Problem 1 Solution

The arc-length s is $l\theta$, so

$$V(s) = mgl\,(1-\cos\theta) = mgl\,[1-\cos(s/l)]$$

Expanding $\cos(s/l)$ we find

$$mgl\,[1-\cos(s/l)] \doteq mgl\left[1-\left(1-\frac{1}{2}\frac{s^2}{l^2}+\frac{1}{24}\frac{s^4}{l^4}\right)\right]$$

$$= \frac{mg}{2l}\left[s^2-\frac{s^4}{12l^2}\right] \equiv V^{(0)}+V^{(1)}$$

where
$$V^{(0)} = \frac{1}{2}m\frac{g}{l}s^2 = \frac{1}{2}m\omega^2 s^2 \quad \text{with } \omega=\sqrt{\frac{g}{l}}$$

as expected for the pendulum. So this is a perturbed HO system, with perturbation

$$V^{(1)} = -\frac{1}{2}m\omega^2\left[\frac{s^4}{12l^2}\right]$$

To find the correct ground-state energy to leading order, we must simply evaluate

$$E_0^{(1)} = \langle 0|V^{(1)}|0\rangle$$

From Shankar, we recall the scaled position coordinate

$$q = \frac{1}{\sqrt{2}}\left[a^\dagger + a\right] \quad \text{with } q = \sqrt{\alpha}\, s \text{ and } \alpha = \frac{\omega m}{\hbar}$$

Thus,

$$V^{(1)} = -\frac{1}{2} m \omega^2 \left[\frac{s^4}{12\ell^2}\right] = -\frac{1}{2} m\omega^2 \frac{q^4}{12\ell^2}\frac{1}{\alpha^2} = -\frac{\hbar^2}{24 m \ell^2} q^4$$

$$\equiv \beta q^4 \quad \text{with } \beta = -\frac{\hbar^2}{24 m \ell^2}$$

Expanding q^4 in terms of raising/lowering operators

$$q^2 = \frac{1}{2}(a^\dagger + a)(a^\dagger + a) = \frac{1}{2}(a^\dagger a^\dagger + a a^\dagger + a^\dagger a + a a)$$

and then,

$$q^4 = \frac{1}{4}\Big(a^\dagger a^\dagger a^\dagger a^\dagger + a^\dagger a^\dagger a a^\dagger + a^\dagger a^\dagger a^\dagger a + a^\dagger a^\dagger a a +$$

$$a a^\dagger a^\dagger a^\dagger + a a^\dagger a a^\dagger + a a^\dagger a^\dagger a + a a^\dagger a a +$$

$$+ a^\dagger a a^\dagger a^\dagger + a^\dagger a a a^\dagger + a^\dagger a a^\dagger a + a^\dagger a a a +$$

$$+ a a a^\dagger a^\dagger + a a a a^\dagger + a a a^\dagger a + a a a a\Big)$$

where I've crossed out the terms that will give 0 when finding the ground state expectation value (we need to raise and lower an equal amount, and we can't lower below 0)

So the correction term looks like

$$\langle 0|V^{(1)}|0\rangle = \frac{\beta}{4}\left\{\langle 0|aa^{+}aa^{+}|0\rangle + \langle 0|aaa^{+}a^{+}|0\rangle\right\}$$

and since $a^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle$ $\quad a|n\rangle = \sqrt{n}\,|n-1\rangle$

we have

$$\langle 0|V^{(1)}|0\rangle = \frac{\beta}{4}\left\{1+2\right\} = \frac{3}{4}\beta$$

and so

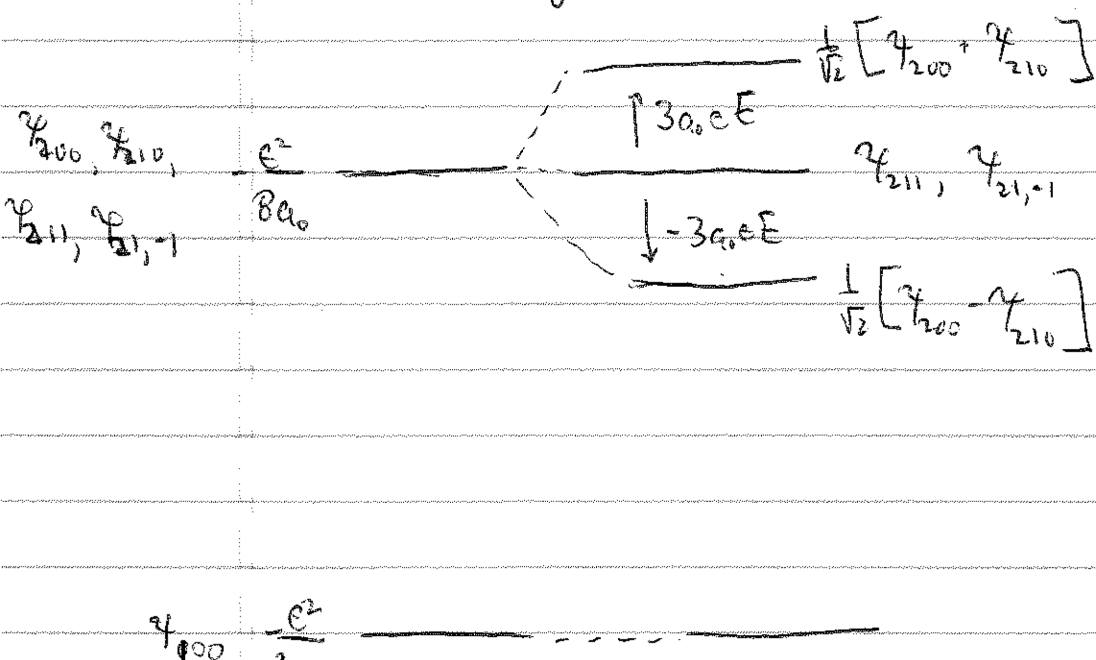
$$H^{(1)} = \frac{1}{2}\hbar\omega - \frac{3}{24}\hbar^2\frac{1}{m\ell^2} \qquad \text{with} \quad \omega = \sqrt{g/\ell}$$

Problem 2 Solution

For this, the solution is found directly in the class notes, at least mostly. (This was an oversight on my part and I will fix it for next year, although going through the problem again on your own might be helpful).

The only state we didn't address was the ground state $\psi_{100}(\vec{r})$, but in class we argued (why?) that all the diagonal elements of the perturbation are 0, so this state remains unperturbed (plus, it's a singlet state, so it has nothing to mix with anyway!)

Thus, we got a state diagram



with transition energies
$$\begin{cases} \frac{3}{8}\frac{e^2}{a_0} + 3a_0 eE & 1 \text{ line} \\ \frac{3}{8}\frac{e^2}{a_0} & 2 \text{ lines} \\ \frac{3}{8}\frac{e^2}{a_0} - 3a_0 eE & 1 \text{ line} \end{cases}$$

Bruce Schumm
Quantum Problem 3 Solution

To apply the variational principle, we must evaluate the hamiltonian with the trial wavefunction:

$$\langle\psi|H|\psi\rangle = \langle\psi|T+V|\psi\rangle = \langle\psi|T|\psi\rangle + \langle\psi|V|\psi\rangle$$

We begin by normalizing $\psi = \beta e^{-r/a}$

$$1 = \langle\psi|\psi\rangle = \iint \beta^2 e^{-2r/a}\, r\,dr\,d\phi = 2\pi\beta^2 \int_0^\infty r e^{-2r/a}\,dr$$

$$= \frac{\pi\beta^2 a^2}{2} \int_0^\infty x e^{-x}\,dx = \frac{\pi\beta^2 a^2}{2} \rightarrow \beta = \frac{\sqrt{2}}{a\sqrt{\pi}}$$

Now, to calculate T, we recall that $T^{op} = -\frac{\hbar^2}{2m}\nabla^2$

$$T^{op}\psi(r,\phi) = -\frac{\hbar^2}{2m}\nabla^2 \beta e^{-r/a}$$

$$= -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right]\beta e^{-r/a} = -\frac{\beta\hbar^2}{2m}\left[\frac{1}{a^2} - \frac{1}{ar}\right] e^{-r/a}$$

and so

$$\langle\psi|T|\psi\rangle = -\frac{\beta^2\hbar^2}{2m}\iint\left[\frac{1}{a^2} - \frac{1}{ar}\right] e^{-2r/a}\, r\,dr\,d\phi =$$

$$= -\frac{\beta^2\hbar^2}{2ma}\, 2\pi \int_0^\infty \left[\frac{r}{a} - 1\right] e^{-2r/a}\, dr$$

$$= -\frac{\pi\beta^2\hbar^2}{ma}\left[\int_0^\infty \frac{r}{a} e^{-2r/a}\, dr - \int_0^\infty e^{-2r/a}\, dr\right]$$

$$= -\frac{\pi\beta^2\hbar^2}{ma}\left[\frac{a}{4\cancel{a}}\int_0^\infty x e^{-x}\, dx - \frac{a}{2}\int_0^\infty e^{-x}\, dx\right]$$

$$= -\frac{\pi\beta^2\hbar^2}{ma}\left[-\frac{a}{4}\right] = \frac{\pi\beta^2\hbar^2}{4m} = \frac{\cancel{\pi}\hbar^2}{4m}\left(\frac{\cancel{2}}{\cancel{\pi}a^2}\right) = \frac{\hbar^2}{2ma^2}$$

To calculate V

$$\langle\psi|V|\psi\rangle = \iint \beta^2 e^{-2r/a}\left(-\frac{ke^2}{r}\right) r\,dr\,d\phi$$

$$= -\beta^2 2\pi \int_0^\infty ke^2 e^{-2r/a}\, dr$$

$$= -\beta^2 2\pi ke^2 \int_0^\infty e^{-2r/a}\, dr = -\frac{\beta^2 \cancel{2}\pi ke^2 a}{\cancel{2}}\int_0^\infty e^{-x}\, dx$$

$$= \left(-\frac{2}{a^{\cancel{2}}\cancel{\pi}}\right)\left(\cancel{\pi} ke^2 \cancel{a}\right) = -\frac{2ke^2}{a}$$

Thus,

$$H_a = \langle \psi | H | \psi \rangle = \frac{\hbar^2}{2m}\left(\frac{1}{a^2}\right) - 2ke^2\left(\frac{1}{a}\right)$$

We now find the value of the parameter a that minimizes H_a:

$$0 = \frac{dH_a}{da} = -\frac{2\hbar^2}{2m}\frac{1}{a^3} + 2ke^2\frac{1}{a^2}$$

$$\Rightarrow \frac{ma^3}{\hbar^2} = \frac{a^2}{2ke^2} \Rightarrow \frac{ma}{\hbar^2} = \frac{1}{2ke^2}$$

$$\Rightarrow a = \frac{\hbar^2}{2mke^2}$$

Plugging back in,

$$H_0 = \frac{\hbar^2}{2m}\left(\frac{4m^2k^2e^4}{\hbar^4}\right) - 2ke^2\left(\frac{2mke^2}{\hbar^2}\right)$$

$$= \frac{2mk^2e^4}{\hbar^2} - \frac{4mk^2e^4}{\hbar^2} = -4\left(\frac{mk^2e^4}{2\hbar^2}\right) = -54.4\,eV$$

This is $\times 4$ more tightly bound that the 3-d H atom. $\times 2$ comes from the fact that a_0 is $\times 2$ smaller than the Bohr radius. The other $\times 2$ comes from the rate at which "volume" increases w/

Problem 4

We have a particle moving in a potential well with a domain $0 \le r < \infty$. The particle moves in three dimensions, with radial motion prescribed by the radial wave equation (12.6.3)

$$-\frac{\hbar^2}{2\mu}\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right] + V(r)\Big\} R_{El} = E R_{El}$$

This is nothing like the 1-D SE for which the WKB formalism was developed, but do not despair, for a simple substitution

$$U_{El} = r R_{El}$$

yields the auxillary radial wave equation

$$\left\{\frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2}\left[E - V(r) - \frac{l(l+1)\hbar^2}{2\mu r^2}\right]\right\} U_{El} = 0$$

which does indeed look like the one-d SE $\chi'' + \frac{2\mu}{\hbar^2}(E-V)\chi = 0$ with effective potential $V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2}$.

But we need not even worry about the "effective potential" point since for S states, $l=0$, yielding

$$\left\{ \frac{d^2}{dr^2} + \frac{2\mu}{\hbar^2}\left[E - V(r)\right] \right\} U_{El} = 0$$

So, U_{El} satisfies the 1-d SE with potential $V(r)$, and we are thus free to apply the WKB formalism.

According to the discussion leading to 16.2.33, in order to generate a coherent standing wave (bound state), the phase shift $\oint p(x')dx'$ from 0 to ∞ and back again must be a multiple of 2π. According to 16.2.31, including the connection to the solution across the classical turning point and into the "forbidden region", we can thus write, for a = turning point,

$$2\left(\frac{1}{\hbar}\int_0^a p(r')\,dr' + \frac{\pi}{4}\right) = 2\pi N' \quad \text{and so}$$

$$2\int_0^a p(r')\,dr' = 2\pi\hbar\left(N' - \tfrac{1}{4}\right) = 2\pi\hbar\left(N + \tfrac{3}{4}\right) \quad \text{letting } N = N' - 1$$

Finally, we realize that $p(r') = \sqrt{2\mu(E - V(r'))}$ and so

$$\boxed{2\int_0^a \sqrt{2\mu(E - V(r'))}\,dr' = \left(N + \tfrac{3}{4}\right)h}$$

Problem 5 (Shankar 16.2.5)

From the previous problem, we derived the following quantization condition for an object moving in a spherically-symmetric potential in 3D, for the case $l=0$

$$2\int_0^a \sqrt{2m(E-V)}\,dr = \left(n+\frac{3}{4}\right)h$$

where a is the classical turning point $E-V(a)=0$. This condition is also known as the "Bohr-Sommerfeld quantization rule".

In our case, let's let $a=r_n$ be the classical turning point for the n^{th} $l=0$ excited state in the potential

$$V(r) = V_0 + kr$$

Thus, r_n is given by the condition

$$0 = E_n - V(r_n) = E_n - V_0 - kr_n$$

$$\Rightarrow r_n = \frac{E_n - V_0}{k}$$

and so our energies are given by the integral equation

$$\left(n+\frac{3}{4}\right)\pi = \frac{\sqrt{2m}}{\hbar}\int_0^{\frac{E_n-V_0}{k}} \sqrt{E_n - V_0 - kr}\,dr$$

We turn this into an algebraic equation by solving the integral via the substitution

$$x = E_n - V_0 - kr \quad \Rightarrow dx = -k\,dr \quad \Rightarrow dr = -dx/k$$

with limits given by

$$r = 0 \Rightarrow x = E_n - V_0 \qquad r = \frac{E_n - V_0}{k} \Rightarrow x = 0$$

where

$$\left(n + \frac{3}{4}\right)\pi = \frac{\sqrt{2m}}{\hbar k} \int_0^{E_n - V_0} \sqrt{x}\, dx$$

$$= \frac{2}{3} \frac{\sqrt{2m}}{\hbar k} \left(E_n - V_0\right)^{3/2}$$

Solving for E_n, we find

$$E_n = \left[\frac{3\hbar k (n + 3/4)\pi}{2\sqrt{2m}}\right]^{2/3} + V_0$$

$$= \underbrace{\left[\frac{3\hbar k \pi}{2\sqrt{2m}}\right]^{2/3}}_{\beta} \left[n + \frac{3}{4}\right]^{2/3} + V_0$$

$$\equiv \beta \left(n + \frac{3}{4}\right)^{2/3} + V_0$$

where we have defined the constant β to be

$$\beta \equiv \left[\frac{3\hbar k\pi}{2\sqrt{2m}}\right]^{2/3}$$

which we will never need to evaluate!

We have the values of the $n=0$ and $n=1$ state energies to be $3.1 = E_0$ and $3.7 = E_1$ (in GeV) respectively. We thus have two equations in the two unknowns β and V_0:

$$\left(\frac{3}{4}\right)^{2/3}\beta + V_0 = 3.1 \qquad \Rightarrow \qquad 3.1 = 0.825\beta + V_0$$

$$\left(\frac{7}{4}\right)^{2/3}\beta + V_0 = 3.7 \qquad\qquad 3.7 = 1.452\beta + V_0$$

Solving, $\beta \doteq 0.96 \quad V_0 \doteq 2.31$ (in GeV)

Thus, we predict

$$E_2 \simeq 2.31 + 0.96\left(\frac{11}{4}\right)^{2/3} = 4.2 \text{ GeV}$$

which, glory be, is in agreement with experiment.

Shankar 16.2.7

From 16.2.32, the quantization condition is

$$\int_{-x_0}^{x_0} p(x)\,dx = (n+\tfrac{1}{2})\pi\hbar$$

where x_0 is the classical turning point. For the harmonic oscillation,

$$V(x) = \tfrac{1}{2} m w^2 x^2$$

and so

$$\int_{-x_0}^{x_0} p(x)\,dx = \int_{-x_0}^{x_0} \sqrt{2m\left(E - \frac{m^2w^2x^2}{2}\right)}\,dx = \sqrt{2mE}\cdot 2\int_0^{x_0}\sqrt{1-\frac{mw^2x^2}{2E}}\,dx$$

But noting that the condition for the turning point is given by

$$E = \tfrac{1}{2} m w^2 x_0^2 \Rightarrow \frac{1}{x_0^2} = \frac{mw^2}{2E}$$

we can then write

$$\int_{-x_0}^{x_0} p(x)\,dx = 2\sqrt{2mE}\int_0^{x_0}\sqrt{1-\frac{x^2}{x_0^2}}\,dx$$

Letting $z = x/x_0$ we get $dx = x_0\,dz$ and

$$\int_{-x_0}^{x_0} p(x)\,dx = 2\sqrt{2mE}\; x_0 \int_0^1 \sqrt{1-z^2}\,dz$$

Letting $z = \sin\theta$ we then have

$$\int_{-x_0}^{x_0} p(x)\,dx = 2\sqrt{2mE}\; x_0 \int_0^{\pi/2} \cos\theta \, d\sin\theta = 2\sqrt{2mE}\, x_0 \int_0^{\pi/2} \cos^2\theta \, d\theta$$

and recalling that $\langle \cos^2\theta \rangle = \frac{1}{2}$ over an integer number of quarter periods, the integral is just $\pi/4$ and so

$$\int_{-x_0}^{x_0} p(x)\,dx = \frac{\pi}{2}\sqrt{2mE}\; x_0 = \frac{\pi}{2}\sqrt{2mE}\sqrt{\frac{2E}{m}}\frac{1}{\omega} = \frac{\pi E}{\omega}$$

Thus, the quantization condition becomes

$$\frac{\pi E}{\omega} = (n+\tfrac{1}{2})\pi\hbar \quad \text{or} \quad \boxed{E = (n+\tfrac{1}{2})\hbar\omega}$$

as hoped for.