Physics 160
Lecture 3

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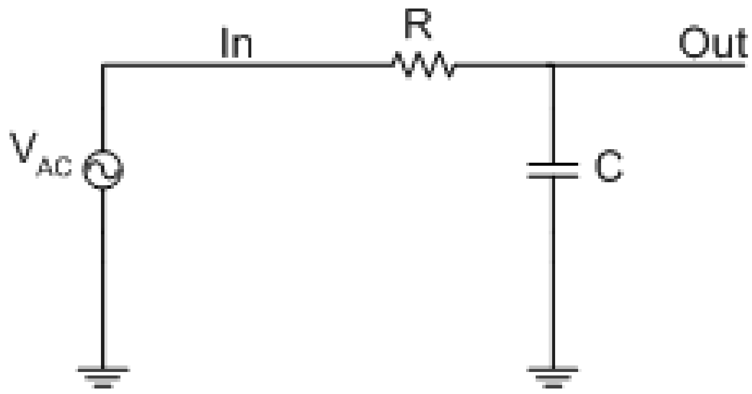
RC Circuit (Low-Pass Filter)

This is the same RC circuit we looked at earlier in the time domain, but here we are interested in the frequency response. So we input a sine wave instead of a step function.



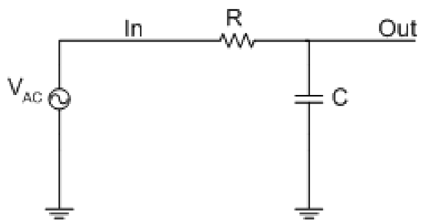
$$\tau \equiv RC$$

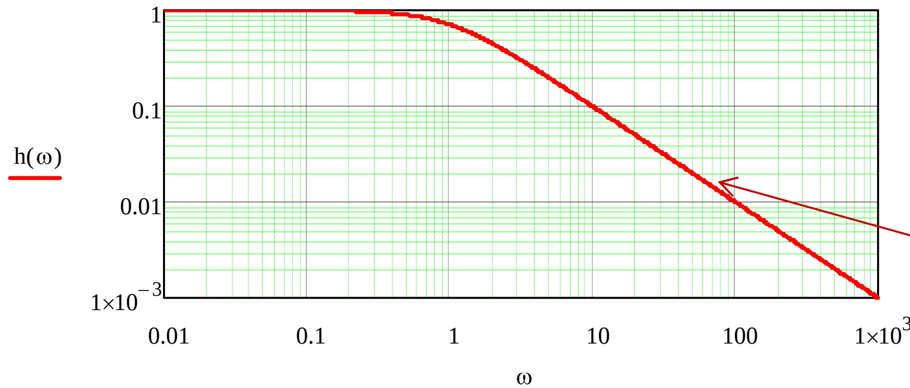
$$I = V_{\text{in}} / Z \qquad \text{where} \qquad Z = R + \frac{1}{j\omega C}$$ *Complex impedance*

$$V_{\text{out}} = I \cdot X_C = V_{\text{in}} \cdot \frac{X_C}{Z} = V_{\text{in}} \times \frac{1/j\omega C}{R + 1/j\omega C}$$

"gain" $$H = \frac{V_{out}}{V_{in}} = \frac{1}{1 + j\omega RC} = \frac{1}{\sqrt{1 + (\omega\tau)^2}} e^{-j\phi} \qquad \text{where} \qquad \phi = \tan^{-1}(\omega\tau)$$

Low Pass Filter Response (V_{out}/V_{in})

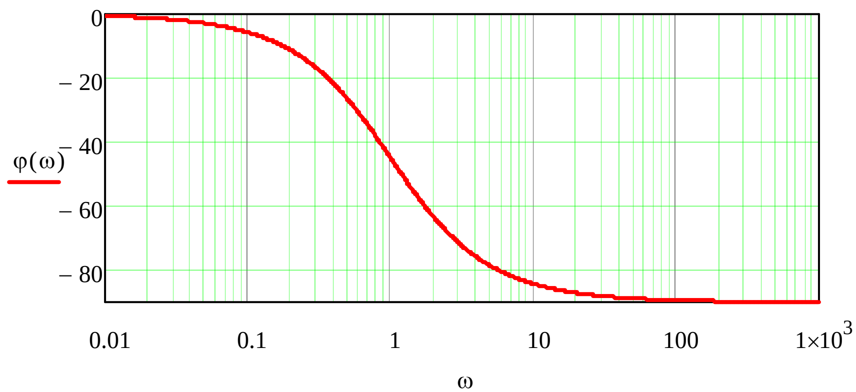




Magnitude

Slope is –6 dB/octave (or –20 dB/decade)

Plotted for $\tau = 1$



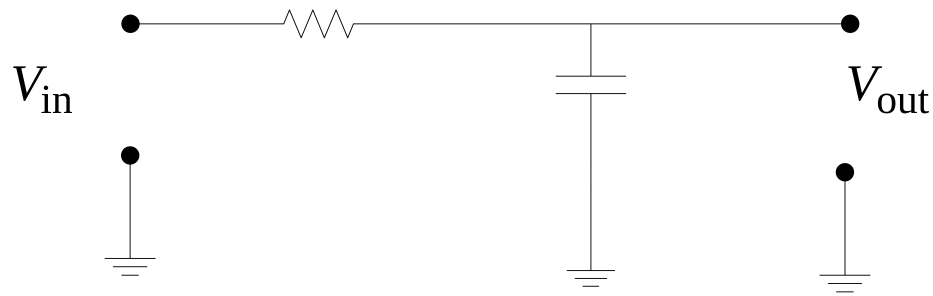
Phase

The "–3dB point" is where the output falls by $\frac{1}{\sqrt{2}}$ (power falls by ½)

$$20\log\left(1/\sqrt{2}\right) = -10\log(2) = -3.01$$

$$-3\text{dB point}: \ \omega = \frac{1}{\tau} \quad \text{and} \quad \phi = -45^\circ$$

Low Pass Filter (single pole) Recap



- *Also known as an "integrator" (time domain)!*
  - *Lousy example; we will make nearly perfect integrators from op amps later in the quarter.*

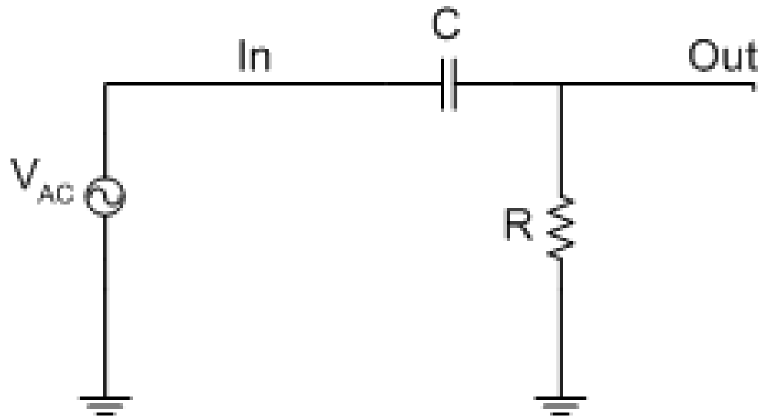
$$V_{out} = \frac{1}{C}\int I(t)\,dt \approx \frac{1}{RC}\int V_{in}\,dt$$

This is valid for only a short time, while V_{out}~0.

- *Frequency response:*
  - *3dB point is at $f=1/(2\pi RC)$*
  - *–6dB/octave falloff at high f*
  - *Phase shift (V_{out} is the voltage across the capacitor):*
    - *V_{out} lags V_{in}*
    - *0 degrees at low frequency*
    - *45 degrees at the 3dB point*
    - *90 degrees at high frequency, far above the 3dB point*

RC Circuit (High-Pass Filter)

Again just an RC circuit, but we swapped the resistor and capacitor, so that now the output voltage is given by the current through the resistor.



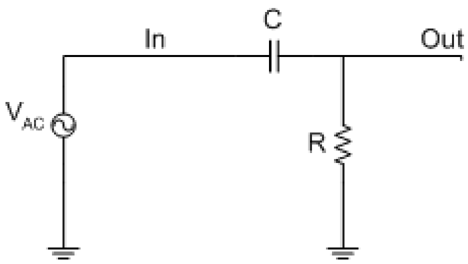
$$\tau \equiv RC$$

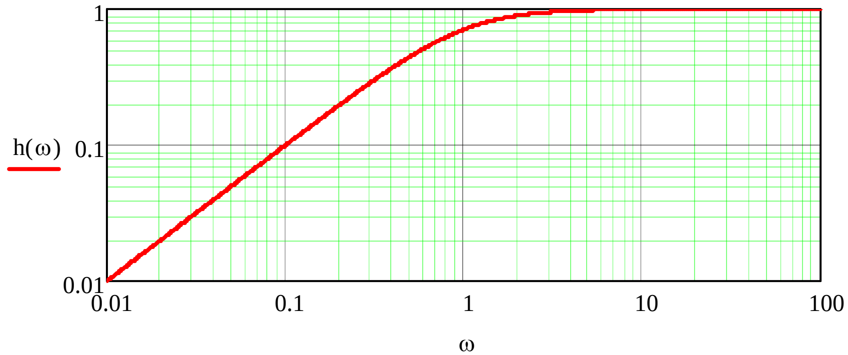
$$I = V_{\text{in}} / Z \qquad \text{where} \qquad Z = R + \frac{1}{j\omega C}$$

$$V_{\text{out}} = I \cdot R = V_{\text{in}} \cdot \frac{R}{Z} = V_{\text{in}} \times \frac{R}{R + 1/j\omega C}$$

$$\frac{V_{out}}{V_{in}} = \frac{j\omega\tau}{1 + j\omega\tau} = \frac{1}{1 - j/\omega\tau} = \frac{1}{\sqrt{1 + (1/\omega\tau)^2}} e^{j\phi} \qquad \text{where} \qquad \phi = \tan^{-1}(1/\omega\tau)$$

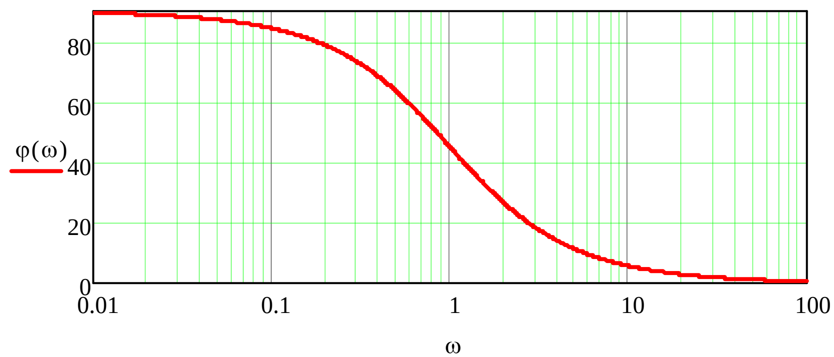
High Pass Filter Response (V_{out}/V_{in})





Magnitude

Plotted for $\tau = 1$



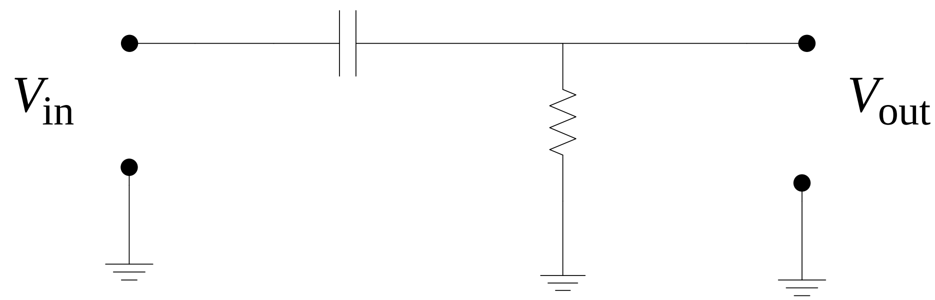
Phase

The "–3dB point" is where the output falls by $\frac{1}{\sqrt{2}}$ (power falls by ½)

$$20\log\left(1/\sqrt{2}\right) = -10\log(2) = -3.01$$

$$-3\text{dB point}: \ \omega = \frac{1}{\tau} \quad \text{and} \quad \phi = 45^\circ$$

High Pass Filter (single pole) Recap



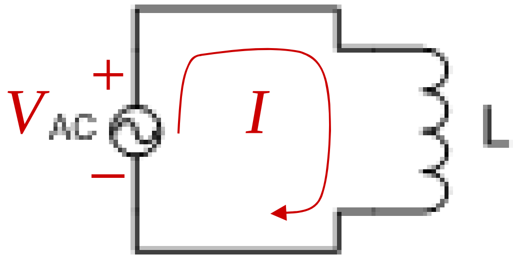
- *Also known as a "differentiator" (time domain)!*
  - *Lousy example; we will make nearly perfect differentiators from op amps later in the quarter.*

$$V_{out} = R \cdot \frac{dQ}{dt} \approx RC \frac{d}{dt} V_{in}$$

This is valid for only a short time, while V_{out}~0.

- *Frequency response*
  - *3dB point is at $f=1/(2\pi RC)$*
  - *6dB/octave (fancy way of saying unity slope) for low f*
  - *Phase shift (V_{out} is proportional to the current in R)*
    - *V_{out} leads V_{in}*
    - *0 degrees at high frequency*
    - *45 degrees at the 3dB point*
    - *90 degrees far below the 3dB point*

Inductor (AC Steady State)



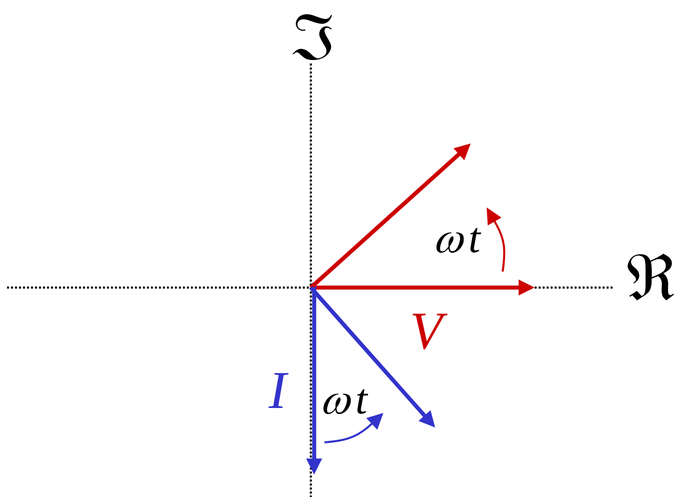
$$I = I_0 \sin \omega t = I_0 \cos(\omega t - \tfrac{\pi}{2})$$

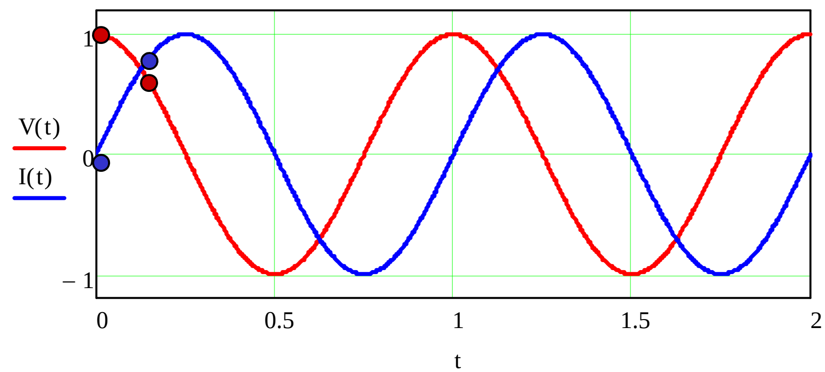
$$V + \left(-L \frac{dI}{dt} \right) = 0$$ **Kirchhoff loop rule**

$$V = L \frac{dI}{dt} = \omega L \cdot I_0 \cos \omega t$$

$$\boxed{V = \omega L \cdot I_0 \cos \omega t}$$

Current ***lags*** voltage by ¼ period (90°)





$\omega = 2\pi$

“Phasors”: The real part traces out the cosine curves as the vector rotates.

Phasors are Complex Numbers

Physical voltage and current are understood to be the real part of the complex quantity.

The real and imaginary parts don't get mixed up by the circuit equations ***as long as the circuit is linear.***

$$I = I_0 \cos(\omega t - \tfrac{\pi}{2}) = \Re\left\{ -j \cdot I_0 e^{j\omega t} \right\} \qquad \left(-j = -\sqrt{-1} = e^{-j\frac{\pi}{2}} \right)$$

$$V = \omega L \cdot I_0 \cos \omega t = \Re\left\{ j\omega L \cdot (-j I_0 e^{j\omega t}) \right\}$$

Considering *I* and *V* to be complex numbers:

⇒ $V = I \cdot Z$ $\boxed{Z = j\omega L}$ "Impedance"

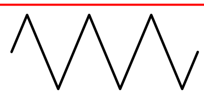
Summary

All 3 linear circuit elements provide an "impedance" (Z) to the flow of current, but one has to specify a phase difference between current and voltage as well as a change in amplitude.

Assume that the voltage is given by $V(t) = V_0 e^{j\omega t}$

$$I(t) = I_{\max} e^{j\phi} e^{j\omega t}$$

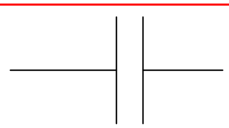
I and V in phase



$$I(t) = \frac{V_0}{R} e^{j\omega t}$$

$$Z = R \text{ and } \phi = 0$$

I leads V by 90°

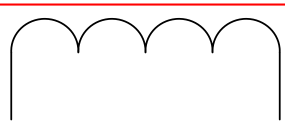


$$I(t) = \frac{V_0}{1/j\omega C} e^{j\omega t}$$

"Reactance:"

$$Z = X_C = \frac{1}{j\omega C} \text{ and } \phi = +\frac{\pi}{2}$$

I lags V by 90°

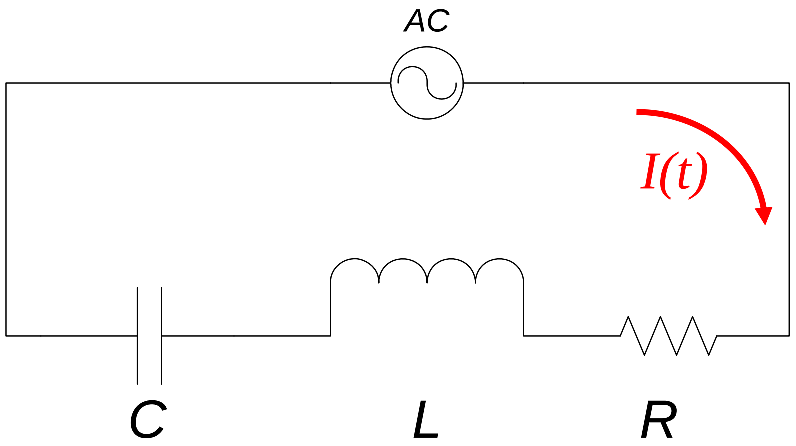


$$I(t) = \frac{V_0}{j\omega L} e^{j\omega t}$$

$$Z = X_L = j\omega L \text{ and } \phi = -\frac{\pi}{2}$$

Series LCR Circuit (Physics 5C)

$$V_s(t) = V_0 \cos \omega t = \Re(V_0 e^{j\omega t})$$



$$I(t) = \Re\left(I_{max} e^{j(\omega t + \phi)}\right)$$

$$I_{max} = \frac{V_0}{|Z|}$$

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

is minimum when $\omega L = \frac{1}{\omega C}$

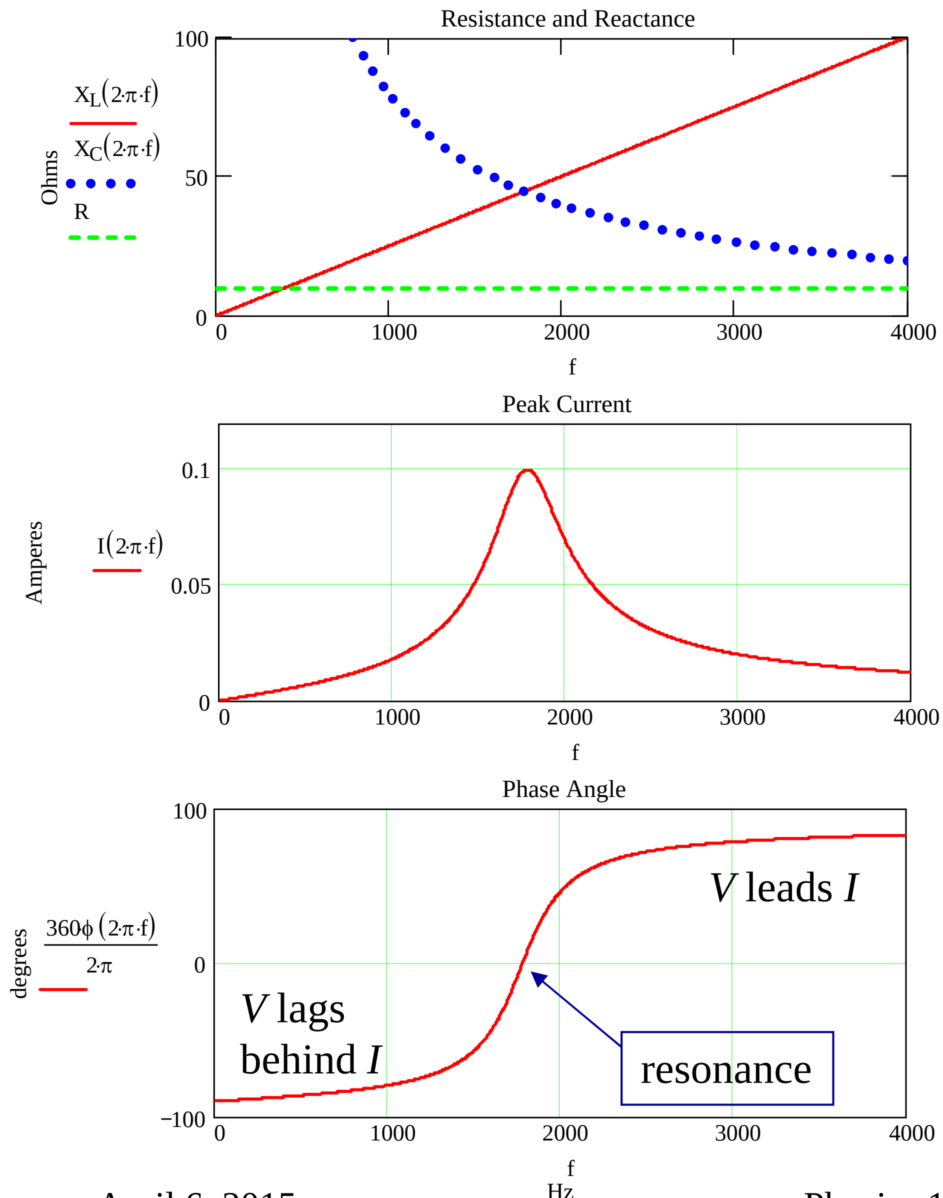
$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\phi = -\tan^{-1}\left(\frac{\omega L - 1/\omega C}{R}\right) = 0$$ *when* $\omega = \frac{1}{\sqrt{LC}}$ → $I(t) = \frac{V_0}{R} \cos \omega t$

Maximum current

http://vnatsci.ltu.edu/s_schneider/physlets/main/rlc.shtml

Resonance in a Series LCR Circuit



At resonance:

1. *The inductive and capacitive reactances are equal in magnitude.*
2. *The current is maximum.*
3. *The phase angle between voltage and current is zero.*

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

At low *f*, the circuit looks *RC*.
At high *f*, the circuit looks *RL*.

Power

$$P(t) = I(t) \cdot V(t)$$

But usually we measure the *average* power.

$$\overline{P} = \frac{1}{T}\int_0^T I(t) \cdot V(t)\,dt = \frac{1}{2}\Re\left(IV^*\right) = I_{\text{rms}} V_{\text{rms}} \cos\phi$$

Complex amplitudes

"Power factor"

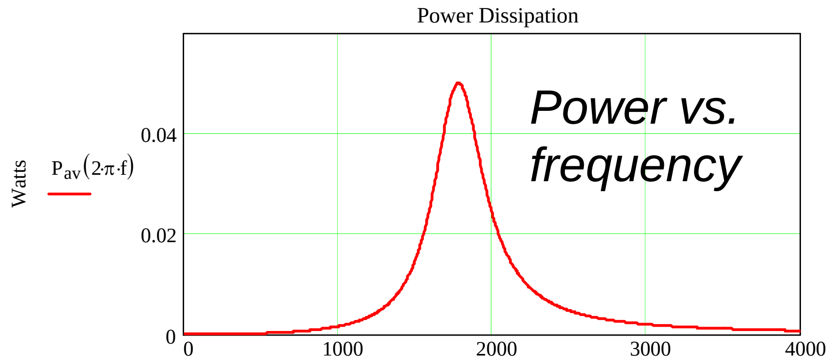
Where phi is the phase difference between the voltage and current.

$\phi = 0$ for resistor (max. power)

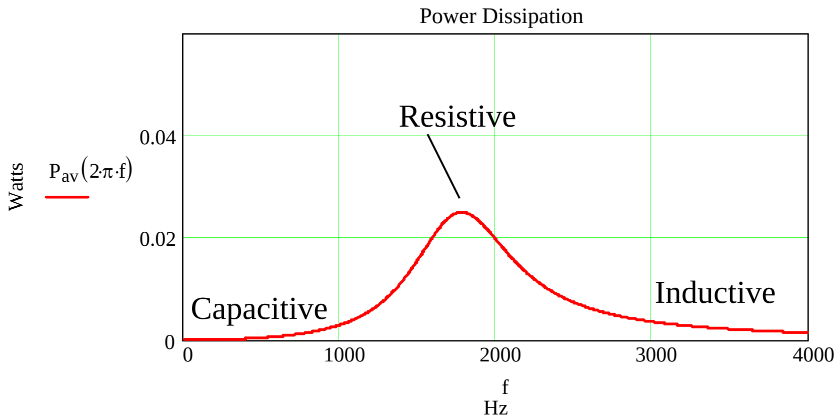
$\phi = \pi/2$ for capacitor (zero power)

$\phi = -\pi/2$ for inductor (zero power)

Resonance in a Series LCR Circuit

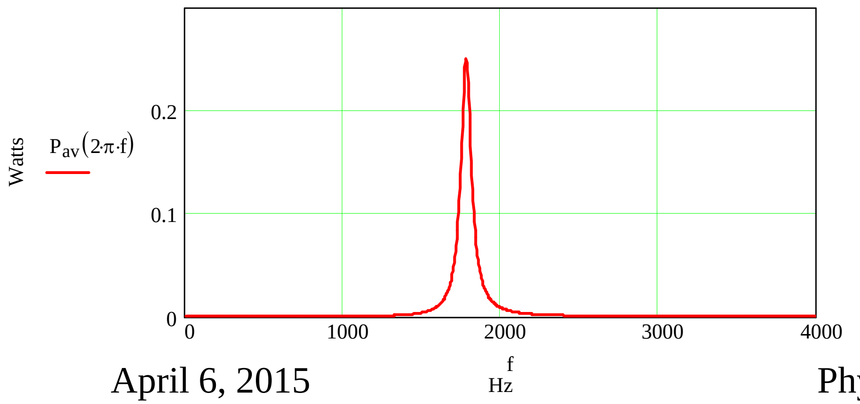


R = 10 ohms



R = 20 ohms

Large damping gives a broad low resonance.

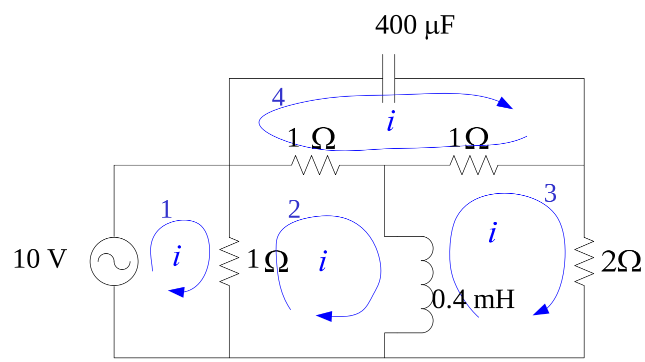


R = 2 ohms

Small damping gives a sharp, high resonance (good for a radio receiver).

More Elaborate LRC Networks

- *Straight-forward application of complex impedance, resulting in simultaneous complex equations.*
  - *Easy to solve with a computer (but messy by hand).*
  - *See example in handout.*
- *But for most of this course we really only need the two simple RC filters!*



V=10 V

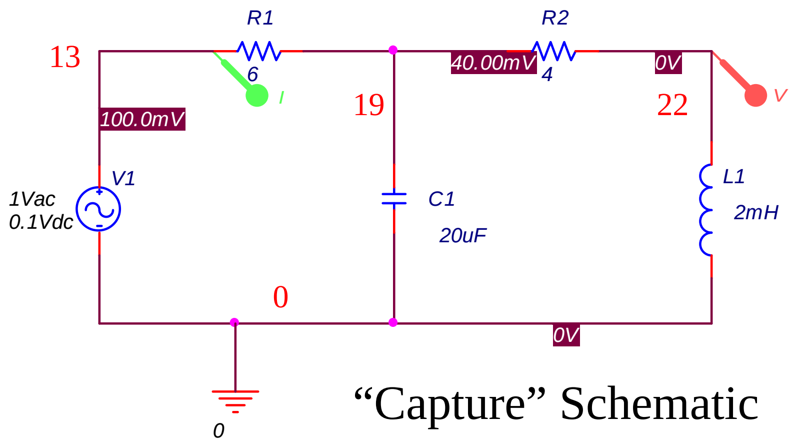
$\omega=10^4$ rad/s

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & -2-4j & 4j & 1 \\ 0 & 4j & -3-4j & 1 \\ 0 & 1 & 1 & -2+0.25j \end{pmatrix} \times \begin{pmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow I = \begin{pmatrix} 15.457-1.787j \\ 5.457-1.787j \\ 4.990+0.652j \\ 5.213+0.084j \end{pmatrix}$$

Impedance seen by the source: $Z_{\mathrm{eq}} = \frac{V}{i_1} = \frac{10}{15.56} \cdot e^{+j0.037\pi}$

AC (linear) Analysis with PSpice

- *See the PSpice tutorial and HW #1.*



"Capture" Schematic

PSpice Netlist

V_V1	*N00013*	*0*	*DC 0.1Vdc*	*AC 1Vac*
R_R1	*N00013*	*N00019*	*6*	
R_R2	*N00019*	*N00022*	*4*	
L_L1	*0*	*N00022*	*2mH*	
C_C1	*0*	*N00019*	*20uF*	

$$\frac{1}{2\pi\sqrt{LC}} = 796\ \text{Hz}$$

PSpice Output

(AC Analysis)