1 Simple Harmonic Oscillator, Hamilton-Jacobi Approach

The difficulty of checking a solution goes like the reciprocal of finding the solution. Let me compute up front the total derivative of $S(q, \alpha, t)$,

$$dS = m\omega q \cos(\omega t) - \alpha \sin(\omega t) \frac{dq}{dt} + \left(m\omega \frac{\alpha \cos(\omega t) - q}{\sin(\omega t)} \frac{d\alpha}{dt} + \frac{m\omega^2}{2} \frac{2\alpha \cos(\omega t) - (q^2 + \alpha^2)}{\sin(\omega t)^2} \frac{dt}{dt} \right)$$

As with any generating function, the differential action

$$p \, dq - \mathcal{H} \, dt = \beta \, d\alpha - K \, dt + dS$$

must be preserved. The $dq$ term tells me, implicitly, how to transform the coordinate $\alpha$,

$$p = \frac{\partial S}{\partial q} = m\omega \frac{q \cos(\omega t) - \alpha}{\sin(\omega t)}$$

the $d\alpha$ terms tells me how to transform the momentum $\beta$,

$$\beta = -\frac{\partial S}{\partial \alpha} = -m\omega \frac{\alpha \cos(\omega t) - q}{\sin(\omega t)} \quad (1.1)$$

and the $dt$ term tells me how to transform the Hamiltonian $K$,

$$K = \mathcal{H}\left(q, p = \frac{\partial S}{\partial q}\right) + \frac{\partial S}{\partial t}$$

$$= \frac{1}{2m} \left[m\omega \frac{q \cos(\omega t) - \alpha}{\sin(\omega t)}\right]^2 + \frac{m\omega^2}{2} q^2 + \frac{m\omega^2}{2} \frac{2\alpha \cos(\omega t) - (q^2 + \alpha^2)}{\sin(\omega t)^2}$$

When I expand the square, all terms cancel out, leaving the new Hamiltonian identically zero. So $S(q, \alpha, t)$ indeed solves the Hamilton-Jacobi equation. Moreover, both $\alpha$ and $\beta$ and conserved, so I can invert the definition of $\beta$ (1.1) to find the physical coordinate

$$q(t) = \alpha \cos(\omega t) + \frac{\beta}{m\omega} \sin(\omega t)$$

which I recognize as the general solution to the simple harmonic oscillator. In particular, $\alpha = q(0)$ is the initial position and $\beta = m\dot{q}(0)$ is the initial momentum.

2 Damped Harmonic Oscillator, Canonical Approach

Part (a) The Hamiltonian

Newton’s Second Law directly yields the equation of motion,

$$-V'(q) - 2m\gamma \dot{q} = m\ddot{q}$$

(2.1)
Now I take the Lagrangian
\[ L(q, \dot{q}, t) = e^{2\gamma t} \left[ \frac{m\dot{q}^2}{2} - V(q) \right] \]
and compute the conjugate momentum
\[ p = \frac{\partial L}{\partial \dot{q}} = e^{2\gamma t} m\dot{q} \quad (2.2) \]
its rate of change
\[ \dot{p} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = e^{2\gamma t} (2\gamma m\dot{q} + m\ddot{q}) \]
and the generalized force
\[ \frac{\partial L}{\partial q} = -e^{2\gamma t} V'(q) \]
Clearly the resulting Euler-Lagrange equation
\[ \ddot{q} = -\frac{V'(q)}{m} - 2\gamma \dot{q} \]
is equivalent to Newton’s Second Law (2.1), so the proposed Lagrangian is valid. Next, to derive the Hamiltonian, I solve equation (2.2) for the velocity \( \dot{q} = \left( \frac{p}{m} \right) e^{-2\gamma t} \), and take the Legendre transform,
\[ H(q, p, t) = e^{2\gamma t} \left[ \frac{m\dot{q}^2}{2} + V(q) \right] - \gamma e^{-2\gamma t} qP \]
Part (b) A Constant of Motion  Given \( F_2(q, P, t) = e^{\gamma t} qP \), I compute its total derivative,
\[ dF_2 = e^{\gamma t} P dq + e^{\gamma t} q dP + \gamma e^{\gamma t} qP dt \]
and require the differential action
\[ pdq - \mathcal{H}dt = PdQ - Kdt + dF_2 \]
to be preserved. The caveat is \( F_2 \) has a \( dP \) rather than a \( dQ \), so I must subtract the total differential \( d(PQ) = PdQ + QdP \) from the right-hand side—that amounts to an integration by parts. Then I collect differentials,
\[ (p - e^{\gamma t} P) dq + (Q - e^{\gamma t} q) dP + (K - H - \gamma e^{\gamma t} qP) dt = 0 \]
and set each term to zero because \( q, P, t \) are independent variables. The \( dq \) term tells me the new momentum \( P = e^{-\gamma t} p \), the \( dP \) terms tells me the new coordinate \( Q = e^{\gamma t} q \), and the \( dt \) term tells me the new Hamiltonian,
\[ K = H + \gamma e^{\gamma t} qP = e^{-2\gamma t} \left[ \frac{P^2}{2m} + e^{2\gamma t} V(q) \right] + \gamma e^{\gamma t} qP \]
\[ K(Q, P, t) = \frac{P^2}{2m} + e^{2\gamma t} V(q) = Q e^{-\gamma t} + \gamma Q P \]
In the case of a harmonic oscillator potential

$$V(q) = \frac{m\omega^2 q^2}{2} \quad \rightarrow \quad V(q = Q e^{-\gamma t}) = \frac{m\omega^2}{2} e^{-2\gamma t} Q^2$$

the transformed Hamiltonian becomes independent of time,

$$K(Q, P) = \frac{P^2}{2m} + \frac{m\omega^2}{2} Q^2 + \gamma Q P$$  \tag{2.3}$$

Whenever the Hamiltonian has no explicit time dependence, it is a constant of motion.

**Part (c) The Solution** The Hamiltonian produces two canonical equations of motion,

\begin{align*}
\dot{Q} &= \frac{\partial K}{\partial P} = \frac{P}{m} + \gamma Q \quad \tag{2.4} \\
\dot{P} &= -\frac{\partial K}{\partial Q} = -m\omega^2 Q - \gamma P \quad \tag{2.5}
\end{align*}

Taken together with the statement that $K$ is conserved, one out of three equations is redundant. I am tempted to solve (2.4) and (2.5) since they are first-order linear equations with straightforward initial conditions,

$$\begin{cases} 
Q(0) = q(0) = x_0 \\
P(0) = p(0) = m\dot{q}(0) = mv_0 
\end{cases} \quad \tag{2.6}$$

but I am told to use the constant of motion (2.3), so here it goes. I solve for $P$ in terms of $Q$ and $K$, using the quadratic formula,

$$P = -m\gamma Q \pm m\sqrt{\frac{2K}{m} - (\omega^2 - \gamma^2)Q^2}$$

Two things to notice: first, I don’t know yet which sign to pick. Second, in order for $P$ to be real, the quantity under the square root must be non-negative. An underdamped oscillator with $\gamma < \omega$ is therefore subject to the constraint

$$|Q| \leq \sqrt{\frac{2K}{m(\omega^2 - \gamma^2)}}$$

The maximum allowed $Q$ corresponds to the oscillator’s amplitude, which decays exponentially. Now I substitute this expression for $P$ into the equation of motion for $Q$ (2.4), making it separable,

$$\sqrt{\frac{2K}{m} - (\omega^2 - \gamma^2)Q^2} = \pm 1$$

Integrating the right-hand side over $(0, t)$ just gives me $\pm t$. The system evolves forward in time, so it’s plausible that I’ll have to choose the positive sign. Meanwhile, integrating the left-hand side from $Q(0)$ to $Q(t)$ calls for the change of variable $\sqrt{\omega^2 - \gamma^2} Q = \sqrt{2K/m} \sin \phi$. 

$$\int_{Q(0)}^{Q(t)} \frac{dQ}{\sqrt{\frac{2K}{m} - (\omega^2 - \gamma^2)Q^2}} = \frac{1}{\sqrt{\omega^2 - \gamma^2}} \int_{\phi(0)}^{\phi(t)} d\phi = \frac{\phi(t) - \phi(0)}{\sqrt{\omega^2 - \gamma^2}}$$
So I obtain $\phi(t) = \phi(0) + \sqrt{\omega^2 - \gamma^2} t$, which looks reassuringly like the phase of oscillation. Changing back to $Q$,

$$Q(t) = \sqrt{\frac{2K}{m(\omega^2 - \gamma^2)}} \sin \phi(t)$$

$$= \sqrt{\frac{2K}{m(\omega^2 - \gamma^2)}} \left[ \sin \phi(0) \cos(\sqrt{\omega^2 - \gamma^2} t) + \cos \phi(0) \sin(\sqrt{\omega^2 - \gamma^2} t) \right]$$

The initial phase and energy are of course determined by the initial conditions (2.6).

$$K = \frac{P(0)^2}{2m} + \frac{m\omega^2}{2} Q(0)^2 + \gamma Q(0) P(0) = \frac{m}{2} (v_0^2 + \omega^2 x_0^2) + \gamma mx_0 v_0$$

$$\sin \phi(0) = \sqrt{\frac{m(\omega^2 - \gamma^2)}{2K}} Q(0) = \sqrt{\frac{m(\omega^2 - \gamma^2)}{2K}} x_0$$

$$\cos \phi(0) = \pm \sqrt{1 - \sin^2 \phi(0)} = \pm \sqrt{1 - \frac{m(\omega^2 - \gamma^2) x_0^2}{2K}}$$

Those ugly square roots all cancel out, leaving me in peace.

$$Q(t) = x_0 \cos(\sqrt{\omega^2 - \gamma^2} t) \pm \frac{v_0 + \gamma x_0}{\sqrt{\omega^2 - \gamma^2}} \sin(\sqrt{\omega^2 - \gamma^2} t)$$

Finally, the physical coordinate $q$ is related to the canonical coordinate $Q$ by $q = Q e^{-\gamma t}$,

$$q(t) = e^{-\gamma t} \left[ x_0 \cos(\sqrt{\omega^2 - \gamma^2} t) + \frac{v_0 + \gamma x_0}{\sqrt{\omega^2 - \gamma^2}} \sin(\sqrt{\omega^2 - \gamma^2} t) \right]$$

I’ve decided on the plus sign because it gives the correct initial condition for $\dot{q}$. I’ve also checked that this solution indeed satisfies the equation of motion given by Newton’s Second Law, $\ddot{q} = -\omega^2 q - 2\gamma \dot{q}$. I would much prefer to have solved that equation from the get-go.

3 Anharmonic Oscillator, Action-Angle Approach

The potential is periodic with infinitely tall barriers, which a classical particle can’t tunnel through, so I’ll only consider the piece $0 < x < \pi x_0$. I can immediately write down the Hamiltonian,

$$\mathcal{H}(x, p) = \frac{p^2}{2m} + \frac{a}{\sin(x/x_0)^2}$$

It is completely integrable since there’s once one spatial dimension. The conservation of energy $\mathcal{H} = E$ allows me to write the momentum as a function of position,

$$p(x) = \pm \sqrt{2m \left[ E - \frac{a}{\sin(x/x_0)^2} \right]}$$

where the sign depends on whether the particle is moving right (+) or left (−). The turning points are where momentum approaches zero,

$$x_1 = x_0 \arcsin \sqrt{a/E} \quad \quad x_2 = x_0 (\pi - \arcsin \sqrt{a/E})$$
There's always two turning points for $E > a$. $E = a$ puts the particle at rest in equilibrium; $E < a$ is not allowed. The action variable is defined as the contour integral

$$J = \oint_{H=E} p \, dx = \int_{x_1}^{x_2} (p) \, dx + \int_{x_2}^{x_1} (-p) \, dx = 2 \int_{x_1}^{x_2} dx \sqrt{2m \left[ E - \frac{a}{\sin(x/x_0)^2} \right]}$$

I will not attempt to evaluate this integral. If I did, I would then solve for $E$ as a function of $J$, and calculate the orbital frequency $\nu = dE/dJ$. But I'm more interested in the period of oscillation,

$$T = \frac{1}{\nu} = \frac{dJ}{dE} = \sqrt{\frac{2m}{E}} \int_{x_1}^{x_2} \frac{dx}{\sqrt{1 - \frac{a/E}{\sin(x/x_0)^2}}}$$

This integral is quite hard to solve. By trial and error I found that the substitution

$$\sqrt{1 - \frac{a}{E}} \sin \phi = -\cos(x/x_0) \quad (3.1)$$

maps the interval of integration $x : (x_1, x_2) \mapsto \phi : (-\pi/2, \pi/2)$, and makes the integrand trivial:

$$T = \sqrt{\frac{2m}{E}} \int_{-\pi/2}^{\pi/2} x_0 \, d\phi = \pi x_0 \sqrt{\frac{2m}{E}} \quad (3.2)$$

Interestingly the period of oscillation depends on energy. More energetic, and thus faster, particles have shorter period. I may also calculate the angular frequency,

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{2E}{mx_0^2}} \quad (3.3)$$

I came up with two ways to check this result. First, one-dimensional motion can always be reduced to quadrature. The canonical equation for the coordinate $x$,

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

can be converted to a directly integrable equation for time,

$$\frac{dt}{dx} = \frac{m}{p(x)}$$

If I integrate from the left turning point to the right, I get half the period, so the full period is twice that integral,

$$T = 2 \int_{x_1}^{x_2} \frac{m}{|p(x)|} \, dx = \sqrt{\frac{2m}{E}} \int_{x_1}^{x_2} \frac{dx}{\sqrt{1 - \frac{a/E}{\sin(x/x_0)^2}}} = \pi x_0 \sqrt{\frac{2m}{E}}$$

Same integral, same answer. In fact, I may go one step further and solve the motion completely. Just relax the upper limit of integration and make the same substitution as above,

$$t = \int_{x_1}^{x} \frac{m}{|p(x)|} \, dx = \sqrt{\frac{m}{2E}} \int_{-\pi/2}^{\phi} x_0 \, d\phi = x_0 \sqrt{\frac{m}{2E}} (\phi + \pi/2)$$
and surprisingly enough, $\phi$ turns out to be the phase of oscillation,

$$\phi = t \sqrt{\frac{2E}{mx_0^2}} - \pi/2 = \omega t - \pi/2$$

I then invoke the substitution (3.1) again to find the displacement,

$$x(t) = x_0 \arccos \left[ \sqrt{1 - a/E} \cos(\omega t) \right]$$

under the initial condition that the particle starts at rest with total energy $E$.

The second check is to consider small oscillations $\delta x$ near the equilibrium $x = \pi x_0/2$. If I Taylor-expand the potential $V(x)$ about the equilibrium,

$$V\left(\frac{\pi x_0}{2} + \delta x\right) = V\left(\frac{\pi x_0}{2}\right) + V'\left(\frac{\pi x_0}{2}\right) \delta x + \frac{1}{2} V''\left(\frac{\pi x_0}{2}\right) \delta x^2 + O(\delta x^3)$$

The first term is the constant $a$. The second term is zero. The third term is a harmonic oscillator potential with “force constant” $V''(\pi x_0/2) = 2a/x_0^2$. So the frequency of small oscillations is

$$\omega = \sqrt{\frac{V''}{m}} = \sqrt{\frac{2a}{mx_0^2}}$$

which is the result (3.3) in the limit $E \to a$.

4 Canonical Transformation

The invariance of Poisson brackets is a necessary and sufficient condition for a canonical transformation. It is also easy to check; taking $Q_1 = q_1^2$, $Q_2 = q_1 + q_2$.

1. $[Q_1, Q_1] = [Q_2, Q_2] = [P_1, P_1] = [P_2, P_2] = 0$ is automatically satisfied.

2. $[Q_1, Q_2] = 0$ because there’s no $p$ dependence;

3. $[Q_1, P_1] = 2q_1 \frac{\partial P_1}{\partial p_1} = 1 \to \frac{\partial P_1}{\partial p_1} = \frac{1}{2q_1}$

4. $[Q_1, P_2] = 2q_1 \frac{\partial P_2}{\partial p_1} = 0 \to \frac{\partial P_2}{\partial p_1} = 0$

5. $[Q_2, P_1] = \frac{\partial P_1}{\partial p_1} + \frac{\partial P_2}{\partial p_2} = 0 \to \frac{\partial P_1}{\partial p_2} = -\frac{\partial P_2}{\partial p_1} = -\frac{1}{2q_1}$

6. $[Q_2, P_2] = \frac{\partial P_1}{\partial p_1} + \frac{\partial P_2}{\partial p_2} = 1 \to \frac{\partial P_1}{\partial p_2} = 1 - \frac{\partial P_2}{\partial p_1} = 1$

From the partial derivatives I can glimpse the general form

$$\begin{cases} P_1 = \frac{p_1 - p_2}{2q_1} + f(q_1, q_2) \\ P_2 = p_2 + g(q_1, q_2) \end{cases}$$

where $f$, $g$ are undetermined functions of $q_1$, $q_2$. The last Poisson bracket,

$$[P_1, P_2] = \frac{\partial P_1}{\partial q_1} \frac{\partial P_2}{\partial p_1} - \frac{\partial P_1}{\partial p_1} \frac{\partial P_2}{\partial q_1} + \frac{\partial P_1}{\partial q_2} \frac{\partial P_2}{\partial p_2} - \frac{\partial P_1}{\partial p_2} \frac{\partial P_2}{\partial q_2}$$

$$= -\frac{1}{2q_1} \frac{\partial g}{\partial q_1} + \frac{\partial f}{\partial q_2} + \frac{1}{2q_1} \frac{\partial g}{\partial q_2} = 0$$
is the only constraint on \( f \) and \( g \). Now consider the Hamiltonian

\[
\mathcal{H}(q_1, q_2, p_1, p_2) = \left( \frac{p_1 - p_2}{2q_1} \right)^2 + p_2 + (q_1 + q_2)^2
\]

I think \( f = 0, g = (q_1 + q_2)^2 \) is a good choice, and it satisfies the last Poisson bracket condition as well. So my canonical transformation is

\[
\begin{align*}
Q_1 &= q_1^2 \\
Q_2 &= q_1 + q_2 \\
P_1 &= \frac{p_1 - p_2}{2q_1} \\
P_2 &= p_2 + (q_1 + q_2)^2
\end{align*}
\]

and my new Hamiltonian is

\[
\mathcal{H}(P_1, P_2) = P_1^2 + P_2
\]

Well, \( Q_1 \) and \( Q_2 \) are ignorable, so \( P_1 \) and \( P_2 \) are conserved. The other two canonical equations are trivially solved,

\[
\begin{align*}
\dot{Q}_1 &= \frac{\partial \mathcal{H}}{\partial P_1} = 2P_1 \\
\dot{Q}_2 &= \frac{\partial \mathcal{H}}{\partial P_2} = 1
\end{align*}
\]

\[
\begin{align*}
Q_1(t) &= 2P_1 t + Q_1(0) \\
Q_2(t) &= t + Q_2(0)
\end{align*}
\]

To find the initial conditions for the new variables, I evaluate the transformations (4.1) at \( t = 0 \), which is the same as adding a subscript 0. Then I iteratively solve for the original variables,

\[
\begin{align*}
q_1 &= \sqrt{Q_1} = \sqrt{\frac{p_{10} - p_{20}}{2q_{10}}} t + q_{10}^2 \\
q_2 &= Q_2 - q_1 = q_{10} + q_{20} + t - \sqrt{\frac{p_{10} - p_{20}}{2q_{10}}} t + q_{10}^2 \\
p_2 &= P_2 - (q_1 + q_2)^2 = p_{20} + (q_{10} + q_{20})^2 - (q_{10} + q_{20} + t)^2 \\
p_1 &= 2P_1 q_1 + p_2 = \frac{p_{10} - p_{20}}{q_{10}} \sqrt{\frac{p_{10} - p_{20}}{q_{10}}} t + q_{10}^2 + p_{20} + (q_{10} + q_{20})^2 - (q_{10} + q_{20} + t)^2
\end{align*}
\]

I’ve double-checked that the initial conditions are self-consistent. I have no idea what physical system this describes.