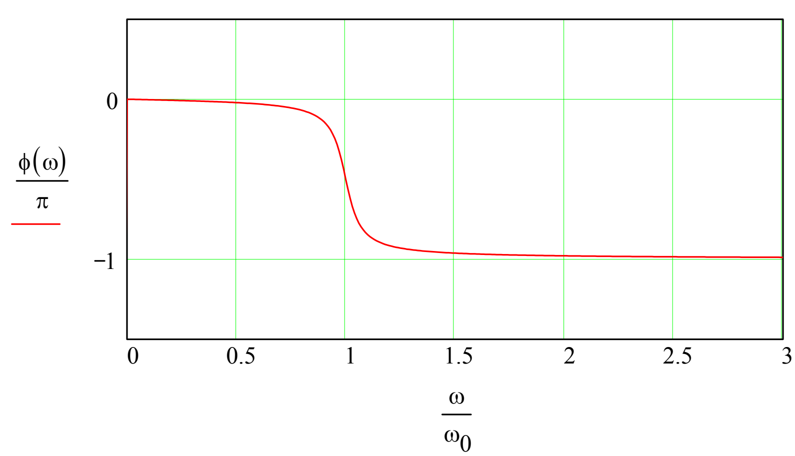
$Q := 10$ $\Gamma := \frac{\omega_0}{Q}$ Q is the "quality", defined to be f0/(f2-f1) where f2-f1 is the width of the peak at 71%

$$A(\omega) := \frac{1}{\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + \Gamma^2 \cdot \omega^2}}$$

$$\phi(\omega) := \text{atan2}\left(\omega^2 - \omega_0^2, \Gamma \cdot \omega\right) - \pi$$



Amplitude Response



Phase Response

Forced damped harmonic oscillator with sine wave input (steady-state solution)

$T := 2$ $\quad \omega_0 := 1 \cdot \frac{2\cdot\pi}{T}$ $\quad \omega_0 = 3.142$ $\quad F := 1$ $\quad m := 1$

$Q := 10$ $\quad \Gamma := \frac{\omega_0}{Q}$

Parameters

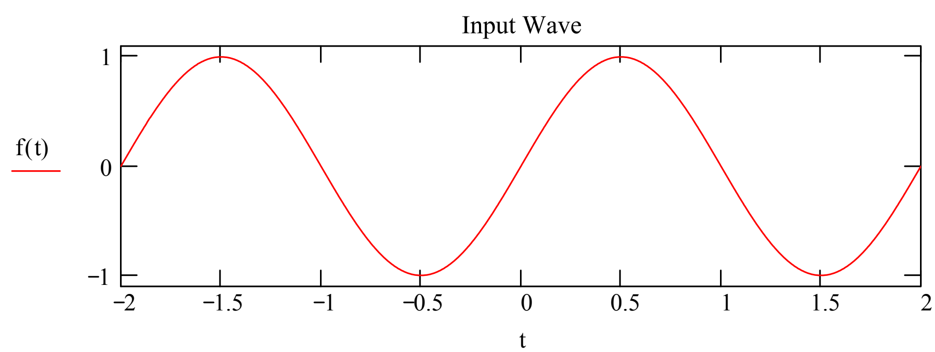
$\omega := \frac{2\cdot\pi}{T}$ $\quad f(t) := F\cdot\sin(\omega\cdot t)$

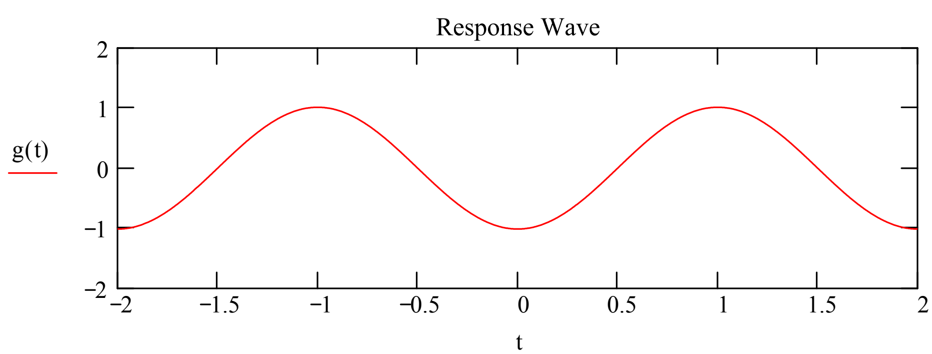
Input sine wave

$$a := \frac{\frac{F}{m}}{\sqrt{\left[(\omega)^2 - \omega_0{}^2\right]^2 + \Gamma^2\cdot(\omega)^2}} \qquad \phi := \operatorname{atan2}\left(\omega^2 - \omega_0{}^2, \Gamma\cdot\omega\right) - \pi$$

$g(t) := a\cdot\sin(\omega\cdot t + \phi)$

Response sine wave





Forced damped harmonic oscillator with square wave input (steady-state solution)

$T := 2$ $\omega_0 := 11 \cdot \frac{2 \cdot \pi}{T}$ $\omega_0 = 34.558$ $F := 1$ $m := 1$ Parameters

$Q := 5$ $\Gamma := \frac{\omega_0}{Q}$

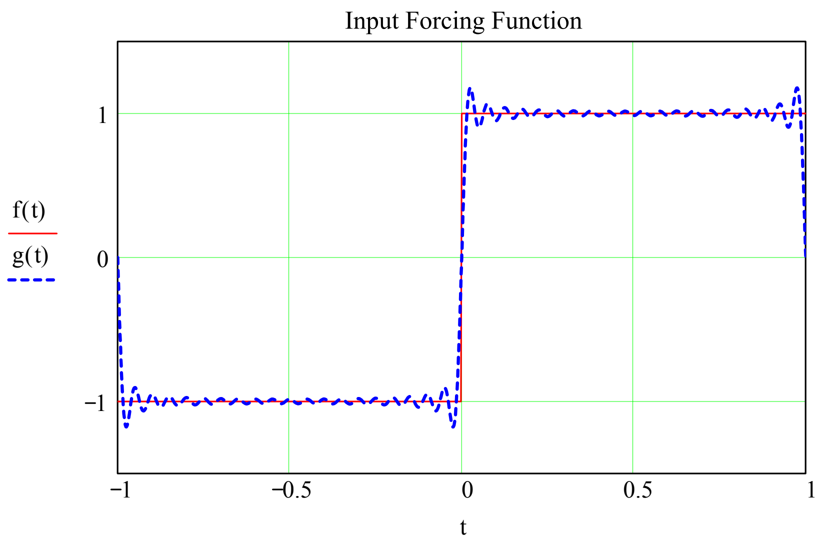
$$\frac{d^2y}{dt^2} + \Gamma \cdot \frac{dy}{dt} + \omega_0{}^2 y := \frac{1}{m} f(t)$$ The ODE

$$f(t) := \begin{cases} -1 & \text{if } \frac{-T}{2} \le t < 0 \\ 1 & \text{otherwise} \end{cases}$$ Input Square Wave

$n := 1, 2 .. 20$

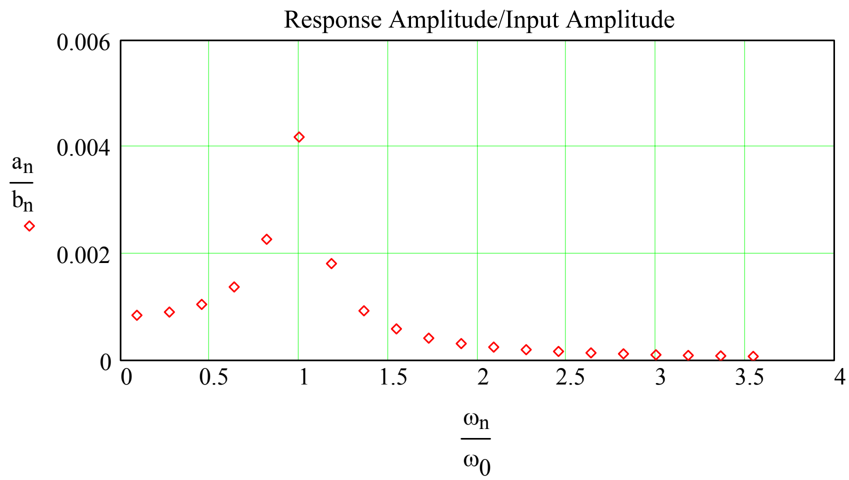
$b_n := \frac{4 \cdot F}{(2 \cdot n - 1) \cdot \pi}$ $\omega_n := (2 \cdot n - 1) \cdot \frac{2 \cdot \pi}{T}$

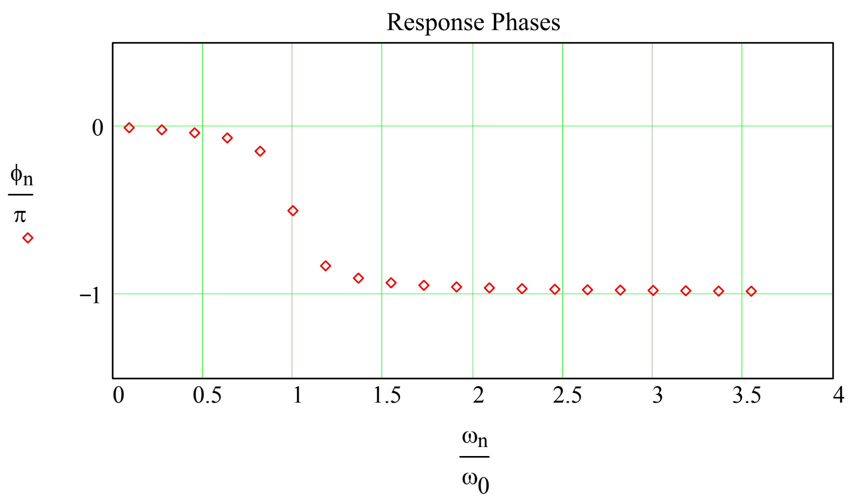
$g(t) := \sum_n \left(b_n \cdot \sin(\omega_n \cdot t)\right)$ Truncated Fourrier series for the input square wave

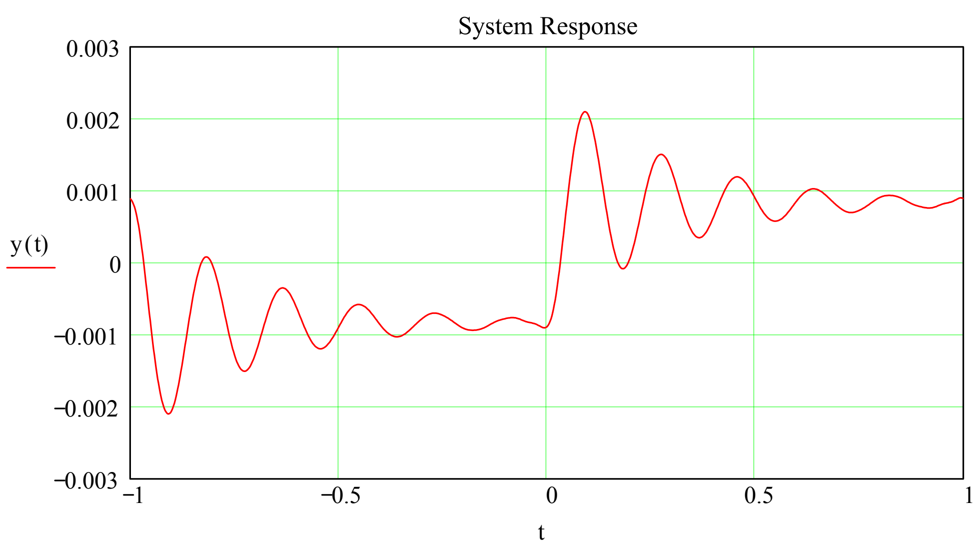


$$a_n := \frac{\frac{1}{m}}{\sqrt{\left[(\omega_n)^2 - \omega_0{}^2\right]^2 + \Gamma^2 \cdot (\omega_n)^2}} \cdot b_n$$ $$\phi_n := \text{atan2}\left[(\omega_n)^2 - \omega_0{}^2, \Gamma \cdot \omega_n\right] - \pi$$

$$y(t) := \sum_n \left(a_n \cdot \sin(\omega_n \cdot t + \phi_n)\right)$$


Response Amplitude/Input Amplitude
0.006
0.004
0.002
0
$\frac{a_n}{b_n}$
0
0.5
1
1.5
2
2.5
3
3.5
4
$\frac{\omega_n}{\omega_0}$


Response Phases
0
−1
$\frac{\phi_n}{\pi}$
0
0.5
1
1.5
2
2.5
3
3.5
4
$\frac{\omega_n}{\omega_0}$


System Response
0.003
0.002
0.001
0
−0.001
−0.002
−0.003
y(t)
−1
−0.5
0
0.5
1
t