HW#1 — SOLUTIONS

1. $$ds = dy\sqrt{1+\left(\frac{dx}{dy}\right)^2} =$$

$$= dy\sqrt{1+\left(\frac{a}{a\sqrt{(2ay-y)^2/a^2}} + \frac{2a-2y}{2\sqrt{2ay-y^2}}\right)^2}$$

$$= dy\sqrt{1+\left(\sqrt{\frac{2a-y}{y}}\right)^2} = dy\sqrt{\frac{2a}{y}}$$

NOW CALCULATE

$$s(y_0) = \sqrt{2a}\int_0^{y_0} dy\,\frac{1}{\sqrt{y}} = 2\sqrt{2ay_0}$$

NOW WRITE THE $\mathcal{L}$ FOR THE GEN. COORDINATE s:

$$\mathcal{L} = \frac{1}{2}m\dot{s}^2 - mg\,y(s) = \frac{1}{2}m\dot{s}^2 - \frac{1}{8a}mgs^2$$

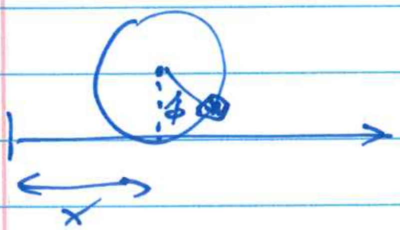
BUT THIS IS THE $\mathcal{L}$ OF A HARMONIC OSCILLATOR WITH $k = \frac{gm}{4a}$

THE PERIOD OF A H.O. IS INDEP. OF THE STARTING AMPLITUDE THUS SINCE $\ddot{s} = -\frac{g}{4a}s$

WE HAVE $s(t) = 2\sqrt{2ay_0}\cos\left(\sqrt{\frac{g}{4a}}\,t\right)$

TIME TO GET TO BOTTOM: $t_0 = \pi\sqrt{\frac{g}{a}}$, INDEP. OF y_0!

2. GEN. COORDINATES: x, ϕ (ANGLE OF BEAD)
(POS. OF HOOP)



ELIMINATE $x = r\theta$
FROM ROLLING W/OUT SLIPPING

USE CONSERVATION OF MOMENTUM

$$m v_0 = m \underbrace{v_f}_{\text{FINAL BEAD VELOCITY}} + M \underbrace{V_f}_{\text{FINAL HOOP VELOCITY}}$$

CONSERVATION OF ENERGY:

$$\frac{1}{2} m v_0^2 + mgr = M V_f^2 - mgr + \frac{1}{2} m v_f^2$$

[... ALGEBRA ...]

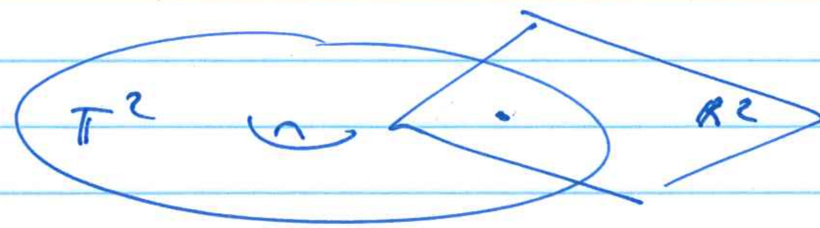
$$V_f = \frac{2 v_0 \frac{m}{M} - \sqrt{\frac{m}{M}\left(v_0^2 \frac{M}{m} + 8 + 4gr\frac{M}{m}\right)}}{\frac{2m}{M} + 1}$$

$$\lim_{\frac{m}{M} \to 0} V_f = -\sqrt{V_0^2 + 4gr}$$

$$\lim_{\frac{m}{M} \to \infty} V_f = V_0$$

3. (a): $Q = S^1 \times S^1 = \mathbb{T}^2$ (2-TORUS)

$TQ = \bigsqcup_{q \in Q} T_q \mathbb{T}^2$ i.e. $\mathbb{R}^2$ ATTACHED TO EACH POINT.



(b) $\alpha, \beta, \dot{\alpha}, \dot{\beta}$ AS GEN COORDINATES.

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2) l_1^2 \dot{\alpha}^2 + \frac{1}{2} m_2 \left(\dot{\beta}^2 l_2^2 + \dot{\alpha}\dot{\beta} l_1 l_2 \sin(\alpha + \beta) \right)$$
$$+ g(m_1 + m_2) l_1 \cos\alpha + g m_2 l_2 \cos\beta$$

(c) $$m_2 l_1 l_2 \ddot{\beta} \sin(\alpha + \beta) + m_2 l_1 l_2 \dot{\beta}^2 \cos(\alpha + \beta)$$
$$+ 2(m_1 + m_2) l_1^2 \ddot{\alpha} + 2(m_1 + m_2) l_1 \sin\alpha = 0$$

AND $$m_2 l_1 l_2 \ddot{\alpha} \sin(\alpha + \beta) + m_2 l_1 l_2 \dot{\alpha}^2 \sin(\alpha + \beta)$$
$$+ 2 m_2 l_2 \ddot{\beta} + g m_2 l_2 \sin\beta = 0$$

4. CALL $\tilde{\vec{r}}$ THE INERTIAL FRAME

$\vec{r}$ ROTATING FRAME

$$R(t)\,\vec{r} = \tilde{\vec{r}}, \quad R(t) = \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}$$

$$\dot{\tilde{x}} = (\dot{x} - \omega y)\cos\omega t - (\dot{y} + \omega x)\sin\omega t$$

etc.

$$\mathcal{L}\left(\tilde{\vec{r}}, \dot{\tilde{\vec{r}}}\right) = \tfrac{1}{2} m \left(\dot{\tilde{\vec{r}}}^{\,2}\right) - V\left(\tilde{\vec{r}}\right)$$

ROTATING FRAME:

$$\mathcal{L}\left(\vec{r}, \dot{\vec{r}}\right) = \tfrac{1}{2} m \left[(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2 + \dot{z}^2 \right) - V(x,y,z)$$

$$= \tfrac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(x,y,z)$$

$$\underbrace{+ \tfrac{1}{2} m\omega^2 (x^2 + y^2) + m\omega (x\dot{y} - y\dot{x})}_{U(x,y,\dot{x},\dot{y})}$$

$$Q_{x,y} = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}} = m\omega^2 \underset{y}{x} \pm m\omega \underset{\dot{x}}{\dot{y}}$$

So $\vec{Q} = m\omega^2 \rho \hat{\rho} + m\omega (\dot{y}\hat{x} - \dot{x}\hat{y})$ ✓