Final Exam

Solve two of the following three problems
(extra points if you attempt a third problem)

1. The transformation equations between two sets of coordinates are:

\[ Q = \ln \left( 1 + q^{1/2} \cos p \right) ; \]
\[ P = 2 \left( 1 + q^{1/2} \cos p \right) q^{1/2} \sin p. \]

(a) Show directly from these transformation equations that \( Q \) and \( P \) are canonical variables, if \( q \) and \( p \) are, using Poisson brackets.

(b) As in (a), but using the symplectic condition \( J = MJM^T \), where \( M \) is the Jacobian of the transformation, \( M^T \) its transpose, and \( J \) the usual nonsingular, skew-symmetric matrix \( ((0, 1), (-1, 0)) \).

(c) Show that the function

\[ F = F(Q, p) = [\exp(Q) - 1]^2 \tan p \]

is a generating function (of the third type), i.e. that

\[ q = \frac{\partial F}{\partial p} \]
\[ P = \frac{\partial F}{\partial Q} \]

[Hint: first solve the transformation equations for \( q = q(Q, p) \) and \( P = P(Q, p) \) and then verify the transformation conditions above]

2. A block of mass \( m \) is attached to a wedge of mass \( M \) by a spring with spring constant \( k \). The inclined frictionless surface of the wedge makes an angle \( \alpha \) to the horizontal. The wedge is free to slide on a horizontal frictionless surface, as in the figure
(a) Given the relaxed length of the spring along is $d$, find the value $s_0$ when both the block and the wedge are at rest.

(b) Find the Lagrangian for the system as a function of the $x$ coordinate of the wedge and the length of the spring $s$. Write the equations of motion.

(c) Find the natural frequency of vibration

3. A disc of mass $M$ and radius $R$ rotates about its center on a horizontal plane. A mass $m$ can slide freely along one of the radii of the disc, and is attached to the center by a spring of natural length $l$ and force constant $k$, as shown in the figure.

(a) Find the moment of inertia of the disc about its center.

(b) Find an expression for the energy of the system in terms of $r$ (the distance of the mass from the center), $\dot{r}$, and the total angular momentum $J$.

(c) Derive Lagrange’s equations.

(d) Suppose the disc initially has a constant angular velocity $\Omega_0$ and the spring has a steady extension $r = r_0$. Find $r_0$ as a function of $\Omega_0$.

(e) Find the frequency of small oscillations around the equilibrium configuration.