PHYSICS 216 WINTER 2018 – HOMEWORK 2

Due in class Tuesday May 1, 2018.

Complementary reading: Shankar Chapter 18, Baym Chapters 12 and 13

Problem 1

Consider a particle of mass m in motion in a one-dimensional harmonic oscillator with a perturbation linear in the coordinate x that grows with a charateristic time τ:

$$H = H^0 + \frac{\lambda t}{\tau} X,$$

where H^0 a pure harmonic oscialltor potential with characteristic frequency ω, and $\lambda \ll 1$. Find the expression for the position operatir X_I in the interaction picture. Conintuing to work the interaction picture, calculate the time-evolution operator $U_I(t, 0)$ to lowest order in the perturbation. Next, assume that the particle is in the ground state of the unperturbed harmonic oscillator at $t = 0$. For the 'sudden approximation' $\omega\tau \ll 1$, make use of the interaction picture to find the leading order approximation to the probability that the particle will find itself in the first excited state of the unperturbed osciallator at time τ.

Problem 2

Consider a charged particle (with charge q) whose motion is confined to a circle of radius R in the $x - y$ plane, with its center at the origin. A thin magentic flux tube of radius $r < R$ is located with its axis along the z-axis. The magnetic field is confined within the flux tube, and the toal magentic flux through the $x - y$ plane is Φ.

It's covenient to work in cylindrical coordinates (ρ, θ, z). In the region where the particle moves, there is no magentic field, and so $\vec{\nabla} \times \vec{A} = 0$, which implies that

$$\vec{A}(\rho, \theta, z) = \vec{\nabla}\chi(\rho, \theta, z).$$

(a) Noting that Stokes' theorem relates Φ to the line integral of $\vec{A}$ taken along the cirlce of radius R, show that the choice

$$\chi(\rho, \theta, z) = \frac{\Phi\theta}{2\pi}$$

satisfies Stokes' theorem and the Coulomb gauge condition.

(b) The wave function for the charged partticle is only a function of θ (since ρ=R and $z = 0$ are fixed due to the contrained motion.). Write down the time-independent Schrodinger equation for the charged particle wave function $\psi(\theta)$ in the cylindrical coordinate representation (simplify your equation as much as possible).

(c) Solve the Shrodinger equation of part (b) for the energy eigenvalues and eigenfunctions. Show that the allowed energies depend on Φ even though the charged particle on the circle never encounters the magentic field. This curious phenomenon is known as the 'Bohm-Aharanov effect'.

HINT: Show that the energy eigenstates are also eigenstates of $d/d\theta$.

Problem 3 (Courtesy JJ Sakurai)

The unperturbed Hamiltonian of a two-state system is represented by

$$H_0 = \begin{pmatrix} E_1^0 & 0 \\ 0 & E_2^0 \end{pmatrix}.$$

There is, in addition, a time-dependent perturbation

$$V(t) = \begin{pmatrix} 0 & \lambda\cos(\omega t) \\ \lambda\cos(\omega t) & 0 \end{pmatrix},$$

where λ is real.

At $t = 0$ the state is set to be in the state

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Using time-dependent PT, derive an expression for the probability that the system will be found in the state

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

at the time t. For values of ω close to a certain value ω_r, this approximation will break down. What is the value of ω_r? Roughly how close does ω need to be to ω_r in order to be considered "too close" for use wtih this approach?

Problem 4 (Courtesy Howie Haber)

This problem provides a basic model for the photoelectrc effect. Those that go on to take Physics 221A with me next fall will see more about the role that the PE effect plays in the absorption of high-energy particles (particularly photons and electrons) by matter.

Neglecting the spins of the electron and proton, consider the hydrogen atom in its ground state. At $t = 0$, the atom is subjected to a high-frequency, uniform electric field that points in the z-direction,

$$\mathbf{E} = E_0 \sin \omega t \hat{\mathbf{z}}.$$

In this problem, we'll compute both the differention and total transition rate for the electron being freed from the atom.

(a) Determine ω_0, the minimum frequency of the field needed to ionize the atom.

(b) Using Fermi's Golden Rule to first order in time-dependent perturbation theory, obtain an expression for the transition rate per unit solid angle as a function of the polar angle θ (measured with respect to the direction of the electric field).

HINT: The matrix element that appears in Fermis golden rule describes a transition of the negative-energy bound electron in its ground state to a positive-energy free electron. The wave function of the latter is actually quite complicated, since one cannot really neglect the effects of the long-range Coulomb potential. Nevertheless, you should simplify the computation by assuming the wave function of the ejected electron is a free-particle plane wave, with wave number vector $\vec{k}$. (Note that the direction of $\vec{k}$ corresponds to that of the ejected electron).

(c) Integrate the result of part (b) over all solid angles to obtain the total ionization rate as a function of the frequency of the field. Determine the value of ω, (in terms of ω_0) for which the total ionization rate is maximal.

HINTS: With a bit of thought, You should be able to set up the answer to this problem making use of what we have learned in class. Working out the arithmetic steps may require a bit of ingenuity though. Here are some hints that may prove to be of help along the way. We can expand a generic plane wave in terms of spherical waves as follows.

$$e^{i\vec{k}\cdot\vec{x}'} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l j_l(kr')[Y_l^m(\theta')(\phi')]^*[Y_l^m(\theta)(\phi)]$$

where $j_l(kr')$ is the l^{th} spherical Bessel function. Second, the differential density of states at energy E, for plane wave in a box of volume V, is given by (extra kudos if you can show this)

$$\frac{d\rho(E)}{d\Omega} = \frac{Vm\hbar k}{(2\pi\hbar)^3}$$

Problem 5 (Courtesy Howie Haber)

Consider the spontaneous emission of an E1 photon (i.e., a photon emitted by an electric dipole transition) by an excited atom. The magnetic quantum numbers (m and m′) of the initial and final atomic state are measured with respect to a fixed z-axis. Assume that the magnetic quantum number of the atom decreases by one unit.

(a) Compute the angular distribution of the emitted photon.

(b) Determine the polarization of the photon emitted in the z-direction.

(c) Verify that the result of part (b) is consistent with angular momentum conservation for the whole (atom plus photon) system.

HINTS: It may prove useful to know that

$$\sum_{\lambda} (\epsilon_\lambda^*)_i (\epsilon_\lambda^*)_j = \delta_{ij} - \hat{k}_i \hat{k}_j$$

for polarization states ϵ_1, ϵ_2 that are mutually perpendicular to some arbitrary vector $\vec{k}$. Finally, the problem can be tackled most simply by recognizing that the operator $\vec{x}$ is a spehrical tensor of rank-one, i.e., it is a linear combination of tensor operators of the form $rY_{1m}(\theta, \phi)$; $m = -1, 0, 1$. The Wigner-Eckart theorem (or equivalently just common-sense application of angular momentum conservation) then allows you to establish constraints on the polarization from transitions you know to be forbidden.