TRANSVERSE STABILITY

VERTICAL MOTION IN STRONG FOCUSSING LATTICES

Variables to consider:

y	vertical coordinate	where $y, y' = 0$ for particle on design orbit.
s	coordinate along path ($s = ct$)	
$y' = \frac{dy}{ds}$	= vertical angle	

$$y' = \theta_y = p_y/p \quad \text{for } p_y \ll p$$

Note that since $= \theta_y$, that y, y' define the vertical phase space, i.e., if we ~~define~~, at any ~~points in the ring~~, the product

$$\sigma_y \, \sigma_{y'} \quad \text{or rms}$$

is proportional to the occupied volume of phase space, and is by Liouville's Thm constant under the application of any time-independent (conservative) force (re-beam coasting in storage ring).

For later. For now, let's recall that our archetypal strong focussing lattice is a series of focus, bend, defocus, bend. But, in vertical, bend ⇒ drift, so

FODO ~~FODO~~ FODOFODOFODO

"FODO" Lattice

What do these elements do to some particle entering them w/ coordinates y, y'?

(A) DRIFTS

No bending, so $y' \to y'$

$y \to y + l y'$ where l is length of drift space.

So, define 2×2 "transfer matrix" $M(s_1, s_0)$

$$\begin{pmatrix} y(s_1) \\ y'(s_1) \end{pmatrix} = M(s_1, s_0) \begin{pmatrix} y(s_0) \\ y'(s_0) \end{pmatrix}$$

so for our drift of length l,

$$M_D(s+l, s) = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

Check:

$$\begin{pmatrix} y(s_1) \\ y'(s_1) \end{pmatrix} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y(s_0) \\ y'(s_0) \end{pmatrix} = \begin{pmatrix} y(s_0) + l y'(s_0) \\ y'(s_0) \end{pmatrix} \quad \checkmark$$

$\boxed{\text{NOTE}}$ True transfer matrix is 6x6, acting on vector

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ t \\ \varepsilon \end{pmatrix}$$

for our purposes, though, $M = \begin{pmatrix} M_x & & \\ & \boxed{M_y} & \\ & & M_t \end{pmatrix}$

block diagonal is 2x2 blocks. We've observed that there is coupling from transverse into longitudinal phase space, but have "decoupled" this by sweeping all under the rug of the dispersion $\eta(s)$.

Ⓑ QUADRUPOLES (thin lens approx)

Just introduces a kink proportional to y. Consider charge moving at c going through a focusing quadrupole of length l_q ($B_x = -G y$)

$$F_y = +q v B_x = -e c G y \qquad (G \text{ gradient})$$

$$\Rightarrow \Delta p_y = F \Delta t = -e c G y \left(\frac{l_q}{c}\right) = -e G y l_q$$

$$\Rightarrow \Delta y' = \Delta \theta = \frac{\Delta p_y}{p_s} = -\frac{e G l_q}{p} y \equiv -\frac{1}{f} y \qquad \boxed{f_{quad} = \frac{p}{e G l_q}}$$

(AP 305)

$\circledast$ Derive 2x2 transfer matrix for thick quad

$\circledast$ Possibly: Calculate dispersion for ~~two~~ bends, then instead quad half-way. Does this use matrix formalism?

Here, we have ignored the fact that, as the particle traverses the quad, y is not constant $\Rightarrow$ thin lens approximation. So ~~at quad no ...~~

So, we approximate quad as simply a kink at $s = s_q$, where s_q is s coordinate of quad center. What is transfer matrix?

~~$z \to z$~~
~~$z' \to z' + \Delta z' = z' + \frac{1}{f} y$~~

$y \to y$
$y' \to y' + \Delta y' = y' - \frac{1}{f} y$

and so the transfer matrix becomes

$$\begin{pmatrix} y(s_q+\delta) \\ y'(s_q+\delta) \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}}_{M_{\text{quad (focusing)}}} \begin{pmatrix} y(s_q-\delta) \\ y'(s_q-\delta) \end{pmatrix}$$

for thin lens approx.

Note that this is ~~first~~ focussing quad F (+ displacement y in quad $\Rightarrow$ increased angle). For defocussing quad D of same strength $-f \to +f$

$$M_{\text{defocus}} = \begin{pmatrix} 1 & 0 \\ +\frac{1}{f} & 1 \end{pmatrix}$$

· Note that "f" makes sense as a focal length. See page facing AP30? $\boxed{\text{AP31}}$

(*) What is tune of 1 TeV machine w/ $G = 10$ T/m & ~~mrro~~ $r = 1$ km?
Assuming single focussing quad around entire diameter

General Features of Vertical Motion: Hill's Equation

Let's derive some general aspects of vertical motion in strong focussing lattices which we can apply to help us understand our thin-lens FODO lattice special case.

Consider motion of particle in infinitely long focussing quad (opposite of thin lens approximation) of ~~strong~~ gradient $-G$ T/m. So, from before

$$\Delta y' = -\frac{eG}{p} \underbrace{l_q}_{\Delta s} y$$

This is l_q, length of quad.

$$\Rightarrow \frac{dy'}{ds} = -\frac{eG}{p} y$$

Since $y' = dy/ds$, then this is an equation of the form

$$\frac{d^2 y}{ds^2} = -ky \qquad \text{with} \qquad k = \frac{eG}{p}$$

and so, as a function of s, y exhibits simple harmonic motion with a "period" ~~"wavelength"~~ in s of

$$\tau_s = 2\pi\sqrt{\frac{p}{eG}}$$

Some obvious terminology which we'll generalize in a ~~latter~~ moment for more realistic lattices:

① Tune : Let $\mathcal{C}$ = circumference of machine. Then, we define the "TUNE" ν via

$$\nu = \frac{\mathcal{C}}{\mathcal{I}_s} = \text{number of oscillations slightly off-orbit particle undergoes per cycle (real number)}$$

② Fractional Tune:

$$\tilde{\nu} = \text{Remainder}\left(\frac{\mathcal{C}}{\mathcal{I}_s}\right)$$

It turns out, empirically, that fractional tunes

$\tilde{\nu} = 0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \ldots$ (common rationals)

are BAD! This is roughly because beam samples same location in quad/dipole/etc field on every orbit ⇒ defects add + beam is lost

{cf p 33 of Bjorken's article}

MAJOR DRIVING FORCE OF ACCELERATOR DESIGN

⇒ AVOID INTEGER, COMMON RATIONAL TUNES ⇐
("Resonances")

All accelerators are designed w/ "tuneable tunes" (higher multipole fields) to make sure you can run at irrational tunes.

③ Phase Advance

$$\mu(s,s_0) \equiv \phi(s) - \phi(s_0) = 2\pi \frac{(s-s_0)}{T_s}$$

⇒ change in phase of sinusoid associated w/ ~~travel~~ propagation of particle from s_0 to s.

Note that All of these, including T_s, are properties of the lattice, not of the trajectory of particular particles through the phase space contained by the lattice. (i.e., particle's initial conditions at s_0: $y(s_0)$, $y'(s_0)$)

Hill's Equation

As we know, a realistic lattice can't continually focus in one dimension. Thus, in the most general lattice,

$$k \rightarrow k(s)$$

constant restoring force

variable restoring force; changes from − to +, and is 0 for bends.

Thus, the behavior of vertical motion is governed by

Hill's Eqvation

$$\boxed{\frac{d^2y}{ds^2} + k(s)y = 0}$$

really, think of

$$\frac{d}{ds}(y'(s)) = -k(s)\,y$$

relates phase space parameters y' + y!

with the obvious constraint $k(s+C) = k(s)$ for a storage ring.

Motivated by the form of the transfer matrix, and our understanding of beamline elements in ~~By analogy w/ the constant k case~~, we propose two forms of transfer matrix, solution : a "sine like" and a "cosine-like" solution :

$S(s,s_0)$ where $S(s_0,s_0)=0$ $S'(s_0,s_0)=1$ ~~$S_y(s_0)=0$ $S'_y(s_0)=1$~~

$C(s,s_0)$ where $C(s_0,s_0)=1$, $C'(s_0,s_0)=0$ ~~$C_y(s_0)=1$ $C'_y(s_0)=0$~~

Really, we say that, in terms of transfer function, solution must look like this.

and so in general, given initial conditions $y(s_0)$, $y'(s_0)$

$$y(s) = C(s,s_0)\,y(s_0) + S(s,s_0)\,y'(s_0)$$

Note: if $k(s)=k$ = constant, then $C(s,s_0)$ = cos of $S(s,s_0)$ = sin and this behaves correctly. So this generalized the nature of the oscillations!

Differentiating, we have

$$y'(s) = C'(s,s_0)\,y(s_0) + S'(s,s_0)\,y'(s_0)$$

and so, in terms of these general solutions, our general transfer matrix $M(s,s_0)$ becomes

$$\xi(s) \doteq \begin{pmatrix} y(s) \\ y'(s) \end{pmatrix} = \overbrace{\begin{pmatrix} C(s,s_0) & S(s,s_0) \\ C'(s,s_0) & S'(s,s_0) \end{pmatrix}}^{M(s,s_0)} \begin{pmatrix} y(s_0) \\ y'(s_0) \end{pmatrix}$$

where we have defined the "phase space vector" $\xi(s) = \begin{pmatrix} y(s) \\ y'(s) \end{pmatrix}$

⇒ Note that C is <u>not</u> a cosine fnc, S is <u>not</u> a sine fnc, and

$$\frac{dS(s,s_0)}{ds} \neq C(s,s_0).$$

We can derive a couple of conditions which hold generally on M:

"Wronskian" - yes, but don't need to mention.

(1) $\det M = CS' - SC' = 1$

To see this, note that

$$\frac{d\left[\det M\right]}{ds} = \cancel{C'S'} + CS'' - \cancel{S'C'} - SC'' = CS'' - SC'' = 0$$

by Hill's equation.

Then, since clearly $\det\left[M(s_0,s_0)\right] = \det\left[\mathbb{1}\right] = 1$, it is shown.

two eigenvalues related in this way.

② The eigenvalues of $M(s_0+C, s_0)$ are of the form $\lambda = e^{\pm i\mu}$

Let $\xi_E(s_0)$ be an eigenvector of $M(s_0+C, s_0)$, so

$$M(s_0+C, s_0)\xi_E(s_0) = \lambda \xi_E(s_0).$$

i.e., n^{th} turn same as first turn since M property of ring. Not true if we consider $M(s_0+s, s_0)$ instead!

After n turns, (which is n application of $M(s_0+C, s_0)$, which is why you must use s_0+C, and ~~not~~ not an arbitrary $s_0+\Delta s$!!)

$$M^n(s_0+C, s_0)\xi_E(s_0) = \lambda^n \xi_E(s_0).$$

two eigenvalues, related in this way!

$\Rightarrow$ λ must be of modulus 1 for stability. But since $\det M(s_0+C, s_0) = 1$ by ① then $\lambda = e^{\pm i\mu}$, for some μ.

It ~~will turn out~~ turns out that $\mu = 2\pi\nu$ is the total phase advance associated w/ 1 turn around the ring, i.e., ν is the tune.

$$\boxed{\nu = \mu/2\pi}$$

ROAD MAP

It's now time to solve Hill's Equation. Armed w/ the above info, this will allow us to calculate tunes, beam spots, & divergences, ~~and ... stability conditions~~, given $k(s)$. We can then plug in our thin-lens approximate ~~k(s)~~ FODO $k(s)$ and derive stability conditions (which will be a homework problem).

Here we go...

note:- could posit $\sqrt{\beta(s)\varepsilon}\cos\phi(s)$ also, would derive same constraints.

General Properties of Hill's Eqn Solutions

$$\boxed{\frac{d^2y}{ds^2} + k(s)\,y = 0}$$

Posit the following solution to Hill's Equation

$$y(s) = \sqrt{\beta(s)\,\varepsilon}\,\sin\phi(s)$$

SAVE ON BOARD

w/ $\beta(s+C) = \beta(s)$ → compare to $A\sin(kx)$

This is just a sinusoid w/ varying amplitude and rate of phase advance. NOTE that $\beta(s)$ is an absolute property of the ring, $(\beta(s+C) = \beta(s))$, but $\phi(s)$ also depends upon the particles initial state $\phi(s) = \mu(s,s_0) + \phi(s_0)$, where the phase advance from s_0 to s $\mu(s,s_0)$ is an absolute property of the ring, but ~~$\phi(s_0)$~~ $0 < \phi(s_0) < 2\pi$ is an arbitrary phase at s_0 specified by the particle's initial condition. Finally, the <u>constant</u> scale factor $\sqrt{\varepsilon}$ is thrown in for reasons which will become apparent later.

Amp A varies, and $\phi(s)$ is not constant advance. $A(s)$ still prop of <u>ring</u> tho!

Initial conditions $\varepsilon, \phi(s_0)$. $\beta(s)$ and $\phi(s-s_0)$ are machine props

In order to solve Hill's Equation, we'll need to plug this solution back in to derive constraints on β, ϕ. To do this, we'll need (dropping the explicit arguments of s)

← this factor will later be shown to be $\equiv 1$.

$$y' = \sqrt{\frac{\varepsilon}{\beta}}\left(\overset{\beta\phi'}{\cos\phi} + \frac{1}{2}\beta'\sin\phi\right) \quad \ldots \quad (1)$$

SAVE ON BOARD

Differentiating again, & plugging back into Hill's Equation $0 = y'' + ky$ yields after a little algebra

$$0 = \left\{ -\frac{\sqrt{\varepsilon}}{\beta^{3/2}} \frac{1}{4} (\beta')^2 + \sqrt{\frac{\varepsilon}{\beta}} \frac{1}{2} \beta'' + k\sqrt{\beta\varepsilon} - \sqrt{\beta\varepsilon} (\phi')^2 \right\} \sin\phi$$

$$+ \left\{ \phi' \beta' \sqrt{\frac{\varepsilon}{\beta}} + \sqrt{\beta\varepsilon}\, \phi'' \right\} \cos\phi$$

Since sin and cos are linearly independent fncs, their coefficients must vanish independently, giving us the 2 constraints we need to solve for β, ϕ.

<u>$\cos\phi$ TERM</u>

$\to \ln\beta = -\ln\phi' + C$

$\Rightarrow \ln\beta = \ln 1/\phi' + C \Rightarrow \beta = \frac{1}{\phi'} \cdot e^{C}$ (A)

$\Rightarrow A/\beta = \phi'$

$$\sqrt{\beta\varepsilon}\, \phi'' + \beta'\phi' \sqrt{\frac{\varepsilon}{\beta}} = 0 \implies \frac{\beta'}{\beta} = -\frac{\phi''}{\phi'} \Rightarrow \text{integrating}$$

$$\Rightarrow \boxed{\frac{A}{\beta} = \phi' = \frac{d\phi(s)}{ds}}$$

{ Don't panic - you get a ln on both sides - just exponentiate both sides to see this!

THE RATE OF PHASE ADVANCE ($\dot{\phi}'$) IS PROPORTIONAL TO THE INVERSE OF THE "β-FUNCTION" $\beta(s) \Rightarrow \beta(s)$ is "local wavelength"

In fact, we can ignore the constant A via the rescaling $\beta \to \beta/A \quad \varepsilon \to A\varepsilon$, and so ~~integrating~~ $d\phi(s) = \frac{ds}{\beta(s)}$. Integrating to get phase advance $\mu(s,s_0)$ between s_0 & s,

$$\mu(s,s_0) = \int_{s_0}^{s} \frac{ds'}{\beta(s')} \quad , \text{ or}$$

$$\boxed{\phi(s) = \int_{s_0}^{s} \frac{ds'}{\beta(s')} + \phi(s_0)}$$

<u>$\sin\phi$ TERM</u>

Substituting ~~for~~ $\phi' = 1/\beta$, after some algebra

$$\boxed{-\frac{1}{4}(\beta')^2 + \frac{1}{2}\beta\beta'' + k\beta^2 = 1}$$

which is a 2nd order D.E. for β in terms of $k(s)$ ~~of ...~~ which can be solved for β in principle. In practice, our explicit knowledge of $M(s, s_0)$ for the FODO lattice will preclude the necessity of solving this eqn, and we'll just forget about it.

<u>RELATION BETWEEN β FNC AND TRANSFER MATRIX</u>

then, physical int. of β fnc. + parameter ε. Then, <u>rotate etc.</u> + summarize! Mention Twiss params.

As mentioned above, we can avoid solving the messy diffeq for β since we explicitly know the transfer fnc. for the FODO lattice. In order to do this, then, we must ~~relate the β fnc. to~~ write the transfer matrix in terms of the β fnc.

There's a little trick used to do this, which is to calculate the transfer matrix

$$M(s_0+C, s_0) = \begin{pmatrix} C(s_0+C, s_0) & S(s_0+C, s_0) \\ C'(s_0+C, s_0) & S'(s_0+C, s_0) \end{pmatrix}$$

for ~~a full~~ one complete turn around the ring. We start by writing down the sine-like solution, which has the ~~property $S(s_0, s_0) = 0$~~ properties

$$S(s_0, s_0) = 0 \qquad (1)$$
$$S'(s_0, s_0) = 1 \qquad (2)$$

and so, including this time explicit dependences on s,

$$S(s, s_0) = \sqrt{\varepsilon\beta(s)}\,\sin\left[\int_{s_0}^{s}\frac{ds'}{\beta(s')} + \overset{\to 0 \text{ by } (1)}{\cancel{\phi(s_0)}}\right]$$

Note that $\phi(s_0)$ here is the ϕ specific to the S solution! In principle ε is too, but $\varepsilon = \beta(s_0)$ fixes this to be same for C, S sol'n!

$$S'(s, s_0) = \frac{\beta'(s)}{2}\sqrt{\frac{\varepsilon}{\beta(s)}}\,\sin\left[\int_{s_0}^{s}\frac{ds'}{\beta(s')}\right] + \underbrace{\sqrt{\frac{\varepsilon}{\beta(s)}}\,\cos\left[\int_{s_0}^{s}\frac{ds'}{\beta(s')}\right]}_{\text{by fund'l thm of calculus } \frac{d}{dx}\int_{x_0}^{x} f(x')dx' = f(x)}$$

which tells us that by (2) (i.e., at point $s = s_0$, so only cos term survives)

$$\boxed{\varepsilon = \beta(s_0)}$$

Note (?): This is not full solution, so this is not ε of courant invariant?!!?

Yes! This is because the use of $S(s, s_0)$ chooses initial phase of 0, and sets $\varepsilon = \beta$. ~~at which point $\beta = \varepsilon$~~

It (ε) doesn't show up after this. The ε that shows up is related to full sol'n.

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Finally, we make use of the fact that

$$\int_{s_0}^{s_0+C} \frac{ds'}{\beta(s')} = \text{total phase advance } \mu \text{ incurred in}$$

one trip around ring $= 2\pi\nu$, where ν is the tune

$$\boxed{\int_{s_0}^{s_0+C} \frac{ds'}{\beta(s')} = 2\pi\nu}$$

Putting this together, and using that $\beta(s_0+C) = \beta(s_0)$ (ring periodicity)

$$S(s_0+C, s_0) = \beta(s_0) \sin 2\pi\nu$$

$$S'(s_0+C, s_0) = \left[\beta'(s_0)/2\right] \sin 2\pi\nu + \cos 2\pi\nu$$

and so

$$M(s_0+C, s_0) = \begin{pmatrix} C(s_0+C, s_0) & \beta(s_0)\sin 2\pi\nu \\ C'(s_0+C, s_0) & \left[\beta'(s_0)/2\right]\sin 2\pi\nu + \cos 2\pi\nu \end{pmatrix}$$

To get the explicit form for M_{11} and M_{21}, recall that

$\det[M] = 1$ and $\mathrm{Tr}[M] = \Sigma$ eigenvalues $= e^{+2\pi i\nu} + e^{-2\pi i\nu}$

$= 2\cos 2\pi\nu$

(*) Use this to show $M_{11} = \cos 2\pi\nu - \frac{\beta'}{2} \sin 2\pi\nu$

$M_{21} = -\frac{1}{\beta}\left(1 + \frac{(\beta')^2}{4}\right) \sin 2\pi\nu$

~~yielding~~ $\det[M] = \underline{1}$ (A)

$Tr[M] = \Sigma \text{eigenvalues} = e^{+2\pi i\nu} + e^{-2\pi i\nu} = 2\cos 2\pi\nu$ (B)

$= 2\cos\mu$

In fact, we need not explicitly calculate M_{11}, M_{21}, since with (B) we already have enough info to calculate ~~plus~~ the beta-fnc $\beta(s)$. Thus, ~~we~~ In fact, we see that

To get $\beta(s)$ at any point s in the ring, knowing the explicit transfer matrix M for all beamline elements

(*)

① Compute $M(s+C, s)$

② $Tr[M(s+C,s)] = 2\cos 2\pi\nu$ ← gives ν

③ $\beta(s) = \dfrac{M_{12}}{\sin 2\pi\nu}$

NOTE: So β is non-local! Change M at 1 point and β changes everywhere in Ring!
~~Because M changes~~

NOTE that $-1 < \dfrac{Tr[M]}{2} < 1$ for all this to work

$\boxed{\text{THIS IS CONDITION FOR TRANSVERSE STABILITY!!}}$

Note that M is simply calculable w/ our two beamline elts: drift + quads, so we never have to explicitly solve Hill's eqn!

A) Derive stability condition for FODO lattice
B) Calculate β_{min}, β_{max} as a func. of phase advance per cell. By how much does beam size vary for $\mu = 90°$?
IMPORTANT HOMEWORK

Final note about our FODO lattice

In this development, since the bend enters nowhere, the only thing special about considering $M(s_0+C, s_0)$ is that ~~the general~~ any lattice repeats itself on this scale, not that you return to the same point in space. Thus, all those same arguments apply to the ~~matrix~~ transfer matrix $M(s_0 + C/N, s_0)$, where N is the no. of FODO cells in the lattice, ~~Thus~~ with μ ~~now~~ now the phase advance per cell. This makes it relatively easy to calculate β at any point on the ring, and to derive stability conditions.

INSERTIONS → more detail in Bjorken notes, 10/30 lecture.

A ring composed of N FODO lattice elements, while ideal for calculations, is useless. There are places in the ring where you have to ~~install special~~ interrupt the lattice & install special optics, such as insertion & ejection points. The classic insertion is the "low β insertion" region" ~~(we'll see shortly that beam~~ (we'll see shortly that ~~which~~ beam size is prop. to $\sqrt{\beta}$) at which detectors are placed (more lum if beams crossed here). If the β fnc is identical at the entrance & exit of the insertion, you simply get some extra phase advance (tune shift) which you can take out w/ your "tune tuning" mechanism, but otherwise optics for rest of ring is unchanged and lattice picks up where it left off!

⇒ "β-matching"

MORE ON THE β-FUNCTION; THE COURANT INVARIANT

Recall our solution to Hill's Equation

$$y(s) = \sqrt{\beta(s)\varepsilon}\,\sin\phi(s)$$

We've talked at some length about the interpretation of $\phi(s)$. What more can we say about $\beta(s)$ and this funny normalization factor ε? ~~Differentiate~~ Recall $y'(s)$

$$y'(s) = \sqrt{\frac{\varepsilon}{\beta}}\left(\cos\phi + \frac{1}{2}\beta'\sin\phi\right)$$

Turning these around

$$\sin\phi = y/\sqrt{\beta\varepsilon} \qquad \cos\phi = \sqrt{\frac{\beta}{\varepsilon}}\left(y' - \frac{\beta'}{2\beta}y\right)$$

but since $\sin^2 + \cos^2 = 1$

$$\frac{y^2}{\beta} + \beta\left(y' - \frac{\beta'}{2\beta}y\right)^2 = \varepsilon$$

With some playing around, you can convince yourself that this is the equation for a rotated ellipse in y,y' space with area $A = \pi\varepsilon$, INDEPENDENT OF THE VALUES OF $\beta + \beta'$!! (This is, in fact, a necessary consequence of Louiville's Thm, as we'll see)

Notes on Bjorken notes (a facing page) on 10/23.

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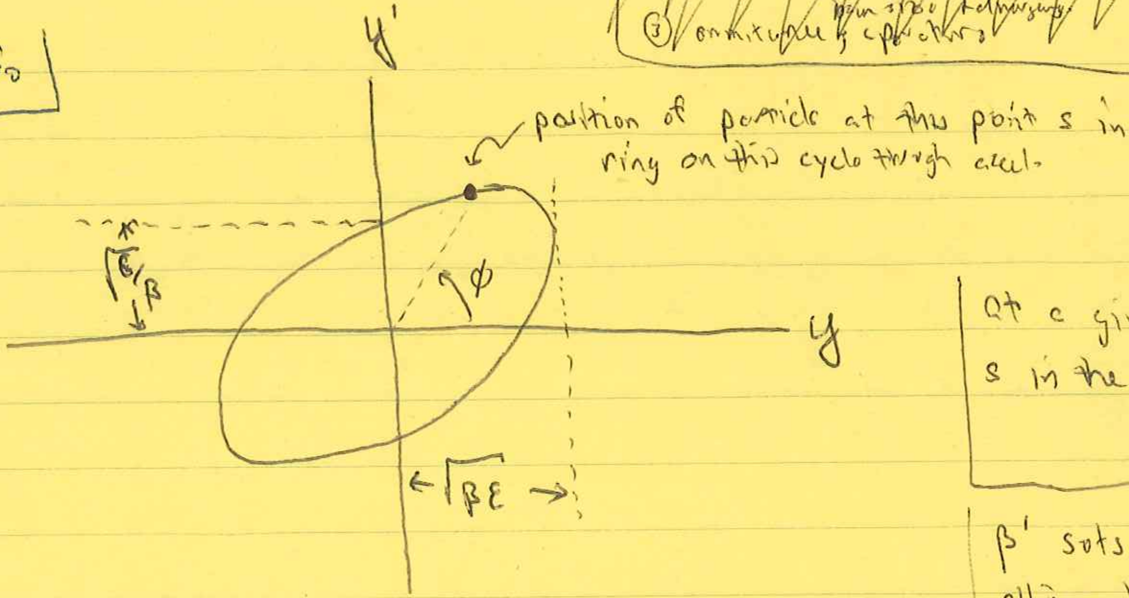
$\boxed{s = s_0}$



at a given point s in the ring!

β' sets cant of ellipse, but not intercepts!

Some important points to be made

① This ellipse specifies the possible phase space location for a particle as it circulates about the ring. By ~~construction~~, ϵ is independent of s, but $\beta(s)$ varies $\Rightarrow$ this ellipse changes length & orientation as you progress around ring, but always has constant area ϵ. ~~In other words~~ Two particles w/ different values of ϵ will never occupy the same phase space point, even at different times. Only true at a given pt s! not really important!

$\epsilon \equiv$ Courant Invariant

Particle travels around surface of invariant torus specified by ϵ

② Since $\beta(s+C) = \beta(s)$, a particle will trace out the entire ellipse, adding a point at an additional phase advance of $\Delta\phi = 2\pi\nu$ every time it goes around ring, as long as ν is not close to a fairly rational number.

(2')

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(3) NOTE that up until now, we have ~~talk~~ talked only about single particle orbits. Obviously, at some point we need to introduce the concept of an ensemble of particles, and this is a convenient place to do it.

[3A] Emittance

(note that $\varepsilon = 0 \Rightarrow$ on-orbit particle!)

Each particle in the ensemble (beam) can be characterized by the value of its Courant Invariant ε (note that more than 1 particle can have any given value of $\varepsilon \neq 0$). Since ε is constant for each particle in the orbit, then $\langle \varepsilon \rangle$ for the ensemble ~~is a~~ ~~good~~ characterizes the constant phase space area occupied by the beam (~~too~~ exactly Liouville's th'm), and is called the emittance. Specifically, say in y

$$\text{emittance} \equiv \langle \varepsilon \rangle_y = \pi \sigma_y \sigma_{y'} \quad \text{rms}$$

~~[3B] Beam characteristics~~

Usually, the brackets are dropped, and the emittance (the ensemble property) is just written ϵ.

[3B] Beam Characteristics

With this definition of the emittance ϵ, then

rms spot size $\cong \sqrt{\beta \epsilon}$
rms beam divergence $\cong \sqrt{\frac{\epsilon}{\beta}}$
} for $\beta(s)$ at any point s in ring

[3C] Aperture or Admittance

Maximum value of Courant Invt ϵ accepted by Lattice.

(4) During acceleration, transverse variables (y) are preserved, but longitudinal variables are boosted. Thus, y' decreased by $1/\gamma$ during acceleration (not a time-independent force $\Rightarrow$ no Liouville)

$$\epsilon = \frac{\epsilon_0}{\gamma}$$

ϵ_0 ← "normalized emittance"

ϵ ← true emittance

Acceleration naturally reduces emittance, linearly w/ γ. (But point x-sections tend to go as $1/s \sim 1/E_{cm}^2$, so you still need tighter focussing!!)

Horizontal motion
~~Dispersion~~
Chromaticity
Lum / Beam-beam int

Word about linear accel: no beam/beam int.
no synch radiation

(5) TWISS PARAMETERS (DISCARD IF TIME IS SHORT)

The value ε of the Courant Inv. corresponding to the emittence $\langle \varepsilon \rangle$ defines an ellipse in phase space

$$\varepsilon = \beta(s)\, y'^2 + \gamma(s)\, y^2 + 2\alpha(s)\, y y'$$

with β just the β-fnc, $\alpha = -\beta'/2$, and

$$\gamma = \frac{(\beta')^2}{4\beta} + \frac{1}{\beta} = \frac{1+\alpha^2}{\beta}.$$

The functions α, β, γ are known as the TWISS PARAMETERS, and sometimes the transfer matrix is written as a 3×3 operator on this 3-dim vector; since only 2 are linearly independent though, there is nothing more to this than our ~~2A~~ 2×2 transfer matrices which operate on y, y'.

Now, we'll close out our discussion of accel physics with a few important effects

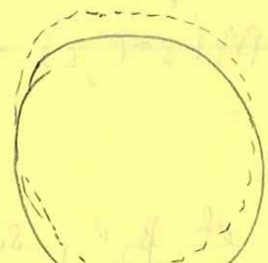
Horizontal motion (bending)
~~Dispersion~~ ~~Dispersion~~
Chromaticity
Beam-beam effects

} can limit machine luminosity.

→ discard maybe?

HORIZONTAL MOTION

Consider a particle lattice which is a single long bend magnet and a slightly off-orbit particle



Thus, the bend magnets do provide some (rather weak) ~~focussing~~ additional focussing, with a period

$\tau_s = 2\pi\rho = 2\pi/\sqrt{k_{bend}}$ ← from p. $\boxed{AP32}$ $\tau_s = \frac{2\pi}{\sqrt{k}}$ ⇒ $\boxed{k_b = \frac{1}{\rho^2} = (1/f)/\ell}$ ⇒ quad strength per unit length

quad strength / unit length $k_b = (1/f)/\ell = 1/\rho^2$

so Hill's equation becomes for horizontal motion

$$\frac{d^2 x}{ds^2} + k_{vert}(s)\,x + k_{bend}\,x = 0$$

actually, $-k_{vert}(s)$, since ~~focussing~~ → defocussing, but this corresponds to simply ½ cell ~~translation~~ rotation of ring; ~~no problem~~ transparent

-or-

OK - don't belabor

$$\frac{d^2 x}{ds^2} + k(s)\,x + \frac{x}{\rho^2} = 0$$

$\circledast$ Homework - show this, a la Bjorken notes, 10/25 ← Requires result $\langle\beta\rangle \doteq \frac{2l}{\sin\mu}$ from previous problem!

This can be treated as a small perturbation, and leads to a tune shift

$$\Delta\nu_{curv} \doteq \left(\frac{\mu}{\sin\mu}\right)\frac{1}{\nu}$$

[x tune slightly different than y tune]

where μ is the phase advance per cell. This is a tune shift which is small compared to 1, and so no problem since we already know that the tune must be tunable.

→ sin ν is typically ~100

(CAN THIS!)

→ note: for arbitrarily long bend, displacement due to off momentum ~~(change focal length of quad ... energy to first order!)~~ + long lever arm dominates small change in focal length due to off momentum, Calc. dispersion in this limit.

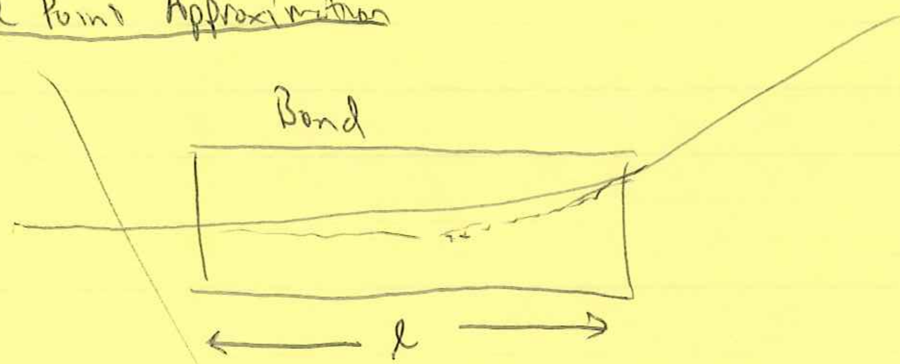
~~Chromaticity~~ Dispersion

Recall - ~~transverse~~ x position depends on momentum. We defined the dispersion fnc. $\eta(s)$ via

$$x = \eta(s)\frac{\Delta p}{p_0} \qquad (\eta(s)=0 \Rightarrow x=0 \Rightarrow \text{on orbit})$$

$\eta(s)$ has units of length, and for a ring w/ no focussing elements, $\eta(s)$ = constant = ρ (radius of ring). For a typical momentum spread of $\Delta p/p \sim 10^{-4}$, this would lead to a huge ($\gtrsim 10$ cm) beam size. Luckily, the focussing substantially reduces $\eta(s)$. We don't have time to work through it, but here's a rough idea of how to treat it...

Bend Point Approximation



To very good approximation, a bend magnet can be treated as a kink of size θ_{bend} at the center of the magnet. Since our coordinate system follows the on energy beam through the bend, we only have to consider the extra kink.

$$\Delta z' = \Delta\theta_{BEND} = \frac{\ell}{\rho}\frac{\Delta p}{p}$$

due to the momentum error Δp, giving us a transfer element

$$M_{KINK} = \begin{pmatrix} 1 & 0 \\ 0 & 1 + \frac{\ell}{\rho}\frac{\Delta p}{p} \end{pmatrix}$$

at the center of bend, and our bend matrix becomes

$$\begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \ell/2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 + \frac{\ell}{\rho}\frac{\Delta p}{p} \end{pmatrix}\begin{pmatrix} 1 & \ell/2 \\ 0 & 1 \end{pmatrix}$$

Realizing that the dispersion is max (min) at an F (D) quad, & working out the transfer matrix, the equation

$$\begin{pmatrix} \eta_{max} \\ 0 \end{pmatrix}_F \begin{pmatrix} M \end{pmatrix} \begin{pmatrix} \eta_{min} \\ 0 \end{pmatrix}_D$$

yields (μ = phase advance / cell)

$$\eta_{\substack{max \\ min}} = \frac{\ell^2}{\rho \sin^2 \mu/2}\left(1 \pm \tfrac{1}{2} \sin \mu/2\right) \; ; \quad \langle \eta \rangle = \frac{\ell^2}{\rho \sin^2 \mu/2}$$

For, say, Tevatron, $\ell \approx 10\,m$, $\rho \approx 1\,km$, $\mu \approx 68°$

$\Rightarrow \langle \eta \rangle \sim 10\,m \Rightarrow \sim 1\,mm$ beam size due to dispersion (except in low β insertion...).

(Also, recall impact on longitudinal phase stability).

Chromaticity

In addition to the ~~dipole bend~~ ~~bend/kick~~ bend strength, the quad focal length also depends upon momentum: $f = P/eG l_q$

$$\frac{\Delta f}{f} = \frac{\Delta p}{p} \qquad (1)$$

Since focussing strength depends on momentum, so does the tune ν. Define chromaticity ξ via

$$\frac{\Delta \nu}{\nu} = \xi \frac{\Delta p}{p}$$ (Beware: sometimes $\Delta \nu = \xi \frac{\Delta p}{p}$)

For FODO lattice w/ bends of length l, quads of strength $1/f$, we'll see from the homework that (μ = phase advance/cell)

$$\cos\mu = 1 - \frac{l^2}{2f^2} \qquad (2)$$

Using (1) + (2) + expanding (after some arithmetic - see next page)

$$\frac{\Delta \nu}{\nu} = \frac{\Delta \mu}{\mu} = \underbrace{-\frac{4\sin^2 \mu/_2}{\mu \sin\mu}}_{\substack{\text{"natural" chromaticity} \\ \xi}} \frac{\Delta p}{p}$$

(see next page for more complete derivation)

$$\sin^2\frac{\mu}{2} = \frac{1}{2}(1-\cos\mu) = \frac{1}{2}\left(1 - 1 + \frac{\ell^2}{2f^2}\right) = \frac{\ell^2}{4f^2} \Rightarrow \frac{\ell^2}{f^2} = 4\sin^2\mu/2$$

$$\cos\mu = 1 - \frac{\ell^2}{2f^2}$$

$$\frac{d\cos\mu}{d\mu} = -\sin\mu \Rightarrow \cos(\mu+\Delta\mu) = \cos\mu - \Delta\mu\sin\mu$$

$$\cos(\mu+\Delta\mu) = 1 - \frac{\ell^2}{2(f+\Delta f)^2} = 1 - \frac{\ell^2}{2f^2}\frac{1}{(1+\Delta f/f)^2}$$

$$\Rightarrow \cancel{\cos\mu} - \Delta\mu\sin\mu = 1 - \frac{\ell^2}{2f^2}\left(1 - 2\frac{\Delta f}{f}\right) = \underbrace{\cancel{1 - \frac{\ell^2}{2f^2}}}_{\cos\mu} + \frac{\ell^2}{f^2}\frac{\Delta f}{f}$$

$$\Rightarrow -\Delta\mu\sin\mu = 4\sin^2\frac{\mu}{2}\frac{\Delta f}{f} = 4\sin^2\frac{\mu}{2}\frac{\Delta p}{p}$$

$$\Rightarrow \boxed{\frac{\Delta\mu}{\mu} = -\frac{4\sin^2\mu/2}{\mu\sin\mu}\frac{\Delta p}{p}}$$

✓

$\left\{ \begin{array}{l} \text{Proton machines: } \Delta\nu \lesssim 2\times10^{-3} \\ \text{Electron machines: } \Delta\nu \lesssim 3\text{-}5\times10^{-2} \end{array} \right.$ ← synch radiation damping tends to mix long. phase space so don't hit resonances as much.

For a typical $\mu \sim 90°$, this is of order ~1

⇒ tune spread of order

$$\Delta\nu = \xi \underset{\sim 100}{\nu} \underset{\sim \text{few}\times10^{-4}}{\frac{\Delta p}{p}} \Rightarrow \sim \text{few}\times .01 \text{ for TEVATRON}$$

Considerable bit large, usually reduced w/ corrective optics (can't go into this here - sextupoles help). Insertions (low β) w/ intense focussing also can add substantial chromaticity if not designed properly!

Luminosity & the Beam-Beam Interaction

We can't get away from accelerator physics w/out defining the luminosity $\mathcal{L}$

$$\#\text{EVENTS} = \text{CROSS-SECTION} \times \int \mathcal{L}\, dt$$

$\int \mathcal{L}\, dt$ usually quoted in b^{-1} (inverse barns)

Typical run (1 yr) of e^+e^- collider ~ 100 pb^{-1}, i.e., 100 events per pico (10^{-12}) barn of cross section.
At Z^0, $\sigma \sim 30$ nb $= 3\times10^4$ pb, so 100 pb^{-1} ⇒ 3×10^6 Z^0's
⇒ LEP sample! (per detector).

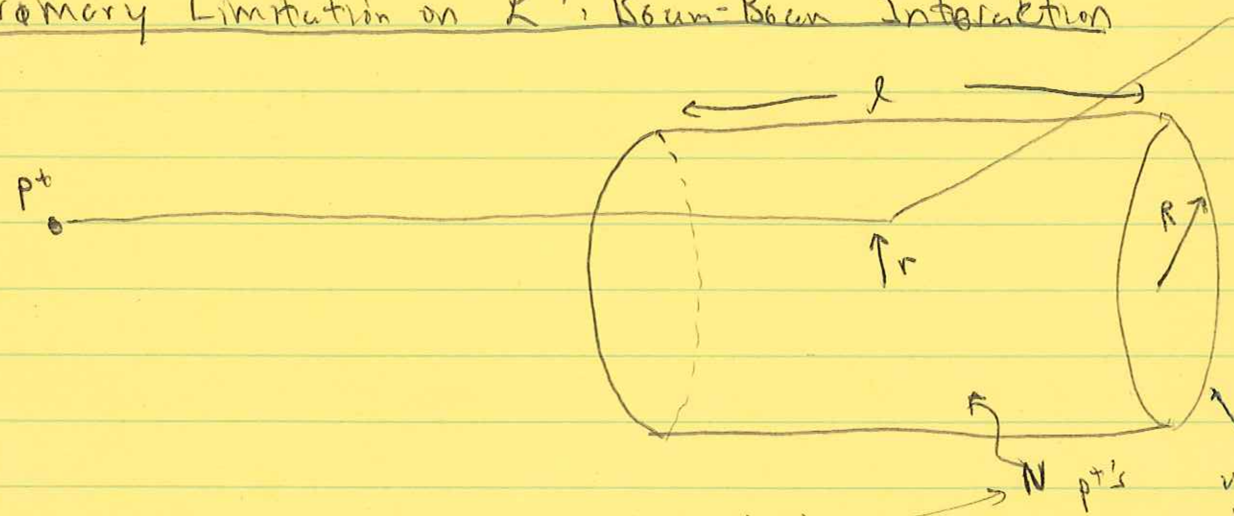
~~$p\bar{p}$ machines typically run @ 10's fb^{-1} per year! ??~~

AP55

$\mathcal{L}$ is ~~often~~ itself a rate, usually quoted in cm^{-2}/s. In these units, $\sim 10^{34}$ is state of the art for e^+e^- colliders (B factory)

$$10^{34}\ cm^{-2}/sec = 10^{10}\ b^{-1}/sec \doteq 10^{17}\ b^{-1}/yr \doteq 100\ fb^{-1}/yr \checkmark$$

<u>Primary Limitation on $\mathcal{L}$: Beam-Beam Interaction</u>



Gauss's Law (uniformly populated cylinder)

$$2\pi r \ell E = \frac{eN r^2}{R^2}$$

$$\Rightarrow \Delta p_\perp^{elec} = eE\ell = \frac{e^2 N}{2\pi R^2}\, r$$

Ampere's Law

Current density is $c \times \frac{eN}{V} = \frac{cNe}{2\pi R^2 \ell}$

Current $= A \cdot \frac{cNe}{2\pi R^2 \ell} = 2\pi r^2 \frac{cNe}{2\pi R^2 \ell} = \frac{eNr^2}{R^2 \ell}$ if $c=1$

$$2\pi r B = \frac{eN r^2}{R^2 \ell}$$

$$\Rightarrow \Delta p_\perp^{mag} = e(\vec{v} \times \vec{B})\ell = \frac{e^2 N}{2\pi R^2}\, r \qquad (c=1)$$

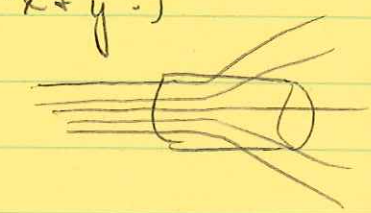
$\boxed{\text{APS6}}$

Both electric + magnetic kick are in same direction

$$\boxed{\Delta p_\perp = \frac{e^2 N}{\pi R^2} r}$$

This is just a kick which goes linearly in the transverse distance from the beam center
⇒ defocussing lens (simultaneously in $x + y$?)

⇒ "PLASMA LENS"



In any case, this is an extra focussing element in lattice, β match, and then
⇒ tune shift. No problem?

But wait ... this was for uniformly populated cylinder. In fact, this is not case - distribution is gaussian or worse

TUNE SPREAD. $\Delta\nu$

sychrotron damping mixes phase-space back up

Recognized that $\Delta\nu \lesssim .03$ (.003) for e^+e^- (pp) machine is fundamental limit on production of dense, highly luminous beams!

* Can't be fixed - no correlated phase space parameter (e.g., as for p for chromaticity) to compensate w/ optics

How to get around this?

① Multiple bunches!

② Linear e^+e^- colliders! (who cares about tune shift, or synch rad, for that matter!)

Finally, note that if two beams have opposite sign, beam-beam effect is focussing! Still produces $\Delta\nu$, but for very dense beams, this can enhance $\mathcal{L}$!

SLC: 5×10^{10} e^+/e^- in ~1μm beams

"pinch effect" ~ ×2.

Can be much higher in NLC, but large disruption is detector background issue!!