PHYSICS 221A - HOMEWORK IV

Problem 1.

Clearly, E_{cm} for two countercirculating beams of energy E is $2E$.
For a highly relativistic beam of energy E colliding with a target of mass m,

$$s = E_{cm}^2 = \left[(E,E,0,0)+(m,0,0,0)\right]^2 = (E+m, E, 0, 0)^2$$

$$= E^2 + 2mE + m^2 - E^2 = 2mE + m^2.$$

For $E \gg m$ (certainly the case for 100 GeV positrons colliding w/ electrons) $2mE \gg m^2$, so

$$E_{cm} = \sqrt{2mE}.$$

Thus,

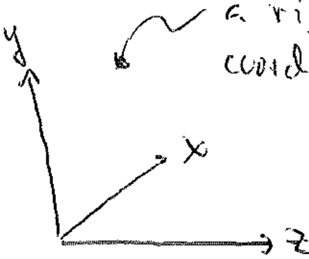
$$R = \frac{E_{cm}^{\text{fixed}}}{E_{cm}^{\text{collide}}} = \frac{\sqrt{2mE}}{2E} = \sqrt{\frac{m}{2E}}$$

For $m = .5\ \text{MeV}/c^2$ and $E = 10^5$ MeV,

$$\boxed{R \cong \sqrt{.25\times10^{-5}} \doteq 1.6\times10^{-3}}$$

substantially less than the Newtonian expectation of ½.

a right-handed coordinate system



Problem 2

Note that, by definition, a quadrupole magnet has a magnetic field of 0 at its center, where the nominal orbit passes through the bore. If this is not the case, we can simply treat the magnet as the combination of a bend (dipole) and an ideal quadrupole, which <u>does</u> have a 0 field at its center.

Now, consider a particle at $y=0$, with a small displacement towards $+x$. If it's a positive particle travelling in the $+z$ direction, the rh rule shows that it will be deflected ~~[illegible]~~ towards the z axis (i.e. focussed) by the field in the $+\hat{y}$ direction, as will be a particle w/ a small displacement towards $-\hat{x}$.

Now, in a source-free region of space such as the beamline vacuum in the quad bore, Maxwell's equations tell us that

$$\vec{\nabla} \times \vec{B} = 0$$

In particular, the $\hat{z}$ component of $\vec{\nabla} \times \vec{B}$ tells us that

$$\frac{\partial}{\partial x} B_y - \frac{\partial}{\partial y} B_x = 0 \qquad \text{-or-} \qquad \frac{\partial}{\partial y} B_x = \frac{\partial}{\partial x} B_y = G$$

Thus, applying the condition $B_x = 0$ at the origin, we have

$$B_x = G y$$

Now, consider the same positively charged particle travelling in the $+\hat{z}$ direction, but this time with a small displacement in the $+\hat{y}$ direction. Again, an application of the rh rule shows that the particle is deflected <u>away</u> from the z axis, ie., it's <u>defocussed</u>! Similarly, a particle w/ a displacement in the $-\hat{y}$ direction will also be driven away from the z axis.

Problem 3

Let R be the beamspot radius, I the intensity (no of particles) in each beam, and f be the crossing frequency. ~~Thus~~ Consider a single particle from beam 1 passing through the bunch from beam 2. The area density seen by this particle is

$$\frac{\#\text{ particles}}{\text{cm}^2} = \frac{I_2}{\pi R^2}$$

which is just the number of interactions the particle from beam 1 will suffer per cm² of interaction cross section. But, there are I_1 particles in beam 1, each with the same interaction probability. In addition, this whole process occurs w/ a frequency f, so

$$\boxed{\mathcal{L} = f\,\frac{I_1 I_2}{\pi R^2}}$$

Note that reducing the size of only one beam won't effect this, for example, the argument is the same if the radius of beam 1 is shrunk - each particle still sees the same area density in beam 2. Similarly, if beam two is reduced, the area density increases for each particle in beam 1 which passes through beam 2, but the number of particles in beam 1 which miss beam 2 altogether compensates for this, leaving the total luminosity unchanged. Thus, for two arbitrary beams, we have in fact that

$$\mathcal{L} = f \frac{I_1 I_2}{\pi R_{max}^2}$$

where $R_{max} = \max\{R_1, R_2\}$.

In any regard, for $f = 120\ s^{-1}$, $I_1 = I_2 = 5\times10^{10}$, and $R = 10^{-4}$ cm

$$\mathcal{L} = 9.5\times10^{30}\ cm^{-2}s^{-1} = 9.5\times10^{6}\ b^{-1}s^{-1}$$

For a 45 nb = 4.5×10^{-8} b cross section, this results in a rate

$$R = \mathcal{L}\sigma = .43\ Hz$$, or about 1 every 2 seconds.

Problem 4

From class, the transport matrix $M(s+C,s)$, where C is the circumference of the ring, is the transport matrix of exactly 1 turn around the ring. Since the β-fnc is a property of the lattice, and not the phase of the beam, then the transport can be characterized by a phase advance ν, given by

$$2\cos 2\pi\nu = Tr[M(s+C,s)].$$

Now, for a perfect FODO lattice w/ an even integer number of cells, by symmetry the β-fnc must have the same symmetry as the lattice, and so the transport matrix $M(s+\alpha,s)$, where α is the length of one FODO cell, must also be characterizable by a cell phase advance μ:

$$2\cos 2\pi\mu = Tr[M(s+\alpha,s)]$$

Now, recall the thin-lens transport matrices

$$F = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \qquad O = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

Thus, since matrices multiply from right to left, we have

$$M(s+a,s) = ODOF = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}\begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix}\begin{pmatrix} 1-l/f & l \\ -1/f & 1 \end{pmatrix} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1-l/f & l \\ \cancel{1/f}-l/f^2-\cancel{1/f} & 1+l/f \end{pmatrix}$$

$$= \begin{pmatrix} 1 - l/f - l^2/f^2 & 2l + l^2/f \\ -l/f^2 & 1 + l/f \end{pmatrix}$$

Thus, $2\cos 2\pi\mu = \mathrm{Tr}[M] = 2 - l^2/f^2$, or

$$\boxed{\mu = \frac{1}{2\pi}\cos^{-1}\left[1 - \frac{l^2}{2f^2}\right]}$$

This condition is satisfied for

$$l^2/2f^2 \leq 2 \Rightarrow \boxed{l < 2f}$$

which is the FODO transverse stability condition. For l larger than $2f$, beams will blow up longitudinally, striking the walls of the beampipe very shortly after injection.

<u>Problem 5</u>

Again, from class notes, we know that

$$\beta(s) = \frac{M_{12}[s+\mathcal{C}, s]}{2\pi\nu}$$

where as before $\mathcal{C}$ is the ring circumference, and ν the single-turn phase advance. Again, exploiting the FODO symmetry, we can write

$$\beta(s) = \frac{M_{12}[s+\alpha, s]}{2\pi\mu}$$

where α is the length of a single cell, and μ its phase advance.

It's clear that $\beta(s)$ will be a maximum at the focussing quad, and a minimum at the defocussing quad. Thus, to get the maximum β, we need to calculate the matrix

$$M^{max} = \sqrt{F}\, O\, D\, O \sqrt{F}$$

where $\sqrt{F}$, "half a quad" is $\begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix}$.

You can easily verify that $\sqrt{F}\sqrt{F} = F$.

Multiplying it out, we find that

$$M^{max} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{2f} & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 - l^2/2f^2 & 2l + l^2/f \\ \frac{l^2}{4f^3} - l/2f^2 & 1 - l^2/2f^2 \end{pmatrix}$$

Notice that we still arrive at the same phase advance per cell of $\cos 2\pi\mu = 1 - l^2/2f^2$, which is reassuring. So,

$$\beta^{max} = \frac{M_{12}}{2\pi\mu} = \frac{2l + l^2/f}{2\pi\mu}.$$

Similarly, $M^{min} = \sqrt{D}\, O F O \sqrt{D}$, so

$$M^{min} = \begin{pmatrix} 1 & 0 \\ \frac{1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{2f} & 1 \end{pmatrix} = \begin{pmatrix} 1 - l^2/2f^2 & 2l - l^2/f \\ -l^2/4f^3 - l/2f^2 & 1 - l^2/2f^2 \end{pmatrix}$$

$$\Rightarrow \beta^{min} = \frac{M_{12}}{2\pi\mu} = \frac{2l - l^2/f}{2\pi\mu}$$

Thus, we see that

$$\boxed{\frac{\beta^{max}}{\beta^{min}} = \frac{2l + l^2/f}{2l - l^2/f} = \frac{1 + l/2f}{1 - l/2f}}$$

Now, from last problem, $\cos(2\pi\mu) = 1 - l^2/2f^2$, so for a $90°$ phase advance per cell,

$$0 = 1 - l^2/2f^2 \Rightarrow l^2 = 2f^2 \rightarrow l = \sqrt{2}\, f.$$

So, in this case

$$\left.\frac{\beta^{max}}{\beta^{min}}\right|_{90°} = \frac{1 + 1/\sqrt{2}}{1 - 1/\sqrt{2}} = 5.82$$

Now, since the beam size is given by $\sqrt{\beta(s)\varepsilon}$, with ε an invariant for coasting beams, we see that

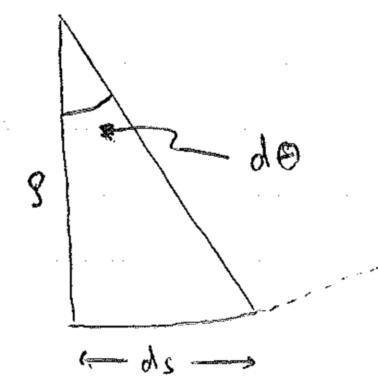
$$\boxed{\frac{\sigma^{max}}{\sigma^{min}} = \sqrt{5.82} = 2.4}$$

Problem #6

The radius of curvature of a singly-charged particle with momentum p GeV/c in a transverse magnetic field of strength B is given by

$$\rho = \frac{pc}{0.3\, B}$$

where B is in Tesla and ρ in meters. Thus, in traversing a distance ds in this field, the particle will be deflected by an angle



$$d\Theta = \frac{ds}{\rho} = \frac{0.3\, B}{pc}\, ds$$

Thus, a particle travelling in the z direction through a magnetic field with X component B_x will experience an angular deflection

$$d\, y' = \frac{0.3\, B_x}{pc}\, ds$$

where we have used the fact that

$$\Theta_y = y' = \frac{dy}{dz}$$

for small Θ_y.

In the case that the x-component of this field has a linear dependence on the displacement in y from the beam axis $B_x = Gy$, we have

$$\frac{dy'}{dz} = \frac{d^2y}{dz^2} = -\frac{0.3G}{pc}\,y$$

where the - sign is necessary if the quadrupole field is to be focussing rather than defocussing; i.e., the effect of the deflection must be to direct particles with a positive y coordinate back towards the beam axis at $y=0$. This is just the differential equation for a sinusoid, with general solution

$$y(z) = A\sin(kz) + B\cos(kz)$$

with $k = \sqrt{\frac{0.3G}{pc}}$

To derive the transfer matrix $M(z_0+l, z_0)$, we need to express A and B in terms of y_0 and y_0', the displacement and angle of the beam particle at the entrance to the quad z_0. Without loss of generality, set $z_0=0$. Then, we see that, at the quad entrance

$$y_0 = y(0) = A\sin(0) + B\cos(0) = B$$

$$y_0' = \left.\frac{dy}{dz}\right|_0 = kA\cos(0) - kB\sin(0) = kA$$

and so we see that

$$y(z) = y_0\cos(kz) + \frac{y_0'}{k}\sin(kz)$$

Differentiating, we find

$$y'(z) = y_0'\cos(kz) - ky_0\sin(kz)$$

From our definition of the transfer matrix

$$\begin{pmatrix} y_1 \\ y_1' \end{pmatrix} = \begin{pmatrix} M \end{pmatrix}\begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

we can substitute $z = l$ in our expressions for y and y' and read off the matrix elements

$$\boxed{\begin{pmatrix} M \end{pmatrix} = \begin{pmatrix} \cos(kl) & \frac{1}{k}\sin(kl) \\ -k\sin(kl) & \cos(kl) \end{pmatrix}}$$

Finally, to recover the thin-lens form for M, we let $\ell \to 0$ while keeping the field integral $B_x \ell = G y \ell$ constant. Thus, the product $G \cdot \ell$ must be constant, and so

$$k^2 \ell = \frac{0.3}{pc} G \ell$$

must also be constant. Thus, $k\ell \to 0$ as $\ell \to 0$, and $\sin k\ell \to k\ell$ as $\ell \to 0$. Thus, since also $\cos(k\ell) \to 1$

$$\lim_{\ell \to 0} (M) = \begin{pmatrix} 1 & \frac{1}{k}(k\ell) \\ -k(k\ell) & 1 \end{pmatrix} = \begin{pmatrix} 1 & \cancelto{0}{\ell} \\ -k^2\ell & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{0.3 G\ell}{pc} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

where $f = \left[\frac{0.3 G \ell}{pc} \right]^{-1}$

as derived in class.