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Matrix Models, $2d$ Gravity and CFT

Abstract

Matrix Models of $2d$ gravity provide important non-perturbative information, to be compared with known results from CFT analysis

After stating the general problem, we will discuss why one expects conformal invariance, in a certain limit, in the random surfaces approach to $2d$ gravity. Then, we will review the matrix models strategy

The one-matrix model solution technique is shortly presented, and it is shown how to compute a scaling exponent of the theory

The same scaling exponent can be calculated in the context of the Liouville theory, and the agreement confirms that the continuum limit of matrix models is a suitable definition for the continuum problem of interest

Finally, we will briefly present the Multi-matrix models and their relevance for the gravitationally dressed Ising model

Statement of the problem

In string theory one wants to compute the all genus partition function

$$Z \;=\; \sum_h \int [\mathcal{D}g][\mathrm{d}X]e^{S(g,X)}$$

$$S \;\equiv\; S_{\text{2d gravity}} \;+\; S_{\text{matter}}$$

Starting with the case of $2d$ *pure gravity* ($S_{\text{matter}} \equiv 0$), i.e. a theory of surfaces, the part. func. takes the form:

$$Z(\beta_0, \gamma_0) = \sum_h Z_h(\beta_0, \gamma_0) \equiv \sum_h \int [\mathcal{D}g]e^{-\beta_0 A + \gamma_0 \chi}$$

$$A(\Sigma) = \int_\Sigma \sqrt{g} \qquad \chi(\Sigma) = \frac{1}{2\pi}\int_\Sigma \sqrt{g}R = 2 - 2h$$

It is possible to compute the p.f. at fixed values of the area, postponing the integration over A

$$Z_h(\beta_0, \gamma_0, A) = \mathrm{Vol}(h, A)\exp[-\beta_0 A(\Sigma) + \gamma_0 \chi(\Sigma)]$$

$\mathrm{Vol}(h, A)$ is either defined by this formula or computed indipendently

A possible regularization scheme is to consider *random triangulations*, modulo topological equivalence classes, via

$$\sum_{\text{genus}} \int [\mathcal{D}g] \to \sum_{\text{triangulations}}$$

We choose to use equilateral triangles of fixed area ε

$N_i \ \to$ number of incident triangles at vertex i

$$\sigma_i = N_i \varepsilon / 3 \ \Leftrightarrow \ \sqrt{g} \qquad A \ = \ \textstyle\sum_i \sigma_i \ = \ n \cdot \varepsilon$$

$$R = R_i \delta^2(z - z_i) = \pi(6 - N_i)/N_i \varepsilon$$

$$\textstyle\int \sqrt{g} R \to \sum_i 2\pi(1 - N_i/6) = 2\pi(V - \frac{1}{2}F)$$

$$= 2\pi(V - E + F) = 2\pi\chi$$

Let $V(h, n)$ be the number of equivalence classes of triangulations of Σ with n triangles

In the relevant large n limit it is known that

$$V(h, n) \sim e^{cn} n^{\tilde{\gamma}\chi - 1} b_h [1 + O(1/n)]$$

Regarding ε as a cutoff when $n = A/\varepsilon$, it makes sense to approximate Vol(h, A) by $V(h, n)$

In the $\varepsilon \to 0$, at fixed area A, because $\sum_n \sim \int \mathrm{d}A/\varepsilon$

$$\mathrm{Vol}_\varepsilon(h, A) \sim \frac{1}{\varepsilon} e^{cA/\varepsilon} \left(\frac{A}{\varepsilon}\right)^{\tilde{\gamma}\chi - 1} b_h [1 + O(\varepsilon)]$$

where $\tilde{\gamma} < 0$ for $h > 1$

$$Z_{\varepsilon,h}(A) = \mathrm{Vol}_\varepsilon(h, A) \exp[-\beta_0 A + \gamma_0 \chi] =$$

$$\tfrac{1}{\varepsilon} e^{cA/\varepsilon} \left(\tfrac{A}{\varepsilon}\right)^{\tilde{\gamma}\chi - 1} b_h \exp[-\beta_0 A + \gamma_0 \chi] =$$

$$b_h A^{\tilde{\gamma}\chi - 1} \exp[A \left(\tfrac{c}{\varepsilon} - \beta_0\right) - \chi \left(\tilde{\gamma} \ln \varepsilon - \gamma_0\right)]$$

The renormalization of the parameters should imply the convergence of $Z_{\varepsilon,h}(A)$ to a well defined function $Z_h(A)$

This implies choosing

$$\beta_0 \to \beta_c = c/\varepsilon \qquad \gamma_0 \to \gamma_c = -\tilde{\gamma} \ln(A_0/\varepsilon)$$

where A_0 results from renormalization ambiguity

$$Z_h(A) = \frac{1}{A} \left(\frac{A}{A_0}\right)^{\tilde{\gamma}\chi} b_h$$

Absorbing $A_0^{-\tilde{\gamma}\chi}$ into b_h and integrating

$$Z(\beta_0, \gamma_0) \equiv \sum_h \int_0^\infty \mathrm{d}A\ Z_h(A) =$$

$$= \sum_h \int_0^\infty \mathrm{d}A A^{\tilde{\gamma}\chi - 1} b_h \exp\left(-(\beta_0 - \beta_c)A + (\gamma_0 - \gamma_c)\chi\right) =$$

$$= \sum_h \frac{\Gamma(\tilde{\gamma}\chi)}{(\beta_0 - \beta_c)^{\tilde{\gamma}\chi}} b_h \exp[(\gamma_0 - \gamma_c)\chi]$$

which can be rewritten as a function of only

$$Z(z) = \sum_h z^{\tilde{\gamma}\chi} f_h, \qquad z \equiv e^{-(\gamma_0 - \gamma_c)/\tilde{\gamma}}(\beta_0 - \beta_c)$$

The limit $\beta_0 \to \beta_c$ and $\gamma_0 \to \gamma_c$ at fixed z is known as **double scaling limit**

At β_c the partition function diverges, but in the d.s.l. the singularity is simultaneously offset for all genera h

One can therefore expect the *conformal symmetry* at this singularity of the partition function

Matrix Models

The sum over genus presented above is divergent, and it is not even Borel-summable ($f_h = \Gamma(\tilde{\gamma}\chi) b_h \propto (2h)!$)

However, the advantage of the 'random triangulations' is that the starting integrals can be *explicitly calculated* in their discretized form

The basic trick is to introduce some 'matrix integrals' and to interpret them as generating functionals of the random triangulations (*MM universality*)

Let M^i_j be an $N \times N$ hermitian matrix

Via Wick theorem one can write the perturbative contributions to $e^Z \equiv \int \mathrm{d}M \exp\left[-\frac{1}{2}\mathrm{Tr}M^2 - \frac{g}{N}\mathrm{Tr}M^4\right]$

$$\langle M^{i_1}_{j_1} \cdots M^{i_n}_{j_n} \rangle \equiv \int \mathrm{d}M \; e^{-\mathrm{Tr}M^2/2} M^{i_1}_{j_1} \cdots M^{i_n}_{j_n} =$$

$$\frac{\partial}{\partial J^{i_1}_{j_1}} \cdots \frac{\partial}{\partial J^{i_n}_{j_n}} \int \mathrm{d}M \; e^{-\mathrm{Tr}M^2/2 + \mathrm{Tr}JM}|_{J=0} = \frac{\partial}{\partial J^{i_1}_{j_1}} \cdots \frac{\partial}{\partial J^{i_n}_{j_n}} e^{\mathrm{J}^2/2}|_{J=0}$$

where J^i_j is a matrix source and the measure

$$\mathrm{d}M = \prod_i \mathrm{d}M^i_i \prod_{i<j} \mathrm{d}\mathbb{R}\mathrm{e}M^i_j \mathrm{d}\mathbb{I}\mathrm{m}M^i_j$$

We also ask the norm. $\int \mathrm{d}M e^{\mathrm{Tr}M^2/2} = 1$

The propagator $\langle M^i_j M^k_l \rangle = \delta^i_j \delta^k_l$ can be visualized in the usual double lines representation

This specify the dual *tetragonization* of the surface

As for the triangulations, one just has to set

$$\frac{1}{N}\mathrm{Tr}M^4 \;\rightarrow\; \frac{1}{\sqrt{N}}\mathrm{Tr}M^3$$

and rescale the variables $M \rightarrow M\sqrt{N}$

The N-dependence of a generic Feynman graph is

$$N^{V-E+F} \;=\; N^{\chi} \;=\; N^{2-2h}$$

A surface with n triangles and Euler ch. χ (whose area is $A = n \cdot \varepsilon$) is weighted with a factor $(-g)^n N^{\chi}$

We thus have some natural identifications

$$(-g) = e^{-\beta_0 \varepsilon} \qquad N = e^{\gamma_0}$$

Different *polygonizations* correspond to more generic potentials, whose coupling constants, in the eventual continuum limit, reflect additional degrees of freedom (coupled conformal matter)

Calculating MM integrals of the type

$$e^Z = \int \mathrm{d}M \; e^{-\mathrm{Tr}V(M)}$$

yields the perturbative definition of quantum $2d$ gravity, and may be able to provide non-perturbative information

We can expand the MM free energy in terms of both g and N

$$Z(g,N) = N^2 Z_0(g) + Z_1(g) + N^{-2} Z_2(g) + ... = \sum N^{2-2h} Z_h(g)$$

What was found above implies that one expects the following behaviour for the N leading term

$$Z_0(g) \sim \sum_n n^{\gamma-3}(g/g_c)^n \sim (g-g_c)^{2-\gamma}, \qquad \tilde{\gamma} = \frac{\gamma-2}{2}$$

In fact, from $\quad Z_0 \sim \int \mathrm{d}A \; A^{\tilde{\gamma}\chi-1} e^{A(\beta_c-\beta_0)}$

and from $\quad A = n \cdot \varepsilon$, $(-g) = e^{-\beta_0 \varepsilon}$, $(-g_c) = e^{-\beta_c \varepsilon}$

we have $\quad Z_0(g) \sim \sum_n n^{2\tilde{\gamma}-1}(g/g_c)^n$

The critical behaviour of the p.f. shows why and how the desired continuum limit is obtained at $g \to g_c$

$$\langle A \rangle = \varepsilon \langle n \rangle = \varepsilon \frac{\partial}{\partial g} \ln Z(g) \sim \frac{\varepsilon}{g - g_c}$$

MM Feynman graphs with a very large number of vertices dominate

MM integrals solution for Z_h also predict

$$Z_h(g) \sim \sum_n n^{(\gamma-2)\chi/2-1} (g/g_c)^n \sim (g_c - g)^{(2-\gamma)\chi/2}$$

Contributions from higher genus ($\chi < 0$) are enhanced as $g \to g_c$

If we take the two limits $N \to \infty$ and $g \to g_c$ together in a correlated manner, we may compensate the large N genus suppression with a $g \to g_c$ enhancement

Rewriting the singular piece of $Z_h(g)$ as

$$Z_h(g) \sim f_h (g - g_c)^{(2-\gamma)\chi/2}$$

we can express Z in terms of the fixed parameter κ ('renormalized' coupling constant)

$$\kappa^{-1} \equiv N(g - g_c)^{(2-\gamma)/2}$$

$$Z = \kappa^{-2} f_0 + f_1 + \kappa^2 f_2 + ... = \textstyle\sum_h \kappa^{2h-2} f_h$$

String Susceptibility

One of the properties derivable via the continuum Liouville approach is a "critical exponent" γ_{str}, defined as

$$Z(A) \;\sim\; A^{(\gamma_{\mathsf{str}}-2)\chi/2-1}$$

In the MM approach, a standard method is to use an infinite set of **orthogonal polynomials**

$$\int_{-\infty}^{+\infty} \mathrm{d}\lambda e^{-V(\lambda)} P_n(\lambda) P_m(\lambda) = h_n \delta_{n,m} \qquad \lambda \in \mathbb{R}$$

Some algebra shows that

$$e^Z = N! \prod_{i=0}^{N-1} h_i = N! h_0^N \prod_{k=1}^{N-1} f_k^{N-k} \qquad f_k \equiv h_k / h_{k-1}$$

In the large N limit $k/N \to \xi$, $0 < \xi < 1$ and (up to an irrelevant additive constant)

$$\frac{1}{N^2} Z = \frac{1}{N} \sum_k (1 - k/N) \ln f_k \sim \int_0^1 \mathrm{d}\xi (1-\xi) \ln f(\xi)$$

The particular $f(\xi)$ is model-dependent
It follows from some recursion relations between f_k's

The leading singular behaviour is however always of the type $f(\xi) - f_c \sim (g_c - g\xi)^{-\gamma}$

Therefore it is straightforward to prove that this γ is the string susceptibility $\gamma_{\rm str}$

$$\frac{1}{N^2} Z \sim \int_0^1 \mathrm{d}\xi (1-\xi)(g_c - g\xi)^{-\gamma} \sim (1-\xi)(g_c - g\xi)^{-\gamma+1}\Big|_0^1 +$$

$$+ \int \mathrm{d}\xi (g_c - g\xi)^{-\gamma+1} \sim (g_c - g)^{-\gamma+2} \sim \sum_n n^{\gamma-3} (g/g_c)^n$$

From above we know that $\gamma = 2\tilde{\gamma} + 2$, so that

$$Z(A) \sim A^{\tilde{\gamma}\chi - 1} \ \Rightarrow Z(A) \ \sim A^{(\gamma-2)\chi/2-1} \ \Rightarrow \ \gamma \equiv \gamma_{\rm str}$$

In the particular case where

$$V(\lambda) = \frac{1}{2g} \left(\lambda^2 + \frac{\lambda^4}{N} + b \frac{\lambda^6}{N^2} \right)$$

one gets $\gamma = 1/2$, corresponding to *pure gravity*

When a more generic potential is chosen, one obtain higher order critical behaviours, associated with $\gamma = -1/m$

String Susceptibility: Liouville theory

The Liouville action can be cast in the form

$$S_L = \frac{1}{8\pi}\int \mathrm{d}^2\xi\sqrt{g}\left(g^{ab}\partial_a\varphi\partial_b\varphi + QR\varphi\right), \qquad Q \equiv \sqrt{\frac{25-D}{3}}$$

and the partition function at fixed area

$$Z(A) = \int \mathcal{D}\varphi\mathcal{D}X\ e^S\delta\left(\int \mathrm{d}^2\xi\sqrt{g}e^{\alpha\varphi} - A\right) \quad \alpha \equiv \frac{Q}{2} - \frac{1}{2}\sqrt{Q^2-8}$$

where the action $S = S_L + S_{\mathrm{matter}} + S_{\mathrm{gh}}$

We define the string susceptibility by

$$Z(A) \sim A^{(\gamma-2)\chi/2-1}, \qquad A \to \infty$$

Under the shift $\varphi \to \varphi + \rho/\alpha$ the measure does not change; the change in the action is

$$\frac{Q}{8\pi}\int \mathrm{d}^2\xi\sqrt{g}R\varphi \to \frac{Q}{8\pi}\int \mathrm{d}^2\xi\sqrt{g}R\varphi + \frac{Q}{8\pi}\frac{\rho}{\alpha}\int \mathrm{d}^2\xi\sqrt{g}R$$

Using the Gauss-Bonnet formula

$$\frac{1}{4\pi}\int \mathrm{d}^2\xi\sqrt{g}R = \chi$$

and the identity $\delta(\lambda x) = \delta(x)/|\lambda|$

$$Z(A) \ = \ e^{-Q\rho\chi/2\alpha-\rho}Z(e^{-\rho}A)$$

We may now choose $e^{\rho} = A$, which gives

$$Z(A) = A^{-Q\chi/2\alpha-1}Z(1) = A^{(\gamma-2)\chi/2-1}Z(1)$$

and finally get

$$\gamma = 2 - \frac{Q}{\alpha} = \frac{1}{12}\left(D - 1 - \sqrt{(D-25)(D-1)}\right) = \frac{-2}{p+q-1}$$

in the case of minimal conformal field theories (p, q)

If we take $(p, q) = (2m-1, 2)$ we get the same critical behaviour as in the one matrix models

Multi Matrix Models (MMM)

The generalization of the one Matrix Model is the MMM

New degrees of freedom are represented by new matrices

A tractable form is the 'matrix chain'
(*Itzykson-Zuber*)

$$e^Z = \int \prod_{i=1}^{q-1} \mathrm{d}M_i \exp\left[- \mathrm{Tr} \left(\sum_{i=1}^{q-1} V_i(M_i) - \sum_{i=1}^{q-2} c_i M_i M_{i+1} \right) \right]$$

A specific example is the **two-matrix model**

$$e^Z = \int \mathrm{d}U \mathrm{d}V \exp\left[- \mathrm{Tr} \left(U^2 + V^2 - 2cUV + \frac{g}{N}(e^H U^4 + e^{-H} V^4) \right) \right]$$

where U and V are $N \times N$ hermitian matrices and
c and H are real parameters

The diagrammatic expansion contains two different types of 4-points vertices, corresponding to U or V

The 'thick' propagator is given by

$$\begin{pmatrix} 1 & -c \\ -c & 1 \end{pmatrix}^{-1} = \frac{1}{1-c^2} \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}$$

Similar vertices get a factor $1/(1-c^2)$
Different vertices a factor $c/(1-c^2)$

This is the structure needed to realize the
Ising model on a random lattice

$$Z = \sum_{\text{lattices}} \sum_{\text{spin config.}} \exp\left[\beta \sum_{<ij>} \sigma_i \sigma_j + H \sum_i \sigma_i\right]$$

To make contact between the constants c and β

$$e^{+\beta} \leftrightarrow \tfrac{1}{1-c^2} \qquad e^{-\beta} \leftrightarrow \tfrac{c}{1-c^2}$$

$$\Downarrow$$

$$e^{2\beta} = 1/c$$

The solution to this model is based on double series of orthogonal polynomials, and it yields

	$h[\varepsilon]$	$h[\sigma]$	ν	η	α	β	γ
Ising	$\frac{1}{2}$	$\frac{1}{16}$	1	$\frac{1}{4}$	0	$\frac{1}{8}$	$\frac{7}{4}$
Ising+$2d$ gravity	$\frac{2}{3}$	$\frac{1}{6}$	$\frac{3}{2}$	$\frac{2}{3}$	-1	$\frac{1}{2}$	2

The model is solvable even in the presence of a magnetic field

Over random lattices there is much more *symmetry*, since the details of any particular lattice (e.g. square) are effectively integrated out

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