# Classical Mechanics Homework 2

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## 1 Problem 1

### 1.a

We begin with our favorite equation, $F = ma$, with $a = -G_N \frac{m_{\text{dust}}}{r^2}$, where $G_N$ is Newton's gravitational constant, $m_{\text{dust}}$ is the mass of the dust acting on the system, and $r$ is the distance from the center of mass (which we will assume is the center of the sun). Because we are concerned about the gravitational force due to a uniform distribution of dust (so a uniform distribution of mass) we may use the shell theorem to conclude that we need only concern ourselves with the mass enclosed in a sphere of radius $r$. The density of the dust is $\rho$, so $m_{\text{dust}} = \rho\frac{4}{3}\pi r^3$.
Therefore,

$$F' = -mG_N \frac{\rho\frac{4}{3}\pi r^3}{r^2} = -mG_N \frac{4}{3}\rho\pi r = -mkr$$

with $k = \frac{4}{3}\rho\pi G_N$.

### 1.b

We showed in class that the Lagrangian of the system gives us this equation of motion

$$m\ddot{r} - \frac{L^2}{mr^3} = \frac{\partial V}{\partial r}$$

In this problem, we have two attractive potentials, one due to the sun, and one due to the dust, so

$$V = -G_N \frac{M}{r} + \frac{kr^2}{2}$$

Because we are assuming a circular orbit, $\ddot{r} = \dot{r} = 0$, $r = r_0$. Thus,

$$-\frac{L^2}{mr_0^3} = -G_N \frac{M}{r_0^2} - kr_0$$

Which we can rewrite as

$$G_N M m r_0 + k r_0^4 - L^2 = 0$$

### 1.c

We can think of this as a Kepler problem with a single dust particle, of mass $\mu$, coming in with a velocity $V$. We now consider the orbit equation

$$\frac{1}{r} = \frac{\mu^2 M G_N}{L^2}\left(1 + \sqrt{1 + \frac{2EL^2}{\mu^3 M^2 G_N^2}}\cos(\theta - \theta')\right)$$

We are interested in the case where the minimum distance between the dust particle and the sun is $r = R$, so we set $\cos(\theta = \theta') = 1$. The angular momentum of the dust particle is $L = \mu V s$, where s is the impact parameter of the dust and the center of the sun. At infinity, the energy of the particle is $E = \frac{1}{2}\mu V^2$.

$$\frac{1}{R} = \frac{M G_N}{V^2 s^2}\left(1 + \sqrt{1 + \frac{V^4 s^2}{M^2 G_N^2}}\right)$$

We solve for $s$ to find

$$s^2 = R^2 + \frac{2MRG_N}{V^2}$$

We are interested in how the mass changes with time; $\frac{dM}{dt} = \rho \frac{dV_{ol}}{dt}$. The volume, $V_{ol}$, is of a cylinder with an effective radius $s$. We plug in our previous results into this to find:

$$\frac{dM}{dt} = \rho \pi R\left(RV + \frac{2MG_N}{V}\right)$$

### 1.d

We go back to our best friend, $\mathbf{F} = \dot{\mathbf{p}} = \dot{M}\mathbf{V} + M\dot{\mathbf{V}}$. We will assume $\mathbf{V}$ is constant, so that $\dot{\mathbf{V}} = 0$. Thus,

$$\mathbf{F} = \dot{M}\mathbf{V} = \pi\rho R(RV^2 + 2MG_N)\hat{V}$$

## 2 Problem 2

In the laboratory frame, consider an incident particle with mass $m_1$, that then scatters off of a stationary particle of mass $m_2$, at an angle $\vartheta$. By conservation of energy,

$$\frac{1}{2}m_1v_0^2 - \frac{1}{2}m_1v_1^2 + Q = \frac{1}{2}m_2v_2^2$$

where $Q$ is the energy lost due to the inelasticity of the scatter, and $v_2$ is the velocity of $m_2$ after scattering. We can impose momentum conservation to get two more relations.

$$m_1v_1\sin(\vartheta) = m_2v_2\sin(\alpha)$$

$$m_1v_0 = m_1v_1\cos(\vartheta) + m_2v_2sin(\alpha)$$

where $\alpha$ is the angle of deflection of $m_2$ after scattering. We now square both momentum equations and add them together to get rid of the dependence on $\alpha$.

$$m_1^2v_1^2\sin^2(\vartheta) + m_1^2v_1^2\cos^2(\vartheta) + m_1v_0^2 - 2m_1^2v_0v_1\cos(\vartheta) = m_2^2v_2^2$$

We now solve for everything in terms of $\cos(\vartheta)$.

$$\cos(\vartheta) = \frac{-m_2^2v_2^2 + m_1^2v_0^2 + m_1^2v_1^2}{2m_1^2v_1v_1}$$

Let $E_0$ be the energy of $m_1$ before the scatter, and $E_1$ be the energy of $m_1$ after the scatter. We can now plug in $v_0 = \sqrt{\frac{2E_0}{m_1}}, v_1 = \sqrt{\frac{2E_1}{m_1}}$, and $v_2^2 = \frac{2(E_0-E_1+Q)}{m_2}$.

$$\cos(\vartheta) = \frac{m_1(E_0+E_1) - m_2(E_0+E_1+Q)}{2m_1\sqrt{(}E_0E_1)}$$

which gives us the desired result

$$\cos(\vartheta) = \frac{m_2+m_1}{2m_1}\sqrt{\frac{E_1}{E_0}} - \frac{m_2-m_1}{2m_1}\sqrt{\frac{E_0}{E_1}} - \frac{m_2Q}{2m_1\sqrt{E_0E_1}}$$

## 3 Problem 3

We will begin by solving for the impact parameter, $s$. In class we derived,

$$\Theta(s) = \pi - 2\int_0^{u_m} \frac{s du}{\sqrt{1-(\frac{k}{2E}+s^2)u^2}}$$

(after plugging in $V(u) = \frac{1}{2}ku^2$). This evaluates to

$$\Theta(s) = \pi - \frac{2s}{\sqrt{\frac{k}{2E}+s^2}} \sin^{-1}\left(\frac{\sqrt{\frac{k}{2E}+s^2}}{r_m}\right)$$

Using conservation of angular momentum, we set $r_m = \frac{v_0}{v_m}s$. We plug this into our relation for conservation of energy to find

$$\frac{mv_0^2}{2} = \frac{mv_m^2}{2} + \frac{k}{2r_m^2} \rightarrow E = E\frac{s^2}{r_m^2} + \frac{k}{2r_m^2}$$

Solving for $r_m$, we get $r_m = \sqrt{s^2 + \frac{k}{2E}}$. Putting this result into our expression for $\Theta(s)$ we find

$$\Theta(s) = \pi\left(1 - \frac{s}{\sqrt{\frac{k}{2E}+s^2}}\right)$$

Let $x = \Theta/\pi = 1 - \frac{s}{\sqrt{\frac{k}{2E}+s^2}}$. We now solve for $s$ to find

$$s = \sqrt{\frac{k(1-x)^2}{2Ex(2-x)}}$$

The cross section $\sigma(x) = \frac{s}{\sin \pi x}\left|\frac{ds}{dx}\right|$, so we take the derivative of $s$ with respect to $x$

$$\left|\frac{ds}{dx}\right| = \left|\frac{1}{s}\frac{k(x-1)}{2E(x-2)^2x^2}\right|$$

We finally recover the desired result

$$\sigma(\Theta)\mathrm{d}\Theta = \frac{k(1-x)\mathrm{d}x}{2Ex^2(2-x)^2\sin\pi x}$$