PRACTICE FINAL #1

PROBLEM # 1

CONVERGENCE OF $\displaystyle\sum_{n=1}^{\infty}{}' \frac{1}{n \cdot n^{1/n}}$

NOTE THAT, CALLING $\dfrac{1}{n \cdot n^{1/n}} = a_n$

AND $\dfrac{1}{n} = b_n$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = \lim_{n\to\infty} \frac{1}{\exp\left(\frac{1}{n}\log n\right)} \underset{\text{l'HÔPITAL}}{=} \lim_{n\to\infty} \frac{1}{\exp\left(\frac{1}{n}\right)} = 1$$

BY THE SPECIAL COMPARISON CRITERION, GIVEN THAT $\sum_{n=1}^{\infty} 1/n$ DIVERGES AND THAT $a_n > 0$,

WE CONCLUDE THAT THE SERIES <u>DIVERGES</u>

PROBLEM # 2

$$-\underbrace{(4\pi\beta R^3)E}_{} = x^2 + x - x\left[1 + \frac{2x}{2} - \frac{(2x)^2}{8} + o(x^3)\right] =$$

TAYLOR SERIES OF $(1+x)^p$

$$= x^2 + x - x - x^2 + \frac{x^3}{2} + o(x^4)$$

$$= \frac{x^3}{2} + o(x^4)$$

THEREFORE

$$E = -\frac{k^3}{8\pi\beta} + o(k^4)$$

PROBLEM # 3

NOTE THAT $(1+i) = \sqrt{2}\, e^{i\left(\frac{\pi}{4} + 2\pi n\right)}$ $n=0,\pm1,\pm2,\ldots$

$$(1+i)^i = \exp\left(i \ln(1+i)\right) =$$

$$= \exp\left(i\left(\ln\sqrt{2} + i\left(\frac{\pi}{4} + 2\pi n\right)\right)\right) =$$

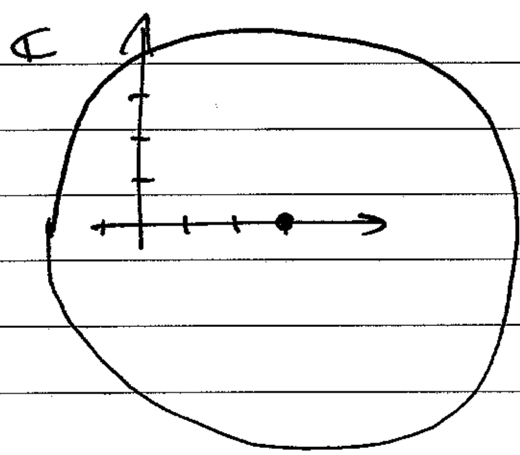
$$= e^{i \ln\sqrt{2}} \cdot e^{-\frac{\pi}{4}} \cdot e^{-2\pi n}, \quad n = 0, \pm1, \pm2, \ldots$$

PROBLEM # ~~3~~ 4

RATIO TEST: $\lim\limits_{n\to\infty} \dfrac{|z-3|^{n+1}}{(n+1)5^n\cdot 5} \cdot \dfrac{n\,5^n}{|z-3|^n} = \lim\limits_{n\to\infty} \dfrac{|z-3|}{(n+1)}\,\dfrac{n}{5} =$

$$= \frac{|z-3|}{5}$$

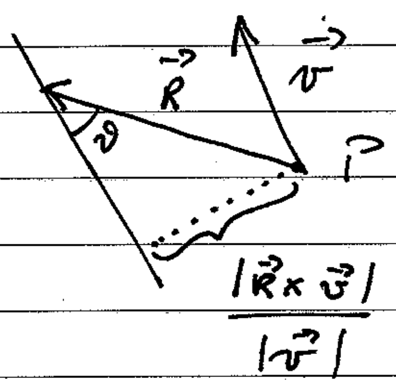
NOW, THE DISK OF CONVERGENCE IS A DISK OF RADIUS 5 CENTERED AT $z = 3$ $\left(|z-3| < 5\right)$



PROBLEM # 5

THE LINE JOINING $\vec{R_2}$ AND $\vec{R_3}$ IS PARALLEL TO THE VECTOR

$$\vec{v} = \vec{R_3} - \vec{R_2} = (0, -1, 2)$$



NOW, THE ⊥ DISTANCE FROM A POINT P TO A GIVEN LINE IS

$$d = |\vec{R}| \sin\vartheta = \frac{|\vec{R} \times \vec{v}|}{|\vec{v}|}$$

LET $\vec{R} = \vec{R_2} - \vec{R_1} = (0, -2, 1)$

$$\vec{R} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -2 & 1 \\ 0 & -1 & 2 \end{vmatrix} = -3\hat{i}$$

AND $d = \dfrac{|\vec{R} \times \vec{v}|}{|\vec{v}|} = \dfrac{3}{\sqrt{1+4}} = \dfrac{3}{\sqrt{5}}$

PROBLEM # 6

CONSIDER $\left(AB - BA\right)^{+} = (AB)^{+} - (BA)^{+} =$

$= B^{+}A^{+} - A^{+}B^{+} = BA - AB = -iC$

A, B HERMITIAN: $A = A^{+}$ AND $B = B^{+}$

BUT $\left(iC\right)^{+} = -iC^{+}$, SO $C = C^{+}$

PROBLEM # 7

YOU WILL NEED DO SHOW THIS IN THE FINAL!!

ASSOCIATED E. VALUE PROBLEM:

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 3 & 7-\lambda & -9 \\ 0 & 2 & -1-\lambda \end{vmatrix} = 0 \qquad \text{... ELEMENTARY ALGEBRA}$$

$$\ldots \lambda = 1, 2, 3, \quad \vec{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \vec{u}_2 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \quad \vec{u}_3 = \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}$$

AND $\vec{X} = c_1 \vec{u}_1 e^{t} + c_2 \vec{u}_2 e^{2t} + c_3 \vec{u}_3 e^{3t}$.

PROBLEM #8

THE j-TH COLUMN OF A IS GIVEN BY $A\hat{e}_j$, SO

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

$\det A = 2$, SO A IS NON SINGULAR

$$A\vec{v} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1+3 \\ 2+2+3 \\ -1+2 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \\ 1 \end{pmatrix}$$

PROBLEM #9

$$\begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0 \qquad \lambda^2 - 25 = 0 \qquad \lambda = \pm 5$$

E. VECTORS: $\frac{1}{\sqrt{5}}(2,1)$ AND $\frac{1}{\sqrt{5}}(-1,2)$, SO

$$S = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \; ; \quad S^t S = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$D = S^t A S = \underset{\text{ELEM. ALGEBRA}}{\ldots} = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix} \quad \checkmark$$

THAT YOU WILL NEED TO SHOW IN THE FINAL!

PROBLEM #10

LET $\ln \frac{1}{x} = u$, i.e. $x = e^{-u}$

$$\int_0^1 \frac{dx}{\sqrt{x \ln\left(\frac{1}{x}\right)}} = \int_\infty^0 e^{u/2} u^{-1/2} \left(-e^{-u} du\right) =$$

$$= \int_0^\infty u^{-1/2} e^{-u/2} du$$

NOW, LET $\frac{u}{2} = z$, TO WRITE

$$\int_0^1 \frac{dx}{\sqrt{x \ln(1/x)}} = \frac{2}{\sqrt{2}} \int_0^\infty z^{-1/2} e^{-z} dz = \sqrt{2}\,\Gamma\left(\frac{1}{2}\right) = \sqrt{2\pi}$$