Physics 216 (2018) Exam Problem 1 Solution

If we write

$$H = H^{0} + H' \quad \text{with} \quad H^{0} = p^2/2m + \tfrac{1}{2} m\omega^2 x^2 \ \text{and}$$

$H' = \beta m^2 \omega^3/\hbar \; x^4$, then $H^0$ is just the standard HO w/ energy eigenstates $(n+\tfrac{1}{2})\hbar\omega$, and ground state $|0\rangle$ with energy $\tfrac{1}{2}\hbar\omega$. We need to calculate

$$\Delta E = \langle 0|H'|0\rangle = \frac{\beta m^2 \omega^3}{\hbar} \langle 0|x^4|0\rangle$$

We can write $x^4 = \gamma(a^{+}+a)$ with $\gamma = \sqrt{\hbar/2m\omega}$, and then easily argue that the only terms in $x^4 = \gamma^4 (a+a^{+})^4$ that won't give an expectation value of 0 are

$\gamma^4 a a a^{+} a^{+}$ and $\gamma^4 a a^{+} a a^{+}$ giving

$$\Delta E = \frac{\beta m^2 \omega^3}{\hbar} \frac{\hbar^2}{4m^2\omega^2} \langle 0| a a a^{+} a^{+} + a a^{+} a a^{+} |0\rangle = \frac{\beta}{2}\left(\frac{\hbar\omega}{2}\right)(2+1)$$

$$= \frac{3\beta}{2}\left(\frac{\hbar\omega}{2}\right) \quad \text{so}$$

$$\frac{\Delta E}{E_0} = \frac{3\beta/2\,\left(\frac{\hbar\omega}{2}\right)}{\left(\hbar\omega/2\right)} = \frac{3}{2}\beta$$

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Physics 216 $_{\wedge}$ MT 1 Problem 2 Solution

The condition will be solved when the phase advance over the slightly-impeded path is $\pi$ greater than the phase advance over the unimpeded path. By the WKB method,

$$\Delta\phi = \int_{-a}^{a} \frac{1}{\hbar}\sqrt{2m(E-V(x))}\, dx \qquad \text{with}$$

$$V(x) = 0 \quad \text{path 1} \qquad V(x) = \begin{cases} V_0/a\,(x+a) & -a \le x \le 0 \\ V_0/a\,(a-x) & 0 \le x \le a \end{cases}$$

It's easy to see that the phase advance from $-a$ to $0$ is identical to the phase advance from $0$ to $a$ in both cases, so what we require is that

$$\frac{\hbar\pi}{2} = \int_0^a \sqrt{2mE}\, dx - \int_0^a \sqrt{2m\left(E - V_0 + \frac{V_0}{a}x\right)}\, dx \equiv I_1 - I_2$$

The second of these integrals is the only one we need to work on. Calling it $I_2$,

$$I_2 = \sqrt{2m}\int_0^a dx \left(E - V_0 + \frac{V_0}{a}x\right)^{1/2} = \sqrt{\frac{2mV_0}{a}}\int_0^a dx\left(x + a\,\frac{E-V_0}{V_0}\right)^{1/2}$$

$$= \frac{2}{3}\sqrt{\frac{2mV_0}{a}}\left(x + a\,\frac{E-V_0}{V_0}\right)^{3/2}\Bigg|_0^a$$

$$= \frac{2}{3}\sqrt{\frac{2mV_0}{a}}\;a^{3/2}\left[\left(1+\frac{E-V_0}{V_0}\right)^{3/2} - \left(\frac{E-V_0}{V_0}\right)^{3/2}\right]$$

$$= \frac{2}{3}a\sqrt{2mV_0}\left[\left(\frac{E}{V_0}\right)^{3/2} - \left(\frac{E}{V_0}-1\right)^{3/2}\right].$$ Now, $I_1 = \sqrt{2mE}\,a$ so

$$\frac{\hbar\pi}{2} = \sqrt{2mE}\,a - \sqrt{2mE}\sqrt{\frac{V_0}{E}}\,\frac{2a}{3}\left[\left(\frac{E}{V_0}\right)^{3/2} - \left(\frac{E}{V_0}-1\right)^{3/2}\right] \quad ,\!so$$

$$\frac{\hbar\pi}{\sqrt{8mE}\,a} = 1 - \sqrt{\frac{V_0}{E}}\,\frac{2}{3}\left[\left(\frac{E}{V_0}\right)^{3/2} - \left(\frac{E}{V_0}-1\right)^{3/2}\right] = 1 - \frac{2}{3}\left[\frac{E}{V_0} - \left(\frac{E}{V_0}-1\right)^{3/2}\left(\frac{V_0}{E}\right)^{1/2}\right]$$

$$= 1 - \frac{2}{3}\left[\frac{E}{V_0} - \left(1-\frac{V_0}{E}\right)^{3/2}\frac{E}{V_0}\right] = 1 - \frac{2}{3}\,\frac{E}{V_0}\left[1 - \left(1-\frac{V_0}{E}\right)^{3/2}\right]$$

$=$ for $V_0/E = 0.1 \doteq 1 - 0.9746 = 0.025 \equiv \beta$

Thus, the minimum $a$ is given by

$$\boxed{a = \frac{\hbar\pi}{2\beta\sqrt{2mE}} = \frac{\hbar\pi}{2p\beta} \doteq 20\,\frac{\hbar\pi}{p}}$$

Exam Problem 3

The square-well eigenstates are given by

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) \equiv |n\rangle$$

with $p_n = \hbar k_n = n\pi\hbar/a$ $\quad E_n = \frac{p_n^2}{2m} = \frac{h^2}{8ma^2} n^2 \equiv E_0 n^2$

with $E_0 = \frac{h^2}{8ma^2} = \frac{\pi^2\hbar^2}{2ma^2}$

Since $\delta(x - a/2)$ is only non-zero at $a/2$, $\psi_n(x)$ will only be perturbed for odd values of $n$. Thus, we'll choose for our trial wavefunction

$$\psi(\alpha) = |1\rangle + \alpha|3\rangle$$

where $\alpha$ is the variational parameter. The Hamiltonian for the system is

$$H = H_0 + V^{(1)} = H_0 + \frac{\pi^2\hbar^2}{4ma}\delta(x - a/2)$$

We then have, in terms of the variational parameter $\alpha$

$$E(\alpha) = \langle\psi(\alpha)|H|\psi(\alpha)\rangle = (\langle 1| + \alpha\langle 3|)|H|(|1\rangle + \alpha|3\rangle)$$

$$= E_1 + \alpha^2 E_3 + (\langle 1| + \alpha\langle 3|)|V^{(1)}|(|1\rangle + \alpha|3\rangle)$$

To find the ME of $V^{(1)}$, we calculate

$$\langle n | \delta(x - a/2) | m \rangle = \int_0^a \frac{2}{a} \sin\left(\frac{n\pi}{a} x\right) \sin\left(\frac{m\pi}{a} x\right) \delta(x - a/2)\, dx$$

If either n or m is even, this will be 0 (that's why we didn't choose $|2\rangle$ for part of our trial wf). For both n and m odd this yields

$$\langle n | \delta(x - a/2) | m \rangle = (-1)^{\frac{n-m}{2}} \left(\frac{2}{a}\right)$$

so $\langle 1 | \delta(x-a/2) | 1 \rangle = \langle 3 | \delta(x-a/2) | 3 \rangle = 2/a$

$\langle 1 | \delta(x-a/2) | 3 \rangle = \langle 3 | \delta(x-a/2) | 1 \rangle = -2/a$ and so

$$E(\alpha) = E_1 + \alpha^2 E_3 + \frac{\pi^2 \hbar^2}{4ma}\left(\frac{2}{a}\right)\left[1 - 2\alpha + \alpha^2\right]$$

$$= E_0 + 9\alpha^2 E_0 + E_0 - 2\alpha E_0 + \alpha^2 E_0$$

$$= 2E_0 - 2\alpha E_0 + 10\alpha^2 E_0$$

Minimizing wrt $\alpha$,

$$0 = \frac{dE(\alpha)}{d\alpha} = -2E_0 + 20\alpha E_0 \Rightarrow \alpha = 0.1$$

Plugging this in to $E(\alpha)$, we find

$$E(0.1) = 2E_0 - 0.2E_0 + 0.1E_0 = 1.9E_0$$

Now, we were encouraged to work w/ an un-normalized wavefunction for the minimization, but we really should normalize our minimized wavefunction for the final energy estimate. Thus,

$$\boxed{E_{var} = \frac{E(0.1)}{\sqrt{1+\alpha^2}} \doteq \frac{1.9E_0}{1.005} \doteq 1.89E_0}$$

Physics 211 (2018) Problem 4 Solution

In the interaction picture, the hamiltonian takes the form

$$H_I^{(1)}(t) = e^{iH_0 t/\hbar} H^{(1)}(t) e^{-iH_0 t/\hbar}$$

where $H_0$ in this case is just the HO potential w/ characteristic frequency $\omega$. To leading order in the perturbation $H^{(1)}$, we have

$$|\psi_I(t)\rangle = |1\rangle + \frac{1}{i\hbar}\int_{-\infty}^{0} dt'\, H_I^{(1)}(t')\,|1\rangle$$

where "$|1\rangle$" and "$|2\rangle$" are (will be) the ground and first excited state of the HO. Putting in the explicit form for $H_I^{(1)}(t)$, we find that

$$|\psi_I(t)\rangle = |1\rangle + \frac{1}{i\hbar}\int_{-\infty}^{0} dt'\, e^{iH_0 t'/\hbar}\left(\beta\hbar\omega e^{2\omega t'} a^\dagger\right) e^{-iH_0 t'/\hbar}\,|1\rangle$$

$$= |1\rangle + \frac{\beta\omega}{i}\int_{-\infty}^{0} dt'\, e^{iH_0 t'/\hbar}\left(e^{2\omega t'} a^\dagger\right) e^{-i\omega_1 t'}\,|1\rangle$$

where $\hbar\omega_1$ is the ground-state energy. Acting with the raising operator, this becomes

$$|\Psi_I(t)\rangle = |1\rangle + \frac{\beta w}{i}\int_{-\infty}^{0} dt'\, e^{iH_0t'/\hbar}\, e^{2wt'}\, e^{-iw_1t'}\,|2\rangle$$

$$= |1\rangle + \frac{\beta w}{i}\int_{-\infty}^{0} dt'\, e^{iw_2t'}\, e^{2wt'}\, e^{-iw_1t'}\,|2\rangle$$

$$= |1\rangle + \frac{\beta w}{i}\int_{-\infty}^{0} dt'\, e^{i(w_2-w_1)t'}\, e^{2wt'}$$

$$= |1\rangle + \frac{\beta w}{i}\int_{-\infty}^{0} dt'\, e^{iwt'}\, e^{2wt'} = |1\rangle + \frac{\beta w}{i}\int_{-\infty}^{0} dt'\, e^{(iw+2w)t'}$$

$$= |1\rangle + \frac{\beta w}{i}\left[\frac{e^{(iw+2w)t}}{(iw+2w)}\right]_{-\infty}^{0} = |1\rangle + \frac{\beta w}{2iw - w}$$

To calculate the amplitude for the transition to $|2\rangle$, we dot with the state $\langle 2|$ in the interaction picture:

$$A_{1\to 2}(t=0) = \langle 2|e^{-iw_2 0}\,|\Psi_I(t)\rangle = \frac{\beta w}{2iw-w} = \frac{\beta}{2i-1}$$

Finally, the probability is given by

$$\boxed{P_{1\to 2}(t=0) = \left(\frac{\beta}{2i-1}\right)^2 = \frac{\beta^2}{4+1} = \frac{\beta^2}{5}}$$