

PROBLEM 1 [25 PTS]

A system consists of N *distinguishable* particles. Each of these particles has two states available to it. The ground state, *which is doubly degenerate* (i.e., there's two of 'em), has $E = 0$. The first (and only!) excited state has an energy ϵ .

a) In the limit that $T \rightarrow 0$ (absolute 0), what fraction of the particles are in the excited state?

b) What would your answer to a) be if $T = \epsilon/k$, where k is Boltzmann's constant?

c) How much energy would it take to raise the temperature of this system from $T = 0$ to $T = \epsilon/k$?

d) Use the above results to *crudely* estimate the molar heat capacity C_V for temperatures below ϵ/k .

PROBLEM 2 [25 PTS]

Deuterium is the isotope of Hydrogen that has a nucleus composed of a proton and a single neutron. The binding interaction between a proton and neutron has the rather surprising property that in the ground state, the half-integer spins of the proton and neutron are aligned, i.e., the deuterium nucleus, or ‘deuteron’, has a spin quantum number of $s = 1$. In the following problem, all interaction energies are small compared to the internal binding energy of the deuteron, so that we can treat the deuteron as a fundamental particle of spin 1, ignoring the fact that it is really made up of a proton and a neutron.

Consider a bound state consisting of a deuteron and antideuteron (antiproton and antineutron, also with spin quantum number $s = 1$). The total angular momentum $\vec{J}$ of this system is the sum of the two deuteron spins:

$$\vec{J} = \vec{S}_1 + \vec{S}_2$$

a) What values of the total angular momentum quantum number j are possible for this system?

b) What values of the total angular momentum $|\vec{J}|$ are possible for this system?

Let m_j be the quantum number associated with the projection of the total angular momentum of the system along the z axis (the direction you ascribe to the z axis is, of course, arbitrary). Assuming that there is no interaction between the deuteron and antideuteron spins, there will be a set of degenerate states corresponding to the different possible combinations of $\vec{S}_1$ and $\vec{S}_2$ into the total angular momentum $\vec{J}$, each state given by a unique value of the quantum numbers j and m_j . In other words, no matter how you combine $\vec{S}_1$ and $\vec{S}_2$ into $\vec{J}$, you get the same energy.

c) What is the resulting degeneracy of states, i.e., how many unique combinations of j and m_j are possible?

d) How many of these degenerate states are associated with each possible value of j from part a)? You should have one answer for each possible value of j from part a), with the total adding up to your answer from part c).

PROBLEM 3 [25 PTS]

For a gas of distinguishable particles, we've seen that the relative probability of any particle having the energy E is given by the Boltzmann factor

$$f_B = e^{-E/kT}.$$

Consider a two-dimensional ideal gas of temperature T composed of N distinguishable objects of mass m whose degrees of freedom are translation in the x and y directions.

a) In terms of the velocity components v_x and v_y , how many particles are there with v_x in the infinitesimal range between v_x and $v_x + dv_x$, *and* in the infinitesimal range between v_y and $v_y + dv_y$? Make sure that you use the formulas for gaussian integrals from the equation sheet to normalize the distribution.

b) Using the result of part a), write down an integral expression representing the total number of particles with v_x lying in the finite range between $v_{x,1}$ and $v_{x,2}$, for *any* value of v_y . Your integral expression should thus not contain v_y . You do not need to perform the integration.

c) For this two-dimensional ideal gas, how many particles are there with *speed* v between v and $v + dv$? In other words, what is the Maxwell distribution $n(v)dv$ of speeds for a two-dimensional ideal gas? Assume that the gas consists of N particles.

PROBLEM 4 [25 PTS]

For the following problem, you really don't need to remember anything about the quantum mechanical harmonic oscillator. You do need to know something about the behavior of Fermi-Dirac particles though.

As we have seen, the solutions of the Schrodinger Equation for a particle of mass m moving under the influence of a harmonic oscillator potential are non-degenerate states evenly spaced in energy, with energies

$$E_n = (n + \frac{1}{2})\hbar\omega \quad n = 0, 1, 2, 3, \dots$$

Consider a system of N identical fermions in motion inside of such a potential well. Feel free, if you like, to ignore the $1/2\hbar\omega$ term in the expression for the energy, i.e., to set $E_n = n\hbar\omega$. (Don't neglect the spin of the electron).

a) What is the Fermi energy E_F for this system, i.e., the energy of the highest-energy occupied state of the system at $T = 0$ (absolute 0). Express your answer in terms of N and ω .

b) Write down an expression showing the number density function $n(E)$ for this system, for any temperature T . You may express your result in terms of E_F , rather than substituting in your result from a).

c) What is the average energy per particle at $T = 0$? Again, you may express your result in terms of E_F if you like.

d) As in homework 4, approximate the effect of temperature on the distribution function as moving all particles with $E_F - kT < E < E_F$ (within kT below the Fermi energy) to the range $E_F < E < E_F + kT$ (within kT above the Fermi energy). Find an expression for the total energy in the system as a function of temperature. Again, you may express your answer in terms of E_F if you like.

e) Use this to estimate the heat capacity at constant volume (C_V) of the system.