Solution to Homework Set #8, Problem #1.

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1 Searching for the Higgs Boson at LEP

We are given the amplitude,

$$\mathcal{M}(e^+ + e^- \to H^0 + Z^0) = \frac{e^2 M_Z}{4\sin^2\theta_W}\frac{1}{s - M_Z^2}\epsilon_\mu^\lambda(Z^0)\bar{v}^{s'}(e^+)(4\sin^2\theta - 1 + \gamma^5)\gamma^\mu u^s(e^-) \tag{1}$$

and are asked to sum the matrix element squared over final polarizations (λ) and average over the initial spins (s, s').

1.1 Summing over final polarizations

We can see that the matrix element can be written in the form $\mathcal{M} = \epsilon_\mu^\lambda \mathcal{M}^\mu$. Summing the amplitude squared over polarizations we get

$$\sum_{\lambda=1}^{3} |\mathcal{M}|^2 = \sum_{\lambda=1}^{3} \epsilon_\mu^{*\lambda}\mathcal{M}^{\dagger\mu}\epsilon_\nu^\lambda \mathcal{M}^\nu \tag{2}$$

which gives an overall factor of

$$\sum_{\lambda=1}^{3} \epsilon_\mu^{*\lambda}\epsilon_\nu^\lambda = -g_{\mu\nu} + \frac{p_Z^\mu p_Z^\nu}{M_Z^2} \tag{3}$$

1.2 Averaging initial spins

Before we can do the sum over spins we have to write down $\mathcal{M}^\dagger$. In class we showed that

$$(\bar{v}\gamma^\mu u)^\dagger = \bar{u}\gamma^\mu v \tag{4}$$

We can see from Eq. 1 that we also have to calculate $(\bar{v}\gamma^5\gamma^\mu u)^\dagger$. We will use the identity $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$:

$$\begin{aligned}(\bar{v}\gamma^5\gamma^\mu u)^\dagger &= u^\dagger\gamma^{\mu\dagger}\gamma^5\gamma^0 v \\ &= -u^\dagger\gamma^{\mu\dagger}\gamma^0\gamma^5 v \\ &= -u^\dagger\gamma^0\gamma^\mu\gamma^5 v \\ &= -\bar{u}\gamma^\mu\gamma^5 v\end{aligned} \tag{5}$$

Now we can write down the parts of $|\mathcal{M}|^2$ with spinnor indices:

$$|\mathcal{M}|^2_{\text{Spinnor}} = [\bar{u}_\alpha(\gamma^\nu[4\sin^2\theta_W - 1 - \gamma^5])_{\alpha\beta}v_\beta][\bar{v}_\sigma([4\sin^2\theta_W - 1 + \gamma^5]\gamma^\mu)_{\sigma\rho}u_\rho] \tag{6}$$

Then summing over spins we'll get

$$\sum_{s=1}^{3} v_\beta^{s'}\bar{v}_\sigma^{s'} = (\not{p}_{e+} - m_e)_{\beta\sigma} \tag{7}$$

$$\sum_{s=1}^{3} u_\rho^{s}\bar{u}_\alpha^{s} = (\not{p}_{e-} + m_e)_{\rho\alpha} \tag{8}$$

Putting this into Eq. 6 and taking the ultra-relativistic limit where $m_e \to 0$ we get

$$\sum_{s,s'=1}^{3} |\mathcal{M}|^2_{\text{Spinnor}} = \text{Tr}[\not{p}_{e-}\gamma^\nu(4\sin^2\theta_W - 1 - \gamma^5)\not{p}_{e+}(4\sin^2\theta_W - 1 + \gamma^5)\gamma^\mu] \tag{9}$$

Expanding this expression we get four terms:

1. $(4\sin^2\theta_W - 1)^2\,\text{Tr}[\not{p}_{e-}\gamma^\nu\not{p}_{e+}\gamma^\mu]$
2. $-(4\sin^2\theta_W - 1)\,\text{Tr}[\not{p}_{e-}\gamma^\nu\gamma^5\not{p}_{e+}\gamma^\mu]$
3. $(4\sin^2\theta_W - 1)\,\text{Tr}[\not{p}_{e-}\gamma^\nu\not{p}_{e+}\gamma^5\gamma^\mu]$
4. $-\,\text{Tr}[\not{p}_{e-}\gamma^\nu\gamma^5\not{p}_{e+}\gamma^5\gamma^\mu]$

The second and third terms are proportional to

$$p_\sigma p_\rho\,\text{Tr}[\gamma^\sigma\gamma^5\gamma^\nu\gamma^\rho\gamma^\mu] = p_\sigma p_\rho(-4i\epsilon^{\sigma\nu\rho\mu}) \tag{10}$$

which is antisymmetric under exchange of the free indices. These terms have to be contracted with the symmetric tensor given in Eq. 3 will give zero, so we only have to worry about terms 1 and 4 above.

In the fourth term we can move one of the γ^5's next to the other at the cost of an overall factor of -1, giving the same trace as in the first term, using $\gamma^5\gamma^5 = 1$. Now we only have to calculate

$$\begin{aligned} \mathrm{Tr}[\not{p}_{e-}\gamma^\nu\not{p}_{e+}\gamma^\mu] &= p^\rho_{e-}p^\sigma_{e+}\,\mathrm{Tr}[\gamma^\rho\gamma^\nu\gamma^\sigma\gamma^\mu] \\ &= 4p^\rho_{e-}p^\sigma_{e+}(g^{\rho\nu}g^{\sigma\mu} - g^{\rho\sigma}g^{\nu\mu} + g^{\rho\mu}g^{\nu\sigma}) \\ &= 4(p^\nu_{e-}p^\mu_{e+} - p_{e-}\cdot p_{e+}g^{\nu\mu} + p^\mu_{e-}p^\nu_{e+}) \end{aligned} \tag{11}$$

Putting it altogeter, contracting Eq. 11 with Eq. 3 and putting in the correct prefactors we get

$$\frac{1}{4}\sum_{\text{spins,pols}}|\mathcal{M}|^2 = A\left((p_{e+}\cdot p_{e-}) + 2\frac{(p_Z\cdot p_{e+})(p_Z\cdot p_{e-})}{M_Z^2}\right) \tag{12}$$

where

$$A = \left(\frac{e^2 M_Z}{s - M_Z^2}\right)^2 \frac{1 + (1 - 4\sin^2\theta_W)^2}{(4\sin^2\theta_W)^2}$$

exactly what we set out to show.

1.3 Differential Cross-section

Before we write down the cross-section lets do some relativistic kinematics. We define the 4-momenta in the center of mass frame as

$$p_{e+} = (E, \vec{p}) \qquad p_{e-} = (E, -\vec{p})$$
$$p_Z = (E_Z, \vec{k}) \qquad p_H = (E_H, -\vec{k})$$

where $\vec{p}$ and $\vec{k}$ are related by an angle θ. Since we are ignoring the electron mass, $|p| \approx E$ so $p_{e+}\cdot p_{e-} = 2E^2 = \frac{1}{2}s$. We also get

$$\begin{aligned} (p_Z\cdot p_{e+})(p_Z\cdot p_{e-}) &= \frac{1}{4}s(E_Z^2 - p_Z^2\cos^2\theta) \\ &= \frac{1}{4}s(E_Z^2 - (E_Z^2 - M_Z^2)\cos^2\theta) \\ &= \frac{1}{4}s(M_Z^2\cos^2\theta + (M_Z^2 + \vec{k}^2)\sin^2\theta) \\ &= \frac{1}{4}s(M_Z^2 + |\vec{k}|^2\sin^2\theta) \end{aligned} \tag{13}$$

Then plugging these into Eq. 12 and then into the general expression for the differential cross-section for any $2 \to 2$ process we get

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\vec{k}|}{|\vec{p}|}|\mathcal{M}|^2 = \frac{\alpha^2}{4}A[1 + \frac{1}{2}\frac{\vec{k}^2}{M_Z^2}\sin^2\theta]\frac{|\vec{k}|}{\frac{1}{2}E_{CM}} \tag{14}$$

It is easy to obtain the following expressionsusing relativistic kinematics and turn them into an expression for E_Z^2.

$$\left.\begin{array}{l} E_Z + E_H = E_{CM} \\ E_Z^2 - E_H^2 = M_Z^2 - M_H^2 \\ E_Z - E_H = \frac{M_Z^2 - M_H^2}{E_{CM}} \end{array}\right\} \Rightarrow E_Z^2 = (E_{CM}^2 + M_Z^2 - M_H^2)^2 \tag{15}$$

With some work you can get

$$\vec{k}^2 = E_Z^2 - M_Z^2 = \frac{1}{4s}(s - (M_H + M_Z)^2)(s - (M_H - M_Z)^2) \tag{16}$$

We are asked to find the angular dependence of the differential cross-section for two sets of M_H and E_{CM}. In the first we have $M_H = 113$ GeV and $E_{CM} = 205$ GeV. We can see from Eq. 16 that this corresponds to a Z-momentum that is much smaller than the Z-mass, so the differential cross-section is isotropic.

In the other case $M_H = 125$ GeV and $E_{CM} = 1000$ GeV. Here the Z-momentum is much larger than the Z-mass and correspondingly the differential cross-section's angular dependance will be $\sin^2\theta$.

1.4 Total cross-section

To get the total cross-section we just have to integrate Eq. 14 over the solid angle to obtain

$$\sigma = \pi\alpha^2 A\left[1 + \frac{1}{3}\frac{\vec{k}^2}{M_Z^2}\right]\frac{|\vec{k}|}{\frac{1}{2}E_{CM}} \tag{17}$$

whose dependence on E_{CM} looks like

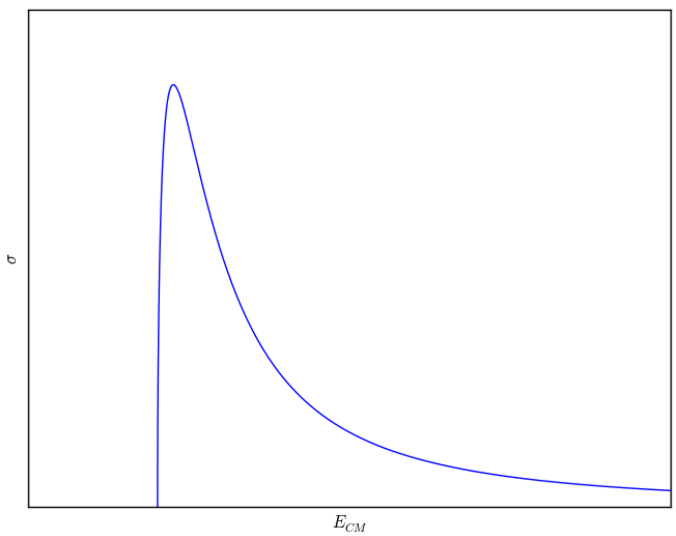

Figure 1: dependence of cross section on E_{CM}