Homework 4 Solutions

Physics 217

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1. n-body final states

i.

Problem:

Prove

$$d\Phi^{(3)}(P_i; m_1, m_2, m_3; \vec{p}_1, \vec{p}_2, \vec{p}_3) = \int \frac{d(\mu^2)}{2\pi} d\Phi^{(2)}(P_i; m_1, \mu; \vec{p}_1, \vec{q})\, d\Phi^{(2)}(q; m_2, m_3; \vec{p}_2, \vec{p}_3)$$

With q a dummy four-momentum, $\mu^2 = q^2$, the integral over $d^3\vec{q}$ and $d(\mu^2)$, and

$$d\Phi^{(n)}(p_i; m_1, ...m_n; \vec{p}_1, ..., \vec{p}_n) \equiv (2\pi)^4 \delta^4(p_i - \sum_{j=1}^{n} p_j) \prod_{j=1}^{n} \frac{d^3\vec{p}_j}{2p_j^0(2\pi)^3}$$

Solution:

$$\begin{aligned}
LHS &= (2\pi)^4\delta^4(P_i - p_1 - p_2 - p_3)\frac{d^3\vec{p}_1}{2p_1^0(2\pi)^3}\frac{d^3\vec{p}_2}{2p_2^0(2\pi)^3}\frac{d^3\vec{p}_3}{2p_3^0(2\pi)^3} \\
&= \int\int d^4q\, d^4q'\, \delta^4(P_i - p_1 - q)\, \delta^4(q' - p_2 - p_3) \times (2\pi)^4\delta^4(q - q')\frac{d^3\vec{p}_1}{2p_1^0(2\pi)^3}\frac{d^3\vec{p}_2}{2p_2^0(2\pi)^3}\frac{d^3\vec{p}_3}{2p_3^0(2\pi)^3} \\
&= \int d^4q\, \delta^4(P_i - p_1 - q)\, \delta^4(q - p_2 - p_3)(2\pi)^4\frac{d^3\vec{p}_1}{2p_1^0(2\pi)^3}\frac{d^3\vec{p}_2}{2p_2^0(2\pi)^3}\frac{d^3\vec{p}_3}{2p_3^0(2\pi)^3}
\end{aligned}$$

But $d^4q = dq_0\, d^3\vec{q} = \frac{d(q_0^2)}{2q_0}\, d^3\vec{q} = (2\pi)^4\frac{d(q_0^2)}{2\pi}\frac{d^3\vec{q}}{2q^0(2\pi)^3}$, so

$$LHS = \int \frac{d(q_0^2)}{2\pi} \times (2\pi)^4\delta^4(P_i - p_1 - q)\frac{d^3\vec{p}_1}{2p_1^0(2\pi)^3}\frac{d^3\vec{q}}{2q^0(2\pi)^3} \times (2\pi)^4\delta^4(q - p_2 - p_3)\frac{d^3\vec{p}_2}{2p_2^0(2\pi)^3}\frac{d^3\vec{p}_3}{2p_3^0(2\pi)^3}$$

Now make the substitution $\mu^2 = q_0^2 - |\vec{q}|^2$. The original bounds in the q_0 integral where $[0, \infty]$, but can be changed to $[|\vec{q}|, \infty]$, since $\delta^4(q - p_1 - p_2) = 0$ for $q_0 < |\vec{q}|$. The μ^2 integral bounds then become $[0, \infty]$, and we get

$$
\begin{aligned}
LHS &= \int \frac{d(\mu^2)}{2\pi} \times (2\pi)^4 \delta^4(P_i - p_1 - q) \frac{d^3\vec{p}_1}{2p_1^0(2\pi)^3} \frac{d^3\vec{q}}{2q^0(2\pi)^3} \times (2\pi)^4 \delta^4(q - p_2 - p_3) \frac{d^3\vec{p}_2}{2p_2^0(2\pi)^3} \frac{d^3\vec{p}_3}{2p_3^0(2\pi)^3} \\
&= \int \frac{d(\mu^2)}{2\pi} d\Phi^{(2)}(P_i; m_1, \mu; \vec{p}_1, \vec{q})\, d\Phi^{(2)}(q; m_2, m_3; \vec{p}_2, \vec{p}_3) \\
&= RHS
\end{aligned}
$$

As was set out to be proven.

ii.

Problem:

Prove

$$
d\Phi^{(4)} = \int \frac{d(\mu^2)}{2\pi} \int \frac{d(\mu'^2)}{2\pi} d\Phi^{(2)}(P_i; \mu, \mu'; \vec{q}, \vec{q}')\, d\Phi^{(2)}(q; m_1, m_2; \vec{p}_1, \vec{p}_2)\, d\Phi^{(2)}(q'; m_3, m_4; \vec{p}_3, \vec{p}_4)
$$

Solution:

$$
\begin{aligned}
LHS &= (2\pi)^4 \delta^4(P_i - p_1 - p_2 - p_3 - p_4) \frac{d^3\vec{p}_1}{2p_1^0(2\pi)^3} \frac{d^3\vec{p}_2}{2p_2^0(2\pi)^3} \frac{d^3\vec{p}_3}{2p_3^0(2\pi)^3} \frac{d^3\vec{p}_4}{2p_4^0(2\pi)^3} \\
&= \int \int d^4q\, d^4q' \delta^4(P_i - p_1 - p_2 - q)\, \delta^4(q' - p_3 - p_4) \\
&\qquad \times (2\pi)^4 \delta^4(q - q') \frac{d^3\vec{p}_1}{2p_1^0(2\pi)^3} \frac{d^3\vec{p}_2}{2p_2^0(2\pi)^3} \frac{d^3\vec{p}_3}{2p_3^0(2\pi)^3} \frac{d^3\vec{p}_4}{2p_4^0(2\pi)^3} \\
&= \int d^4q'\, \delta^4(P_i - p_1 - p_2 - q')\, \delta^4(q' - p_3 - p_4) \times (2\pi)^4 \frac{d^3\vec{p}_1}{2p_1^0(2\pi)^3} \frac{d^3\vec{p}_2}{2p_2^0(2\pi)^3} \frac{d^3\vec{p}_3}{2p_3^0(2\pi)^3} \frac{d^3\vec{p}_4}{2p_4^0(2\pi)^3} \\
&= \int \frac{d(\mu'^2)}{2\pi} \times (2\pi)^4\, \delta^4(P_i - p_1 - p_2 - q') \frac{d^3\vec{p}_1}{2p_1^0(2\pi)^3} \frac{d^3\vec{p}_2}{2p_2^0(2\pi)^3} \frac{d^3\vec{q}'}{2q'^0(2\pi)^3} \\
&\qquad \times (2\pi)^4\, \delta^4(q' - p_3 - p_4) \frac{d^3\vec{p}_3}{2p_3^0(2\pi)^3} \frac{d^3\vec{p}_4}{2p_4^0(2\pi)^3} \\
&= \int \frac{d(\mu'^2)}{2\pi}\, d\Phi^{(3)}(P_i; m1, m2, \mu' \vec{p}_1, \vec{p}_2, q')\, d\Phi^{(2)}(q'; m3, m4, ; \vec{p}_3, \vec{p}_4)
\end{aligned}
$$

But now we can use the result of part (i) on $d\Phi^{(3)}$, and get

$$
\begin{aligned}
LHS &= \int \frac{d(\mu^2)}{2\pi} \int \frac{d(\mu'^2)}{2\pi} d\Phi^{(2)}(P_i; \mu, \mu'; \vec{q}, \vec{q}')\, d\Phi^{(2)}(q; m_1, m_2; \vec{p}_1, \vec{p}_2)\, d\Phi^{(2)}(q'; m_3, m_4; \vec{p}_3, \vec{p}_4) \\
&= RHS
\end{aligned}
$$

As was set out to be proven.

iii.

Problem:

Prove

$$d\Phi^{(n)} = \int \frac{d(\mu^2)}{2\pi} d\Phi^{(2)}(P_i; \mu, m_1; \vec{q}, \vec{p}_1)\, d\Phi^{(n-1)}(q; m_2, ..., m_n; \vec{p}_2, ..., \vec{p}_n)$$

Solution:

$$\begin{aligned}
LHS &= (2\pi)^4 \delta^4(P_i - \sum_{j=1}^{n} p_j) \times \prod_{j=1}^{n} \frac{d^3\vec{p}_j}{2p_j^0(2\pi)^3} \\
&= \int\int d^4q\, d^4q' \delta^4(P_i - p_1 - q)\, \delta^4(\sum_{j=2}^{n} p_j - q') \times (2\pi)^4 \delta^4(q - q') \times \prod_{j=1}^{n} \frac{d^3\vec{p}_j}{2p_j^0(2\pi)^3} \\
&= \int d^4q\, \delta^4(P_i - p_1 - q)\, \delta^4(\sum_{j=2}^{n} p_j - q) \times (2\pi)^4 \prod_{j=1}^{n} \frac{d^3\vec{p}_j}{2p_j^0(2\pi)^3} \\
&= \int \frac{d(q_0^2)}{2\pi} \times (2\pi)^4 \delta^4(P_i - p_1 - q) \frac{d^3\vec{p}_1}{2p_1^0(2\pi)^3} \frac{d^3\vec{q}}{2q^0(2\pi)^3} \times (2\pi)^4 \delta^4(\sum_{j=2}^{n} p_j - q) \prod_{j=2}^{n} \frac{d^3\vec{p}_j}{2p_j^0(2\pi)^3} \\
&= \int \frac{d(\mu^2)}{2\pi} d\Phi^{(2)}(P_i; \mu, m_1; \vec{q}, \vec{p}_1)\, d\Phi^{(n-1)}(q; m_2, ..., m_n; \vec{p}_2, ..., \vec{p}_n) \\
&= RHS
\end{aligned}$$

As was set out to be proven.