Physics 210 Homework #4

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1 Simple Harmonic Oscillator, Hamilton-Jacobi Approach

The difficulty of checking a solution goes like the reciprocal of finding the solution. Let me compute up front the total derivative of $S(q, \alpha, t)$,

$$\mathrm{d}S = m\omega \frac{q\cos(\omega t) - \alpha}{\sin(\omega t)}\,\mathrm{d}q + m\omega \frac{\alpha\cos(\omega t) - q}{\sin(\omega t)}\,\mathrm{d}\alpha + \frac{m\omega^2}{2} \frac{2q\alpha\cos(\omega t) - (q^2 + \alpha^2)}{\sin(\omega t)^2}\,\mathrm{d}t$$

As with any generating function, the differential action

$$p\,\mathrm{d}q - \mathcal{H}\mathrm{d}t = \beta\,\mathrm{d}\alpha - K\mathrm{d}t + \mathrm{d}S$$

must be preserved. The $\mathrm{d}q$ term tells me, implicitly, how to transform the coordinate α,

$$p = \frac{\partial S}{\partial q} = m\omega \frac{q\cos(\omega t) - \alpha}{\sin(\omega t)}$$

the $\mathrm{d}\alpha$ terms tells me how to transform the momentum β,

$$\beta = -\frac{\partial S}{\partial \alpha} = -m\omega \frac{\alpha\cos(\omega t) - q}{\sin(\omega t)} \tag{1.1}$$

and the $\mathrm{d}t$ term tells me how to transform the Hamiltonian K,

$$\begin{aligned} K &= \mathcal{H}\left(q, p = \frac{\partial S}{\partial q}\right) + \frac{\partial S}{\partial t} \\ &= \frac{1}{2m}\left[m\omega \frac{q\cos(\omega t) - \alpha}{\sin(\omega t)}\right]^2 + \frac{m\omega^2}{2} q^2 + \frac{m\omega^2}{2} \frac{2q\alpha\cos(\omega t) - (q^2 + \alpha^2)}{\sin(\omega t)^2} \end{aligned}$$

When I expand the square, all terms cancel out, leaving the new Hamiltonian identically zero. So $S(q, \alpha, t)$ indeed solves the Hamilton-Jacobi equation. Moreover, both α and β and conserved, so I can invert the definition of β (1.1) to find the physical coordinate

$$\boxed{q(t) = \alpha\cos(\omega t) + \frac{\beta}{m\omega}\sin(\omega t)}$$

which I recognize as the general solution to the simple harmonic oscillator. In particular, $\alpha = q(0)$ is the initial position and $\beta = m\dot{q}(0)$ is the initial momentum.

2 Damped Harmonic Oscillator, Canonical Approach

Part (a) The Hamiltonian Newton's Second Law directly yields the equation of motion,

$$\boxed{-V'(q) - 2m\gamma\dot{q} = m\ddot{q}} \tag{2.1}$$

Now I take the Lagrangian

$$\mathcal{L}(q,\dot{q},t) = \mathrm{e}^{2\gamma t}\left[\frac{m\dot{q}^2}{2} - V(q)\right]$$

and compute the conjugate momentum

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \mathrm{e}^{2\gamma t} m\dot{q} \tag{2.2}$$

its rate of change

$$\dot{p} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \mathrm{e}^{2\gamma t}\left(2\gamma m\dot{q} + m\ddot{q}\right)$$

and the generalized force

$$\frac{\partial \mathcal{L}}{\partial q} = -\mathrm{e}^{2\gamma t}\,V'(q)$$

Clearly the resulting Euler-Lagrange equation

$$\boxed{\ddot{q} = -\frac{V'(q)}{m} - 2\gamma\dot{q}}$$

is equivalent to Newton's Second Law (2.1), so the proposed Lagrangian is valid. Next, to derive the Hamiltonian, I solve equation (2.2) for the velocity $\dot{q} = (p/m)\,\mathrm{e}^{-2\gamma t}$, and take the Legendre transform,

$$\mathcal{H} = p\dot{q} - \mathcal{L} = \mathrm{e}^{2\gamma t}\left[\frac{m\dot{q}^2}{2} + V(q)\right]$$

$$\boxed{\mathcal{H}(q,p,t) = \mathrm{e}^{-2\gamma t}\,\frac{p^2}{2m} + \mathrm{e}^{2\gamma t}\,V(q)}$$

Part (b) A Constant of Motion Given $F_2(q,P,t) = \mathrm{e}^{\gamma t}\,qP$, I compute its total derivative,

$$\mathrm{d}F_2 = \mathrm{e}^{\gamma t}\,P\,\mathrm{d}q + \mathrm{e}^{\gamma t}\,q\,\mathrm{d}P + \gamma \mathrm{e}^{\gamma t} qP\,\mathrm{d}t$$

and require the differential action

$$p\,\mathrm{d}q - \mathcal{H}\mathrm{d}t = P\mathrm{d}Q - K\mathrm{d}t + \mathrm{d}F_2$$

to be preserved. The caveat is F_2 has a $\mathrm{d}P$ rather than a $\mathrm{d}Q$, so I must subtract the total differential $\mathrm{d}(PQ) = P\mathrm{d}Q + Q\mathrm{d}P$ from the right-hand side—that amounts to an integration by parts. Then I collect differentials,

$$(p - \mathrm{e}^{\gamma t}\,P)\,\mathrm{d}q + (Q - \mathrm{e}^{\gamma t}\,q)\,\mathrm{d}P + (K - \mathcal{H} - \gamma \mathrm{e}^{\gamma t} qP)\,\mathrm{d}t = 0$$

and set each term to zero because q, P, t are independent variables. The $\mathrm{d}q$ term tells me the new momentum $P = \mathrm{e}^{-\gamma t}\,p$, the $\mathrm{d}P$ terms tells me the new coordinate $Q = \mathrm{e}^{\gamma t}\,q$, and the $\mathrm{d}t$ term tells me the new Hamiltonian,

$$K = \mathcal{H} + \gamma \mathrm{e}^{\gamma t} qP = \mathrm{e}^{-2\gamma t}\,\frac{p^2}{2m} + \mathrm{e}^{2\gamma t}\,V(q) + \gamma \mathrm{e}^{\gamma t} qP$$

$$\boxed{K(Q,P,t) = \frac{P^2}{2m} + \mathrm{e}^{2\gamma t} V(q = Q\,\mathrm{e}^{-\gamma t}) + \gamma QP}$$

In the case of a harmonic oscillator potential

$$V(q) = \frac{m\omega^2 q^2}{2} \quad \longrightarrow \quad V(q = Q\,\mathrm{e}^{-\gamma t}) = \frac{m\omega^2}{2}\,\mathrm{e}^{-2\gamma t}\,Q^2$$

the transformed Hamiltonian becomes independent of time,

$$\boxed{K(Q,P) = \frac{P^2}{2m} + \frac{m\omega^2}{2}Q^2 + \gamma QP} \tag{2.3}$$

Whenever the Hamiltonian has no explicit time dependence, it is a constant of motion.

Part (c) The Solution The Hamiltonian produces two canonical equations of motion,

$$\dot{Q} = \frac{\partial K}{\partial P} = \frac{P}{m} + \gamma Q \tag{2.4}$$

$$\dot{P} = -\frac{\partial K}{\partial Q} = -m\omega^2 Q - \gamma P \tag{2.5}$$

Taken together with the statement that K is conserved, one out of three equations is redundant. I am tempted to solve (2.4) and (2.5) since they are first-order linear equations with straightforward initial conditions,

$$\begin{cases} Q(0) = q(0) = x_0 \\ P(0) = p(0) = m\dot{q}(0) = mv_0 \end{cases} \tag{2.6}$$

but I am told to use the constant of motion (2.3), so here it goes. I solve for P in terms of Q and K, using the quadratic formula,

$$P = -m\gamma Q \pm m\sqrt{\frac{2K}{m} - (\omega^2 - \gamma^2)Q^2}$$

Two things to notice: first, I don't know yet which sign to pick. Second, in order for P to be real, the quantity under the square root must be non-negative. An underdamped oscillator with $\gamma < \omega$ is therefore subject to the constraint

$$|Q| \leq \sqrt{\frac{2K}{m(\omega^2 - \gamma^2)}}$$

The maximum allowed Q corresponds to the oscillator's amplitude, which decays exponentially. Now I substitute this expression for P into the equation of motion for Q (2.4), making it separable,

$$\frac{\dot{Q}}{\sqrt{\frac{2K}{m} - (\omega^2 - \gamma^2)Q^2}} = \pm 1$$

Integrating the right-hand side over $(0, t)$ just gives me $\pm t$. The system evolves forward in time, so it's plausible that I'll have to choose the positive sign. Meanwhile, integrating the left-hand side from $Q(0)$ to $Q(t)$ calls for the change of variable $\sqrt{\omega^2 - \gamma^2}\,Q = \sqrt{2K/m}\,\sin\phi$.

$$\int_{Q(0)}^{Q(t)} \frac{\mathrm{d}Q}{\sqrt{\frac{2K}{m} - (\omega^2 - \gamma^2)Q^2}} = \frac{1}{\sqrt{\omega^2 - \gamma^2}} \int_{\phi(0)}^{\phi(t)} \mathrm{d}\phi = \frac{\phi(t) - \phi(0)}{\sqrt{\omega^2 - \gamma^2}}$$

So I obtain $\phi(t) = \phi(0) + \sqrt{\omega^2 - \gamma^2}\, t$, which looks reassuringly like the phase of oscillation. Changing back to Q,

$$\begin{aligned} Q(t) &= \sqrt{\frac{2K}{m(\omega^2-\gamma^2)}}\, \sin\phi(t) \\ &= \sqrt{\frac{2K}{m(\omega^2-\gamma^2)}}\Big[\sin\phi(0)\cos(\sqrt{\omega^2-\gamma^2}\, t) + \cos\phi(0)\sin(\sqrt{\omega^2-\gamma^2}\, t)\Big] \end{aligned}$$

The initial phase and energy are of course determined by the initial conditions (2.6).

$$\begin{aligned} K &= \frac{P(0)^2}{2m} + \frac{m\omega^2}{2}Q(0)^2 + \gamma Q(0)P(0) = \frac{m}{2}(v_0^2 + \omega^2 x_0^2) + \gamma m x_0 v_0 \\ \sin\phi(0) &= \sqrt{\frac{m(\omega^2-\gamma^2)}{2K}}\, Q(0) = \sqrt{\frac{m(\omega^2-\gamma^2)}{2K}}\, x_0 \\ \cos\phi(0) &= \pm\sqrt{1-\sin^2\phi(0)} = \pm\sqrt{1 - \frac{m(\omega^2-\gamma^2)x_0^2}{2K}} \end{aligned}$$

Those ugly square roots all cancel out, leaving me in peace.

$$Q(t) = x_0\cos(\sqrt{\omega^2-\gamma^2}\, t) \pm \frac{v_0 + \gamma x_0}{\sqrt{\omega^2-\gamma^2}}\, \sin(\sqrt{\omega^2-\gamma^2}\, t)$$

Finally, the physical coordinate q is related to the canonical coordinate Q by $q = Q\, \mathrm{e}^{-\gamma t}$,

$$\boxed{q(t) = \mathrm{e}^{-\gamma t}\left[x_0\cos(\sqrt{\omega^2-\gamma^2}\, t) + \frac{v_0+\gamma x_0}{\sqrt{\omega^2-\gamma^2}}\, \sin(\sqrt{\omega^2-\gamma^2}\, t)\right]}$$

I've decided on the plus sign because it gives the correct initial condition for $\dot{q}$. I've also checked that this solution indeed satisfies the equation of motion given by Newton's Second Law, $\ddot{q} = -\omega^2 q - 2\gamma\dot{q}$. I would much prefer to have solved that equation from the get-go.

3 Anharmonic Oscillator, Action-Angle Approach

The potential is periodic with infinitely tall barriers, which a classical particle can't tunnel through, so I'll only consider the piece $0 < x < \pi x_0$. I can immediately write down the Hamiltonian,

$$\mathcal{H}(x,p) = \frac{p^2}{2m} + \frac{a}{\sin(x/x_0)^2}$$

It is completely integrable since there's once one spatial dimension. The conservation of energy $\mathcal{H} = E$ allows me to write the momentum as a function of position,

$$p(x) = \pm\sqrt{2m\left[E - \frac{a}{\sin(x/x_0)^2}\right]}$$

where the sign depends on whether the particle is moving right $(+)$ or left $(-)$. The turning points are where momentum approaches zero,

$$x_1 = x_0 \arcsin\sqrt{a/E} \qquad\qquad x_2 = x_0(\pi - \arcsin\sqrt{a/E})$$

There's always two turning points for $E > a$. $E = a$ puts the particle at rest in equilibrium; $E < a$ is not allowed. The action variable is defined as the contour integral

$$
\begin{aligned}
J = \oint_{\mathcal{H}=E} p \, \mathrm{d}x &= \int_{x_1}^{x_2} (+p) \, \mathrm{d}x + \int_{x_2}^{x_1} (-p) \, \mathrm{d}x \\
&= 2 \int_{x_1}^{x_2} \mathrm{d}x \sqrt{2m \left[E - \frac{a}{\sin(x/x_0)^2} \right]}
\end{aligned}
$$

I will not attempt to evaluate this integral. If I did, I would then solve for E as a function of J, and calculate the orbital frequency $\nu = \mathrm{d}E/\mathrm{d}J$. But I'm more interested in the period of oscillation,

$$
T = \frac{1}{\nu} = \frac{\mathrm{d}J}{\mathrm{d}E} = \sqrt{\frac{2m}{E}} \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{1 - \frac{a/E}{\sin(x/x_0)^2}}}
$$

This integral is quite hard to solve. By trial and error I found that the substitution

$$
\begin{aligned}
\sqrt{1 - a/E} \, \sin\phi &= -\cos(x/x_0) \qquad (3.1) \\
\sqrt{1 - a/E} \, \cos\phi \, \mathrm{d}\phi &= \sin(x/x_0) \, \mathrm{d}x/x_0
\end{aligned}
$$

maps the interval of integration $x : (x_1, x_2) \mapsto \phi : (-\pi/2, \pi/2)$, and makes the integrand trivial:

$$
T = \sqrt{\frac{2m}{E}} \int_{-\pi/2}^{\pi/2} x_0 \, \mathrm{d}\phi = \pi x_0 \sqrt{\frac{2m}{E}} \qquad (3.2)
$$

Interestingly the period of oscillation depends on energy. More energetic, and thus faster, particles have shorter period. I may also calculate the angular frequency,

$$
\boxed{\omega = \frac{2\pi}{T} = \sqrt{\frac{2E}{m x_0^2}}} \qquad (3.3)
$$

I came up with two ways to check this result. First, one-dimensional motion can always be reduced to quadrature. The canonical equation for the coordinate x,

$$
\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m}
$$

can be converted to a directly integrable equation for time,

$$
\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{m}{p(x)}
$$

If I integrate from the left turning point to the right, I get half the period, so the full period is twice that integral,

$$
T = 2 \int_{x_1}^{x_2} \frac{m}{|p(x)|} \, \mathrm{d}x = \sqrt{\frac{2m}{E}} \int_{x_1}^{x_2} \frac{\mathrm{d}x}{\sqrt{1 - \frac{a/E}{\sin(x/x_0)^2}}} = \pi x_0 \sqrt{\frac{2m}{E}}
$$

Same integral, same answer. In fact, I may go one step further and solve the motion completely. Just relax the upper limit of integration and make the same substitution as above,

$$
t = \int_{x_1}^{x} \frac{m}{|p(x)|} \, \mathrm{d}x = \sqrt{\frac{m}{2E}} \int_{-\pi/2}^{\phi} x_0 \, \mathrm{d}\phi = x_0 \sqrt{\frac{m}{2E}} (\phi + \pi/2)
$$

and surprisingly enough, ϕ turns out to be the phase of oscillation,

$$\phi = t\sqrt{\frac{2E}{mx_0^2}} - \pi/2 = \omega t - \pi/2$$

I then invoke the substitution (3.1) again to find the displacement,

$$\boxed{x(t) = x_0 \arccos\Big[\sqrt{1 - a/E}\,\cos(\omega t)\Big]}$$

under the initial condition that the particle starts at rest with total energy E.

The second check is to consider small oscillations δx near the equilibrium $x = \pi x_0/2$. If I Taylor-expand the potential $V(x)$ about the equilibrium,

$$V\Big(\frac{\pi x_0}{2} + \delta x\Big) = V\Big(\frac{\pi x_0}{2}\Big) + V'\Big(\frac{\pi x_0}{2}\Big)\,\delta x + \frac{1}{2}V''\Big(\frac{\pi x_0}{2}\Big)\,\delta x^2 + \mathcal{O}(\delta x^3)$$

The first term is the constant a. The second term is zero. The third term is a harmonic oscillator potential with "force constant" $V''(\pi x_0/2) = 2a/x_0^2$. So the frequency of small oscillations is

$$\boxed{\omega = \sqrt{\frac{V''}{m}} = \sqrt{\frac{2a}{mx_0^2}}}$$

which is the result (3.3) in the limit $E \to a$.

4 Canonical Transformation

The invariance of Poisson brackets is a necessary and sufficient condition for a canonical transformation. It is also easy to check; taking $Q_1 = q_1^2$, $Q_2 = q_1 + q_2$,

1. $[Q_1, Q_1] = [Q_2, Q_2] = [P_1, P_1] = [P_2, P_2] = 0$ is automatically satisfied.
2. $[Q_1, Q_2] = 0$ because there's no p dependence;
3. $[Q_1, P_1] = 2q_1 \frac{\partial P_1}{\partial p_1} = 1 \longrightarrow \frac{\partial P_1}{\partial p_1} = \frac{1}{2q_1}$
4. $[Q_1, P_2] = 2q_1 \frac{\partial P_2}{\partial p_1} = 0 \longrightarrow \frac{\partial P_2}{\partial p_1} = 0$
5. $[Q_2, P_1] = \frac{\partial P_1}{\partial p_1} + \frac{\partial P_1}{\partial p_2} = 0 \longrightarrow \frac{\partial P_1}{\partial p_2} = -\frac{\partial P_1}{\partial p_1} = -\frac{1}{2q_1}$
6. $[Q_2, P_2] = \frac{\partial P_2}{\partial p_1} + \frac{\partial P_2}{\partial p_2} = 1 \longrightarrow \frac{\partial P_2}{\partial p_2} = 1 - \frac{\partial P_2}{\partial p_1} = 1$

From the partial derivatives I can glimpse the general form

$$\boxed{\begin{cases} P_1 = \dfrac{p_1 - p_2}{2q_1} + f(q_1, q_2) \\ P_2 = p_2 + g(q_1, q_2) \end{cases}}$$

where f, g are undetermined functions of q_1, q_2. The last Poisson bracket,

$$\begin{aligned} [P_1, P_2] &= \frac{\partial P_1}{\partial q_1}\frac{\partial P_2}{\partial p_1} - \frac{\partial P_1}{\partial p_1}\frac{\partial P_2}{\partial q_1} + \frac{\partial P_1}{\partial q_2}\frac{\partial P_2}{\partial p_2} - \frac{\partial P_1}{\partial p_2}\frac{\partial P_2}{\partial q_2} \\ &= -\frac{1}{2q_1}\frac{\partial g}{\partial q_1} + \frac{\partial f}{\partial q_2} + \frac{1}{2q_1}\frac{\partial g}{\partial q_2} = 0 \end{aligned}$$

is the only constraint on f and g. Now consider the Hamiltonian

$$\mathcal{H}(q_1, q_2, p_1, p_2) = \left(\frac{p_1 - p_2}{2q_1}\right)^2 + p_2 + (q_1 + q_2)^2$$

I think $f = 0$, $g = (q_1 + q_2)^2$ is a good choice, and it satisfies the last Poisson bracket condition as well. So my canonical transformation is

$$\boxed{\begin{cases} Q_1 = q_1^2 \\ Q_2 = q_1 + q_2 \\ P_1 = \dfrac{p_1 - p_2}{2q_1} \\ P_2 = p_2 + (q_1 + q_2)^2 \end{cases}} \tag{4.1}$$

and my new Hamiltonian is

$$\boxed{\mathcal{H}(P_1, P_2) = P_1^2 + P_2}$$

Well, Q_1 and Q_2 are ignorable, so P_1 and P_2 are conserved. The other two canonical equations are trivially solved,

$$\begin{cases} \dot{Q}_1 = \dfrac{\partial \mathcal{H}}{\partial P_1} = 2P_1 \\ \dot{Q}_2 = \dfrac{\partial \mathcal{H}}{\partial P_2} = 1 \end{cases} \longrightarrow \begin{cases} Q_1(t) = 2P_1 t + Q_1(0) \\ Q_2(t) = t + Q_2(0) \end{cases}$$

To find the initial conditions for the new variables, I evaluate the transformations (4.1) at $t = 0$, which is the same as adding a subscript 0. Then I iteratively solve for the original variables,

$$\begin{cases} q_1 = \sqrt{Q_1} = \sqrt{\dfrac{p_{10} - p_{20}}{2q_{10}} t + q_{10}^2} \\ q_2 = Q_2 - q_1 = q_{10} + q_{20} + t - \sqrt{\dfrac{p_{10} - p_{20}}{2q_{10}} t + q_{10}^2} \\ p_2 = P_2 - (q_1 + q_2)^2 = p_{20} + (q_{10} + q_{20})^2 - (q_{10} + q_{20} + t)^2 \\ p_1 = 2P_1 q_1 + p_2 = \dfrac{p_{10} - p_{20}}{q_{10}} \sqrt{\dfrac{p_{10} - p_{20}}{q_{10}} t + q_{10}^2} + p_{20} + (q_{10} + q_{20})^2 - (q_{10} + q_{20} + t)^2 \end{cases}$$

I've double-checked that the initial conditions are self-consistent. I have no idea what physical system this describes.