RC Circuit: Charging

This is an example of a “transient” response. The current changes with time after the switch closes and eventually reaches a constant value of zero.

For AC circuits, we are not interested in the transient response, but instead we consider only the behavior after the circuit has been on for a long time.

The charge on the capacitor is given by:

\[ Q(t) = CV_0 \cdot \left(1 - e^{-t/RC}\right) \]

The current is given by:

\[ I(t) = I_0 \cdot e^{-t/RC} \]
Kirchhoff’s loop rule holds at every instant in time:

\[ \mathcal{E}(t) = v_R(t) + v_C(t) \]

We want to find the resulting current, which generally will not be in phase with the voltage:

\[ i(t) = I_{\text{max}} \cos(\omega t + \phi) \]

We have to find both the amplitude \( I_{\text{max}} \) and the phase \( \phi \).

\[ v_R = I_{\text{max}} R \cos(\omega t + \phi) \]

The resistor voltage is in phase with current

\[ v_C = I_{\text{max}} \frac{1}{\omega C} \cos\left(\omega t + \phi - \frac{\pi}{2}\right) \]

The capacitor voltage lags behind the current by 90 degrees.

Assume that the source has been turned on for a long time, so all transients have died out.
The equation for the voltage across a resistor and capacitor in an RC circuit is:

\[ \mathcal{E}(t) = v_R(t) + v_C(t) \]

where \( V_R = \mathcal{E}_0 \cos \omega t \) and \( V_C = I_{\text{max}} \frac{1}{\omega C} \cos(\omega t + \phi - \frac{\pi}{2}) \).

This equation can be solved for both \( I_{\text{max}} \) and \( \phi \) by using trig identities, but it is easier to do it graphically using phasors. The algebra then just looks like vector addition.

\[ \mathcal{E}_0 = \sqrt{V_R^2 + V_C^2} = I_{\text{max}} \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2} \]

\[ \tan \phi = \frac{V_C}{V_R} = \frac{1}{\omega RC} \]

As in a circuit with just a capacitor, the voltage lags behind the current, but by less than 90 degrees. \( \phi > 0 \)

Note: \( Z = \sqrt{R^2 + X_C^2} \)

\[ \mathcal{E}_0 \]

\[ \omega \]

\[ \phi \]

\[ V_R \]

\[ V_C \]

\[ I_{\text{max}} \]
Problem 35-36

a) Evaluate $V_R$ at emf frequencies 100, 300, 1000, 3000, and 10,000 Hz

b) Graph $V_R$ vs frequency.

This is an example of a “high pass filter.”

Note that for $f=0$ (DC) the current must be zero, and therefore $V_{out}$ is 0.
Problem 35-36

The output (voltage across resistor) is in phase with the current, which leads the input voltage by an angle

\[ \phi(f) := \text{atan} \left( \frac{1}{2\pi f RC} \right) \]

Low \( f \), the capacitor impedance dominates, and current leads voltage by 90°

High \( f \), the resistor impedance dominates, and current is in phase with \( V_{in} \)

\[
V_{out}(f) := V_{in} \sqrt{\frac{R}{R^2 + \left( \frac{1}{2\pi f C} \right)^2}}
\]

\[
V_{out}(100) = 0.994 \quad V_{out}(1000) = 7.068 \quad V_{out}(10000) = 9.95
\]

\[
V_{out}(300) = 2.871 \quad V_{out}(3000) = 9.486
\]
Problem 35-36 continued.

This plot demonstrates for one frequency (1500 Hz) how the resistor and capacitor voltages add together at all times to yield the voltage of the source.

\[
\begin{align*}
R &= 100 \\
C &= 1.59 \cdot 10^{-6} \\
V_{in} &= 10 \\
\omega &= 2\pi \cdot 1500 \\
V(t) &= V_{in}\cos(\omega \cdot t) \\
I_{max} &= \frac{V_{in}}{\sqrt{R^2 + \left(\frac{1}{\omega \cdot C}\right)^2}} \\
\phi &= \arctan\left(\frac{1}{\omega \cdot R \cdot C}\right) \\
\phi &= 33.716 \text{ deg} \\
I(t) &= I_{max}\cos(\omega \cdot t + \phi) \\
V_R(t) &= I_{max} \cdot R \cdot \cos(\omega \cdot t + \phi) \\
V_C(t) &= I_{max} \frac{1}{\omega \cdot C} \cdot \cos\left(\omega \cdot t + \phi - \frac{\pi}{2}\right) \\
\end{align*}
\]
Problem 35-37

a) Evaluate $V_C$ at emf frequencies 1, 3, 10, 30, and 100 kilo-Hertz

b) Graph $V_C$ vs frequency.

This is an example of a “low pass filter.”

Note that for $f=0$ (DC) the current must be zero, and therefore $V_{out}$ is $V_{in}$. 
Problem 35-37

The voltage across the capacitor lags behind the current by $90^\circ$, so $V_{\text{out}}$ has a phase angle

\[ \phi(f) := \text{atan} \left( \frac{1}{2\pi f R C} \right) - \frac{\pi}{2} \]

Low $f$, the capacitor looks like an open circuit, so $V_{\text{out}}$ is the same as $V_{\text{in}}$.

High $f$, the resistor dominates, so the current is in phase with $V_{\text{in}}$, while $V_{\text{out}}$ lags by $90^\circ$.
It is very common to plot frequency on a log scale. In that case, the same phase-angle plot looks as shown here.

The point where the phase angle is $-45^\circ$ is called the cross-over frequency. It occurs where

$$\omega_c = \frac{1}{RC} = 62.5 \text{ kHz or } f = 10 \text{ kHz}$$
Kirchhoff’s rule: \( V(t) + \mathcal{E} = 0 \quad \mathcal{E} = -L \frac{dI}{dt} \)

\[
V = L \frac{dI}{dt}
\]

\[
I(t) = \frac{1}{L} \int V(t) dt
\]

\[
I(t) = \frac{V_0}{L} \int \cos \omega t \, dt
\]

\[
I(t) = \frac{1}{\omega L} V_0 \sin \omega t
\]

\[
I(t) = \frac{V_0}{\omega L} \cdot \cos(\omega t - \frac{\pi}{2})
\]

Energy is stored, NOT dissipated!
Phasors for Inductive Circuit

(a) $v_L$ and $i_L$

- $i_L$ peaks $\frac{1}{4}T$ after $v_L$ peaks.
- We say that the current lags the voltage by $90^\circ$.

(b) The current phasor lags the voltage phasor by $90^\circ$. 
Reactance

All 3 elements provide an “impedance” \( Z \) to the flow of current, but one has to specify a phase difference between current and voltage as well as a change in amplitude.

Assume that the voltage is given by

\[
V(t) = V_0 \cos \omega t
\]

\[
I(t) = I_{\text{max}} \cos(\omega t + \phi)
\]

- \( I \) and \( V \) in phase
- \( I \) leads \( V \) by 90°
- \( I \) lags \( V \) by 90°

\[
I(t) = \frac{V_0}{R} \cos \omega t
\]

\[
Z = R \text{ and } \phi = 0
\]

\[
I(t) = \frac{V_0}{1/\omega C} \cos(\omega t + \frac{\pi}{2})
\]

\[
Z = X_C = \frac{1}{\omega C} \text{ and } \phi = +\frac{\pi}{2}
\]

\[
I(t) = \frac{V_0}{\omega L} \cos(\omega t - \frac{\pi}{2})
\]

\[
Z = X_L = \omega L \text{ and } \phi = -\frac{\pi}{2}
\]
RL Circuit with AC Source

- Problem 35-47.

The current is lagging behind the voltage (negative phase). $\phi < 0$

\[ V_0 = \sqrt{V_R^2 + V_L^2} = I_{\text{max}} \sqrt{R^2 + (\omega L)^2} \]

\[ \phi = -\tan^{-1} \frac{V_L}{V_R} = -\tan^{-1} \frac{\omega L}{R} \]

Note: $Z = \sqrt{R^2 + X_L^2}$
Problem 35-47, continued

\[ i(t) = \frac{V_0}{\sqrt{R^2 + (\omega L)^2}} \cos(\omega t + \phi) \quad \phi = -\tan^{-1} \frac{\omega L}{R} \]

\[ v_R(t) = \frac{V_0 \cdot R}{\sqrt{R^2 + (\omega L)^2}} \cos(\omega t + \phi) \]

\[ v_L(t) = \frac{V_0 \cdot \omega L}{\sqrt{R^2 + (\omega L)^2}} \cos(\omega t + \phi + \frac{\pi}{2}) \]

Part b) \( V_R \) goes to zero as the frequency goes to infinity.

Part c) If \( V_{out} = V_R \), then this is a low pass filter

Part d) The cross-over frequency occurs when \( R = X_L \), or equivalently \( V_R = V_L \). At that point, the output voltage is down by \( 1/\sqrt{2} \) and the phase angle is \( -45^\circ \).

\[ \omega_c = \frac{1}{R/L} \]