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Twisted Reduced Chiral Models

Based on:

- S.P. master thesis "*Modelli Chirali Ridotti nel limite a grande N*" (supervisor E. Vicari, 24th Sept. 2001, University of Pisa)
- S.P. and E. Vicari, "*Twisted Eguchi-Kawai Reduced Chiral Models*", JHEP **05** (2002) 014
- S.P., "*Noncommutative Principal Chiral Models*", JHEP **10** (2002) 035

Trieste, 16th January 2003

Lando Caiani Memorial Prize 2002

Summary of the talk

- Large N limit

  and Reduced Models

- A case of *falsification*:

  2d Reduced Chiral Models

- Perspectives: Noncommutative

  Field Theories

The large N limit

Many field theories, whose internal symmetry groups depend on a parameter N, are studied in the limit $N \to \infty$

't Hooft (1974): $SU(3) \ \to \ SU(N)$ for QCD

't Hooft limit (*planar limit*)

$N \to \infty$ with N_f, $g^2 N$ fixed

⇒ **Hadronic phenomenology**
(qualitative features of mesons, glueballs and baryons)

⇒ **Analytical simplifications** (*factorization*):
the number of algebraically independent correlation functions decreases

The absence of an explicit solution for unitary matrix models, even in the large N limit, has lead to various approaches, such as

Reduced Models

Reduced Models

Eguchi-Kawai (EK, 1982):

$$U_\mu(x) \;\rightarrow\; U_\mu$$

in the Wilson lattice action

(*suppression of the space-time dependence*)

Motivations:

- Possible *analytical insights*
- Greater *computational accessibility*

Demonstration of the validity of the prescription:

$\boxed{\text{Schwinger-Dyson equations}}$

(hypothesis: the $[U(1)]^d$ symmetry of the models is not *spontaneously broken*, **FALSE** in $d = 4$)

$$\Downarrow$$

- New Reduced Models, e.g. **Twisted EK**

  (Gonzalez-Arroyo and Okawa, 1983)

- Generalization of the reduction procedure to other theories (Parisi et al., 1984)

Reduced Models: TEK

The reduced theory possesses an *Associated Lattice*

Translations are implemented via

d $SU(N)$ matrices Γ_i obeying the

't Hooft Algebra $\Gamma_\mu \Gamma_\nu = (Z_{\mu\nu}) \Gamma_\nu \Gamma_\mu$

where $\quad Z_{\mu\nu} \;=\; \exp\left(\frac{2\pi\imath}{N} n_{\mu\nu}\right)$

and $n_{\mu\nu}$ is the $d \times d$ antisymmetric **twist**

e.g., in $d = 2$:

$$U(x) \rightarrow D(\vec{x})\ U\ D^\dagger(\vec{x})$$

$$\vec{x} \;=\; (x_1, x_2) \qquad D(\vec{x}) \;=\; \Gamma_1^{x_1} \Gamma_2^{x_2} \qquad \Gamma_i^N = \mathbf{1}$$

$D(\vec{x})$ is the Translation Operator
on the Associated Lattice

Validity of the Model

Functions of the field variables are *translated* as

$$f[U(x)] \ \rightarrow \ f[D(x)\ U\ D^{\dagger}(x)]$$

The **validity** of the reduction prescription means

$$\boxed{\lim_{N\to\infty} \langle f[U(x)]\rangle_{\text{original th.}} \ = \ \langle f[D(x)\ U\ D^{\dagger}(x)]\rangle_{TEK}}$$

$$\langle f[D(x)\ U\ D^{\dagger}(x)]\rangle_{TEK} \ = \ \frac{1}{Z_{TEK}} \int \mathrm{d}U\ f \cdot e^{-S_{TEK}}$$

TEK has been applied in various contexts, e.g.:

- $4d$ Lattice Gauge Theories
- $2d$ Principal Chiral Models
- Superstring Matrix Theories (IKKT theory)
- Noncommutative (Gauge) Theories

Motivations and Interest

- **Validity check** of the TEK approach in the case of $2d$ Chiral Models:
  - *Analytical and numerical high precision data about the parent theory*
  - *Computational tractability (the only d.o.f. is one $SU(N)$ matrix)*

  $\Rrightarrow$ More stringent test than in the case of the reduced lattice gauge theory

- Preliminary investigation of the relevance of twisted reduced models in the context of **Noncommutative Field Theories** on the lattice

Chiral Models in $d = 2$

The continuum action reads (Parent theory):

$$\boxed{S = \frac{N}{2g^2} \int \mathrm{d}^2 x \; \mathrm{Tr} \left(\partial_\mu U(x) \partial^\mu U^\dagger(x) \right)} \qquad U(x) \in SU(N) \; \forall x$$

The lattice action reads (Parent theory):

$$\boxed{S_L = -2N\beta \sum_{x,\mu} \mathrm{Tr} \left[U_x U^\dagger_{x+\mu} + \text{ h.c.} \right]}$$

What we know about the Parent Theory:

- Strong coupling expansions ($\beta \to 0$)
- Weak coupling expansions ($\beta \to \infty$)
- Monte Carlo simulations

The action of the Reduced Models reads:

$$\boxed{S_{TEK} \;=\; -\beta N \sum_{\mu=1}^{2} \mathrm{Tr} \left[U \Gamma_\mu U^\dagger \Gamma^\dagger_\mu \;+\; \text{h.c.} \right]}$$

Symmetries of the Red. Model

$$U \;\to\; z \cdot U, \qquad z \in \mathbb{Z}_N \tag{1}$$

$$U \;\to\; D(x)\; U\; D^{\dagger}(x) \tag{2}$$

(1) $\to$ Residual symmetry from the original Model:

$$SU(N)_L \times SU(N)_R \;\to\; \mathbb{Z}_N$$

(2) $\to$ Translational symmetry on the associated lattice

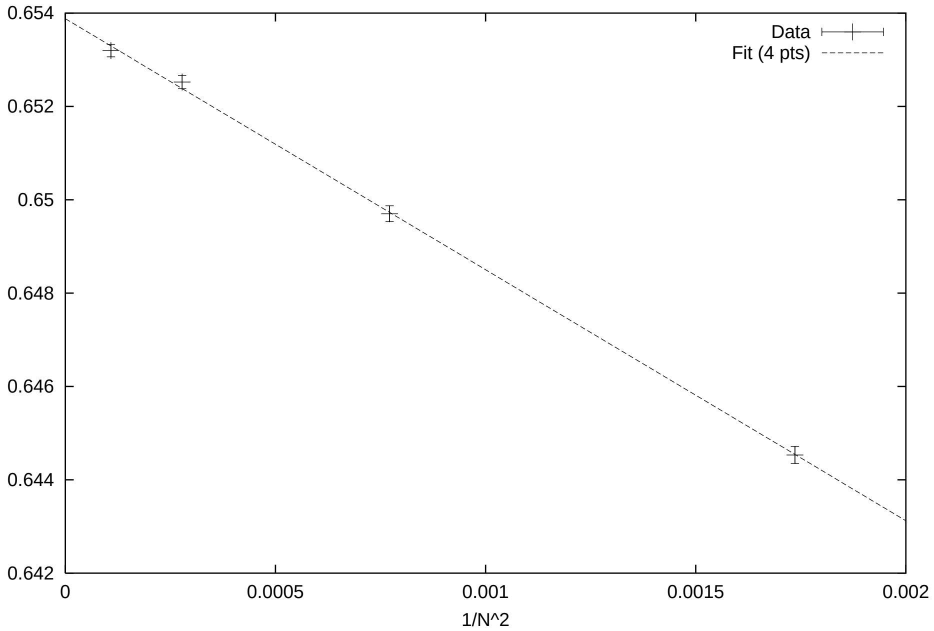
Equivalence of TEK Reduced Model and Original Non Reduced Theory

Computation of the *Schwinger-Dyson equations* for the original and the reduced theories

If (2) **is not spontaneously broken the two sets of SD equations correspond**

Numerical Analysis (I)

- We verified the **validity** of the reduction procedure in the *strong coupling* limit ($\beta \to 0$)



Internal Energy at $\beta = 0.28$

$$\boxed{0.6538(1) \to \text{Reduced Theory}}$$

$$\boxed{0.65373(4) \to \text{Non Reduced Theory}}$$

- The **numerical accuracy** was always found to be worse than in the Parent theory

Numerical Analysis (II)

• We found evidence of the existence, in the Reduced theory, of the large N phase transition occurring in the Parent theory

$$\boxed{\beta_c = 0.3057(3) \text{ Monte Carlo, Non Reduced Th.}}$$

$$\boxed{\beta_c = 0.3058(3) \text{ S.C. series, Non Reduced Th.}}$$

$$\boxed{\beta_c = 0.3061(3) \text{ Reduced Model}}$$

• In the *weak coupling* regime ($\beta > \beta_c$) the expected correspondence between Reduced Model and Parent theory **breaks down**:

- We found observables (e.g. magnetic susceptibility) which show ill defined large N limits

- The correlation functions do not reproduce the expected *exponential behaviour*

- The finite N corrections to observables do not show the *expected scaling* $\propto 1/N^2$
  $\Rightarrow$ ambiguous large N limit

A case of falsification

- We verified that the symmetry invoked for the correspondence of the SD sets of eqns. is **not spontaneously broken** in the Reduced theory

$$\Downarrow$$

- ➪ In the weak coupling regime the TEK reduction prescription is not valid (or its large N limit is singular)
- ➪ The theoretical arguments on which the correspondence is based (SD eqns.) are found to be *not conclusive*
- ➪ The Reduced theory, in the weak coupling regime, could reproduce another model, namely the Noncommutative $U(1)$ lattice chiral model

TEK Reduced Models and Noncommutative Field Theory

$\boxed{\text{Morita equivalence}}$

Reduced Models turn out to be a
non-perturbative matrix regularization of
noncommutative gauge theories

Large N limit $\Leftrightarrow$ Continuum limit

Proper Definition of
Continuum NC Theories
(UV/IR mixing)

$+$

Nonperturbative Renormalizability

NC U(1) Chiral Model

Theory	*TEK model*	*NC lattice*
D.o.f.	U(N)	$\star$U(1)
Lattice	1×1	$N \times N$

The unitarity condition on U is translated into the *star unitarity* of $\mathcal{U}(x)$:

$$\mathcal{U}(x) \star \mathcal{U}(x)^* = \mathcal{U}(x)^* \star \mathcal{U}(x) = 1$$

where the lattice star product is defined as

$$\mathcal{A} \star \mathcal{B} \equiv \frac{1}{N^2} \sum_y \sum_z \mathcal{A}(x+y)\mathcal{B}(x+z) e^{2i(\theta^{-1})_{ij} y^i z^i}$$

and the (dimensionful) noncommutative parameter

$$\boxed{\theta_{\mu\nu} = \frac{a^2 N}{\pi} n_{\mu\nu}}$$

The lattice star product reproduces the usual Moyal product when the lattice spacing $a \to 0$

Double Scaling Limit

Fixing the noncommutative parameter θ
amounts to fixing $a^2 N$

$\boxed{\text{Therefore a } \textbf{double scaling limit} \text{ is required}}$

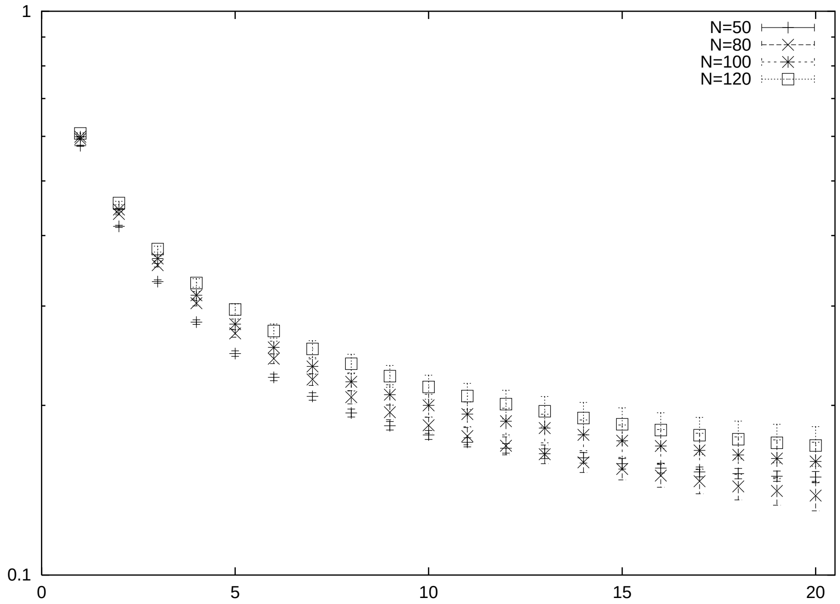
$$N \to \infty \quad \beta \to \infty \quad a^2(\beta)N = \text{fixed}$$

This means keeping into account also
non-planar contribution in the large N limit.

What we expect, if the NC theory is renormalizable,
is that the correlation functions exhibit an
asymptotic convergence in the double scaling limit

Results and Perspectives

Numerical results on correlation functions suggest
that the U(1) 2d chiral theory is
well defined in the continuum limit



Numerical results for the 2-points function $\mathcal{G}(n)$

These results provided the first numerical evidence of nonperturbative renormalizability of a NC field theory

In a recent work we studied the 3-dimensional case of U(1) gauge theory, proposing a matrix formulation for lattice fermions
(F. Bazzocchi, M. Cirafici, C. Maccaferri and S.P., `hep-th/0211060`)

Numerical analysis of NC 3D lattice QED is currently being performed