where **k** is the unit vector along the z axis. Since $\hat{n}=(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, it follows that

$$\hat{\theta}=\frac{1}{\sin\theta}(-\sin\theta\sin\phi, \sin\theta\cos\phi, 0)=(-\sin\phi, \cos\phi, 0) \qquad (14.3.46)$$

The rotation matrix is, from Eq. (14.3.44),

$$\exp\left(-\frac{i\theta}{2}\hat{\theta}\cdot\boldsymbol{\sigma}\right)=\begin{bmatrix}\cos(\theta/2) & -\sin(\theta/2)\,e^{-i\phi}\\ \sin(\theta/2)\,e^{i\phi} & \cos(\theta/2)\end{bmatrix} \qquad (14.3.47)$$

According to our mnemonic, the first column gives the rotated version of $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. We see that it agrees with $|\hat{n}, +\rangle$ given in Eq. (14.3.28) up to an overall phase.

Here is a summary of useful formulas that were derived or simply stated:

$$\mathbf{S}=\frac{\hbar}{2}\boldsymbol{\sigma}$$

$$[\sigma_i, \sigma_j]_+=2I\delta_{ij}$$

$$[\sigma_i, \sigma_j]=2i\sum_k \varepsilon_{ijk}\sigma_k$$

$$(\hat{n}\cdot\boldsymbol{\sigma})^2=I$$

$$\text{Tr}\,\sigma_i=0$$

$$\text{Tr}(\sigma_\alpha\sigma_\beta)=2\delta_{\alpha\beta} \qquad (\alpha, \beta=x, y, z, 0)$$

$$\exp\left(-i\frac{\theta}{2}\hat{\theta}\cdot\boldsymbol{\sigma}\right)=\cos\left(\frac{\theta}{2}\right)I-i\sin\left(\frac{\theta}{2}\right)\hat{\theta}\cdot\boldsymbol{\sigma}$$

$$(\mathbf{A}\cdot\boldsymbol{\sigma})(\mathbf{B}\cdot\boldsymbol{\sigma})=(\mathbf{A}\cdot\mathbf{B})I+i(\mathbf{A}\times\mathbf{B})\cdot\boldsymbol{\sigma}$$

*Exercise 14.3.2.** (1) Show that the eigenvectors of $\boldsymbol{\sigma}\cdot\hat{n}$ are given by Eq. (14.3.28).
(2) Verify Eq. (14.3.29).

*Exercise 14.3.3.** Using Eqs. (14.3.32) and (14.3.33) show that the Pauli matrices are traceless.

*Exercise 14.3.4.** Derive Eq. (14.3.39) in two different ways.
(1) Write $\sigma_i\sigma_j$ in terms of $[\sigma_i, \sigma_j]_+$ and $[\sigma_i, \sigma_j]$.
(2) Use Eqs. (14.3.42) and (14.3.43).

Let us say $\hat{n}$ points in the direction (θ, ϕ), i.e., that

$$\begin{aligned}\hat{n}_z &= \cos\theta \\ \hat{n}_x &= \sin\theta\cos\phi \\ \hat{n}_y &= \sin\theta\sin\phi\end{aligned} \tag{14.3.26}$$

The kets $|\hat{n}, \pm\rangle$ are eigenvectors of

$$\begin{aligned}\hat{n}\cdot\mathbf{S} &= n_x S_x + n_y S_y + n_z S_z \\ &= \frac{\hbar}{2}\begin{bmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{bmatrix} \\ &= \frac{\hbar}{2}\begin{bmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{bmatrix}\end{aligned} \tag{14.3.27}$$

It is a simple matter to solve the eigenvalue problem (Exercise 14.3.2) and to find

$$|\hat{n}\ \text{up}\rangle \equiv |\hat{n}+\rangle = \begin{bmatrix} \cos(\theta/2)\, e^{-i\phi/2} \\ \sin(\theta/2)\, e^{i\phi/2} \end{bmatrix} \tag{14.3.28a}$$

$$|\hat{n}\ \text{down}\rangle \equiv |\hat{n}-\rangle = \begin{bmatrix} -\sin(\theta/2)\, e^{-i\phi/2} \\ \cos(\theta/2)\, e^{i\phi/2} \end{bmatrix} \tag{14.3.28b}$$

You may verify that as claimed

$$\begin{aligned}\langle\hat{n}\pm|\mathbf{S}|\hat{n}\pm\rangle &= \pm(\hbar/2)(\mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta) \\ &= \pm(\hbar/2)\hat{n}\end{aligned} \tag{14.3.29}$$

An interesting feature of $\mathbb{V}_s$ is that not only can we calculate $\langle\mathbf{S}\rangle$ given a state, but we can also go the other way, i.e, deduce the state vector given $\langle\mathbf{S}\rangle$. This has to do with the fact that any element of $\mathbb{V}_s$ has only two (complex) components α and β, constrained by the normalization requirement $|\alpha|^2 + |\beta|^2 = 1$, i.e., three real degrees of freedom, and $\langle\mathbf{S}\rangle$ contains exactly three pieces of information. If we write $\langle\mathbf{S}\rangle$ as $(\hbar/2)\hat{n}$, then the corresponding ket is $|\hat{n}, +\rangle$ or if you want $|-\hat{n}, -\rangle$. Another way to state this result is as follows. Instead of specifying a state by α and β, we can give the operator $\hat{n}\cdot\mathbf{S}$ of which it is an eigenvector with eigenvalue $\hbar/2$. An interesting corollary is that every spinor in $\mathbb{V}_s$ is an eigenket of some spin operator $\hat{n}\cdot\mathbf{S}$ with eigenvalue $\hbar/2$.

Exercise 14.3.1. Let us verify the above corollary explicitly. Take some spinor with components $\alpha = \rho_1 e^{i\phi_1}$ and $\beta = \rho_2 e^{i\phi_2}$. From $\langle\chi|\chi\rangle = 1$, deduce that we can write $\rho_1 = \cos(\theta/2)$ and $\rho_2 = \sin(\theta/2)$ for some θ. Next pull out a common phase factor so that the spinor takes the form in Eq. (14.3.28a). This verifies the corollary and also fixes $\hat{n}$.

it is in fact $n+\frac{1}{2}$ and hence the usual quantization rule, Eq. (16.2.32). If, however, ψ actually vanishes at x_1 and x_2 because the potential barrier there is infinite (as in the case of a particle in a box), Eq. (16.2.40) [and not Eq. (16.3.32)] is relevant.‡ One can also consider an intermediate case where the barrier is infinite at one turning point and not at the other. In this case the quantization rule has an $(n+3/4)$ factor in it.

The WKB method may also be applied in three dimensions to solve the radial equation in a rotationally invariant problem. In the $l=0$ state, there is no centrifugal barrier, and the WKB wave function has the form

$$U(r) \sim \frac{1}{[p(r)]^{1/2}} \sin\left[\frac{1}{\hbar}\int_0^r p(r')\,dr'\right], \qquad p=\{2m[E-V(r)]\}^{1/2} \quad (16.2.41)$$

where the lower limit in the phase integral is chosen to be 0, so that $U(0)=0$. The quantization condition, bearing in mind that the barrier at $r=0$ is infinite, is

$$\int_0^{r_{\max}} p(r)\,dr = \left(n+\frac{3}{4}\right)\hbar\pi, \qquad n=0, 1, 2, \ldots \quad (16.2.42)$$

where $r_{\max}$ is the turning point. This formula is valid only if $V(r)$ is regular at the origin. If it blows up there, the constant we add to n is not 3/4 but something else. Also if $l \neq 0$, the centrifugal barrier will alter the behavior near $r=0$ and change both the wave function and this constant.

*Exercise 16.2.5.** In 1974 two new particles called the ψ and ψ' were discovered, with rest energies 3.1 and 3.7 GeV, respectively (1 GeV$=10^9$ eV). These are believed to be nonrelativistic bound states of a "charmed" quark of mass $m=1.5$ GeV$/c^2$ (i.e., $mc^2=1.5$ GeV) and an antiquark of the same mass, in a linear potential $V(r)=V_0+kr$. By assuming that these are the $n=0$ and $n=1$ bound states of zero orbital angular momentum, calculate V_0 using the WKB formula. What do you predict for the rest mass of ψ'', the $n=2$ state? (The measured value is $\simeq 4.2$ GeV$/c^2$.) [Hints: (1) Work with GeV instead of eV. (2) There is no need to determine k explicitly.]

Exercise 16.2.6. Obtain Eq. (16.2.39) for the λx^4 potential by the scaling trick.

*Exercise 16.2.7** Find the allowed levels of the harmonic oscillator by the WKB method.

Exercise 16.2.8. Consider the $l=0$ radial equation for the Coulomb problem. Since $V(r)$ is singular at the turning point $r=0$, we can't use $(n+3/4)$.

(1) Will the additive constant be more or less than 3/4?

(2) By analyzing the exact equation near $r=0$, it can be shown that the constant equals 1. Using this constant show that the WKB energy levels agree with the exact results.

‡ The assumption that $V(x)$ may be linearized near the turning point breaks down and this invalidates Eq. (16.2.29).

Exercise 19.5.3. (1) Show that $\sigma_0 \to 4\pi r_0^2$ for a hard sphere as $k \to 0$.

(2) Consider the other extreme of kr_0 very large. From Eq. (19.5.27) and the asymptotic forms of j_l and n_l show that

$$\sin^2 \delta_l \xrightarrow[kr_0 \to \infty]{} \sin^2(kr_0 - l\pi/2)$$

so that

$$\sigma = \sum_{l=0}^{l_{\max}=kr_0} \sigma_l \cong \frac{4\pi}{k^2} \int_0^{kr_0} (2l) \sin^2 \delta_l \, dl$$

$$\cong 2\pi r_0^2$$

if we approximate the sum over l by an integral, $2l+1$ by $2l$, and the oscillating function $\sin^2 \delta$ by its mean value of 1/2.

*Exercise 19.5.4.** Show that the s-wave phase shift for a square well of depth V_0 and range r_0 is

$$\delta_0 = -kr_0 + \tan^{-1}\left(\frac{k}{k'} \tan k' r_0\right)$$

where k' and k are the wave numbers inside and outside the well. For k small, kr_0 is some small number and we ignore it. Let us see what happens to δ_0 as we vary the depth of the well, i.e., change k'. Show that whenever $k' \simeq k_n' = (2n+1)\pi/2r_0$, δ_0 takes on the resonant form Eq. (19.5.30) with $\Gamma/2 = \hbar^2 k_n/\mu r_0$, where k_n is the value of k when $k' = k_n'$. Starting with a well that is too shallow to have any bound state, show k_1' corresponds to the well developing its first bound state, at zero energy. (See Exercise 12.6.9.) (Note: A zero-energy bound state corresponds to $k=0$.) As the well is deepened further, this level moves down, and soon, at k_2', another zero-energy bound state is formed, and so on.

Exercise 19.5.5. Show that even if a potential absorbs particles, we can describe it by

$$S_l(k) = \eta_l(k)\, e^{2i\delta_l}$$

where $\eta(<1)$, is called the *inelasticity factor*.

(1) By considering probability currents, show that

$$\sigma_{\text{inel}} = \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1)[1-\eta_l^2]$$

$$\sigma_{\text{el}} = \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1)(1+\eta_l^2 - 2\eta_l \cos 2\delta_l)$$

and that once again

$$\sigma_{\text{tot}} = \frac{4\pi}{k} \operatorname{Im} f(0)$$