HW III

Problem 2

I'll just do the eigenvector w/ the positive eigenvalue; the other eigenvector follows from the same arguments.

First, as usual, find the eigenvalues:

$$0 = \begin{vmatrix} c-\lambda & se^{-\beta} \\ se^{\beta} & -c-\lambda \end{vmatrix} = -(c-\lambda)(c+\lambda) - s^2 = \lambda^2 - 1$$

(where $s = \sin\theta$ $c = \cos\theta$ and $\beta = i\phi$ for ease of notation.

Since $\lambda = \pm 1$, choose $\lambda = +1$

$$\begin{pmatrix} c & se^{-\beta} \\ se^{\beta} & -c \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = + \begin{pmatrix} a \\ b \end{pmatrix}$$ so

$$ca + se^{-\beta} b = a \quad \Rightarrow \quad \frac{b}{a} = \frac{1-c}{se^{-\beta}}$$ and so

$$n_+ = \alpha \begin{pmatrix} se^{-\beta} \\ 1-c \end{pmatrix}$$ where the normalization constant

$$\alpha^2 = s^2 + (1-c)^2 = s^2 + 1 + c^2 - 2c = 2(1-c) = 4\sin^2\theta/2$$

breaking out the half-angle formulas. Hence

$$n_+ = \frac{1}{2\sin\theta/2}\begin{pmatrix} \sin\theta\, e^{-i\phi} \\ 1-\cos\theta \end{pmatrix} = \frac{e^{-i\phi/2}}{2\sin\theta/2}\begin{pmatrix} 2\sin\theta/2\cos\theta/2\, e^{-i\phi/2} \\ 2\sin^2\theta/2\, e^{i\phi/2} \end{pmatrix}$$

$$= e^{-i\phi/2}\begin{pmatrix} \cos\theta/2\, e^{-i\phi/2} \\ \sin\theta/2\, e^{i\phi/2} \end{pmatrix}$$

up to an overall phase of $e^{-i\phi/2}$, which the rules say we may ignore at no cost to life nor limb.

Now,

$$\langle n_+|\vec{S}|n_+\rangle = \frac{\hbar}{2}\langle n_+|\sigma_x\hat{x} + \sigma_y\hat{y} + \sigma_z\hat{z}|n_+\rangle$$

OK, so do term-by-term, with now $c = \cos\theta/2$ $s = \sin\theta/2$ $\beta = i\phi/2$

$$\sigma_x|n_+\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c e^{-\beta} \\ s e^{\beta} \end{pmatrix} = \begin{pmatrix} s e^{\beta} \\ c e^{-\beta} \end{pmatrix}$$

$$\langle n_+|\sigma_x|n_+\rangle = \begin{pmatrix} c e^{\beta} & s e^{-\beta} \end{pmatrix}\begin{pmatrix} s e^{\beta} \\ c e^{-\beta} \end{pmatrix} = cs e^{2\beta} + cs e^{-2\beta} = 2cs\cos 2\beta/i$$

$$= \sin\theta\cos\phi \checkmark$$

$$\langle n_+|\sigma_y|n_+\rangle = \overline{\begin{pmatrix} ce^{\beta} & se^{-\beta}\end{pmatrix}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} ce^{-\beta} \\ se^{\beta} \end{pmatrix}$$

$$= \overline{\begin{pmatrix} ce^{\beta} & se^{-\beta}\end{pmatrix}} \begin{pmatrix} -ise^{\beta} \\ ice^{-\beta} \end{pmatrix} = -icse^{2\beta} + icse^{-2\beta}$$

$$= 2cs\,\frac{e^{2\beta}-e^{-2\beta}}{2i} = \sin\theta\sin\phi \;\checkmark$$

$$\langle n_+|\sigma_z|n_+\rangle = \overline{\begin{pmatrix} ce^{\beta} & se^{-\beta}\end{pmatrix}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} ce^{-\beta} \\ se^{\beta} \end{pmatrix}$$

$$= \overline{\begin{pmatrix} ce^{\beta} & se^{-\beta}\end{pmatrix}} \begin{pmatrix} ce^{-\beta} \\ -se^{\beta} \end{pmatrix} = c^2 - s^2 = \cos\theta \;\checkmark$$

While I think it has been very good for me to work all this out, this ~~explicit~~ also shows explicitly how the spin eigenstate $|\tfrac{1}{2},\tfrac{1}{2}\rangle$ obtains a $-$ sign when the spin is rotated through ordinary space by an amount ~~[illegible]~~ $\theta = 2\pi$. Very interesting, indeed.

Problem ~~B~~ 2

We have the Pauli spin matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a) For $t<0$, we have

$$H = -\mu_B \sigma_z B_z = \begin{pmatrix} -\mu_B B_z & 0 \\ 0 & \mu_B B_z \end{pmatrix}$$

The TDSE is $H\Psi = -i\hbar \frac{\partial \Psi}{\partial t}$. Using the trial function $\Psi(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\omega t}$ we get

$$H\Psi = -\mu_B B_z \Psi(t)$$

$$-i\hbar \frac{\partial \Psi}{dt} = -i\hbar(i\omega)\Psi(t) = \hbar\omega \Psi(t)$$

Which works as long as $\omega = \frac{-\mu_B B_z}{\hbar}$ ✓

For $t>0$

(b) $H = -\mu_B\left[\sigma_x B_x + \sigma_z B_z\right] = \begin{pmatrix} -\mu_B B_z & -\mu_B B_x \\ -\mu_B B_x & \mu_B B_z \end{pmatrix} = \begin{pmatrix} -z & -x \\ -x & z \end{pmatrix}$

[6]

Let's make ansatz $\Psi(t) = \begin{bmatrix} a_1 \cos \Omega t + a_2 \sin \Omega t \\ b_1 \cos \Omega t + b_2 \sin \Omega t \end{bmatrix}$ which represents a vector processing w/ ang. frequency Ω

Imposing $\Psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ on the ansatz, we have

$$\Psi(t) = \begin{pmatrix} \cos \Omega t + a_2 \sin \Omega t \\ b_2 \sin \Omega t \end{pmatrix} \equiv \begin{pmatrix} c + as \\ bs \end{pmatrix}$$

Applying the TDSE

$$H\Psi = \begin{pmatrix} -Z & -X \\ -X & Z \end{pmatrix} \begin{pmatrix} c + as \\ bs \end{pmatrix} = \begin{pmatrix} -Z(c+as) - Xbs \\ -X(c+as) + Zbs \end{pmatrix}$$

and

$$-i\hbar \frac{\partial \Psi}{\partial t} = -i\hbar\Omega \begin{pmatrix} -s + ac \\ bc \end{pmatrix}$$

Equating we find, setting $\alpha = -i\hbar\Omega$

(A) $-Z(c+as) - Xbs = [-s + ac]\alpha$

(B) $-X(c+as) + Zbs = [bc]\alpha$

From (B), treating the sin and cos terms as independent,

$$-Xc = bc\alpha \Rightarrow b = -X/\alpha$$

$$-Xas + Zbs = 0 \Rightarrow a = \frac{Z}{X} b = -Z/\alpha$$

The sine term of (A) gives

$$-Za - Xb = -\alpha$$

Plugging in for a, b gives

$$\frac{Z^2}{\alpha} + \frac{X^2}{\alpha} = -\alpha \Rightarrow -\alpha^2 = Z^2 + X^2$$

But $-\alpha^2 = \hbar^2\Omega^2$ so

$$\boxed{\Omega = \frac{\sqrt{X^2 + Z^2}}{\hbar}} \quad \text{with} \begin{cases} X = \mu_B B_x \\ Z = \mu_B B_z \end{cases}$$

Finally, plugging in for a and b, we have

$$\Psi(t) = \begin{pmatrix} \cos\Omega t + \frac{Z}{i\sqrt{X^2+Z^2}}\sin\Omega t \\ \frac{X}{i\sqrt{X^2+Z^2}}\sin\Omega t \end{pmatrix}$$

which always has a squared amplitude of 1. The maximum contribution to $\begin{pmatrix}0\\1\end{pmatrix}$ comes when $\sin\Omega t = 1$, so

$$\boxed{P_{max} = \left[\frac{X}{i\sqrt{X^2+Z^2}}\right]^2 = \frac{X^2}{X^2+Z^2}}$$

complex square!

Problem 3

We begin by determining the strength of the $n=2$ spin-orbit interaction. From any book, or the notes,

$$W_{so} = \frac{1}{4m^2c^2}\,\frac{1}{r}\,\frac{dV(r)}{dr}\left[J^2 - L^2 - S^2\right]$$

For $L=0$, $J=S$ and $W_{so}=0$, which makes sense since in this case there's no orbital motion for the spin to couple to. Now, we can calculate from the notes

	J^2	L^2	S^2	$J^2-L^2-S^2$	ΔE_{so}
$2P_{1/2}$	3/4	2	3/4	-2	$-\frac{1}{48}\lambda$
$2P_{3/2}$	15/4	2	3/4	$+1$	$+\frac{1}{96}\lambda$

whence we see that

$$\Delta E_{so}^{n=2,\,l=1} = \beta\left[J^2 - L^2 - S^2\right] \quad \text{with}$$

$$\boxed{\beta^{21} = \frac{1}{96}\lambda} \qquad n=2 \quad l=1$$

$$\boxed{\beta^{20} = 0} \qquad n=2 \quad l=0$$

To find the strength β of the spin-orbit interaction for $n=3$, we need to calculate the expectation value of the operator

$$\beta_{so}^{op} = \frac{1}{4m^2c^2}\frac{1}{r}\frac{dV(r)}{dr} \propto \frac{1}{r^3}$$

We'll need to do this for $n=3, l=1$ and $n=3, l=2$, but not $n=3, l=0$ since $\beta=0$ whenever $l=0$, as we noted above. But we can save a lot of headache (worry about constants, etc). by scaling to the $n=2, l=1$ value we know, so we'll calculate that too (as suggested in the statement of the problem).

For this, we'll need the r-dependence of the $n=2$ and $n=3$ wavefunctions. For this we consult pp 356-7 of Shankar. There, we find that

$$R_{nl}(\rho) \sim e^{-\rho}\rho^{l} L_{n-l-1}^{2l+1}(2\rho)$$

where $\rho = \frac{r}{na_0}$ and $L_p^k(x)$ is an associated Laguerre polynomial

The definition of $L_p^k(x)$ is in the footnote on p356, but on p. 357 we are reminded that $L_0^k(x)$ is a constant so we need not worry about that. This immediately gives us that

$$R_{21}(\rho) \sim \rho e^{-\rho} \qquad \rho = r/2a_0$$

$$R_{32}(\rho) \sim \rho^2 e^{-\rho} \qquad \rho = r/3a_0$$

The only $L_p^k(x)$ we need to calculate is for $n=3$ $l=1$, i.e.,

$$L_1^3 = -\left(\frac{d}{dx}\right)^3 L_4^0 = -\left(\frac{d}{dx}\right)^3\left[e^x\left(\frac{d}{dx}\right)^4 e^{-x}x^4\right] = \text{a little scratching}$$

$$= 24[4-x] = 24[4-2\rho] = 48[2-\rho]$$

We discard the 48 and preserve the "$2-\rho$" to get

$$R_{31}(\rho) \sim \rho(2-\rho)e^{-\rho} \qquad \rho = r/3a_0$$

We're now ready to chug away. For each of these, we need

$$\langle r^{-3}\rangle_{nl} = \frac{\int_0^\infty (r^{-3})(r^2)|R_{nl}|^2\,dr}{\int_0^\infty (r^2)|R_{nl}|^2\,dr}$$

and we calculate the numerators and denominators separately (we need the denominators because our R_{nl}'s are not normalized).

$$NUM_{21} = \int \frac{1}{r}|R_{21}|^2\,dr = \int \frac{1}{2a_0\rho}\rho^2 e^{-2\rho}(2a_0\,d\rho)$$

$$= \int \rho e^{-2\rho}\,d\rho = \frac{1}{4}\int x e^{-x}\,dx = \frac{1}{4}$$

$$DEN_{21} = \int r^2|R_{21}|^2\,dr = 8a_0^3\int \rho^4 e^{-2\rho}\,d\rho = \frac{a_0^3}{4}\int x^4 e^{-x}\,dx = 6a_0^3$$

using $\int x^n e^{-x}\,dx = n!$

$$\boxed{\langle r^{-3}\rangle_{21} = \frac{1}{24a_0^3}}$$

Staggering onward,

$$NUM_{31} = \int \frac{1}{3a_0\rho} \rho^2 e^{-2\rho} (2-\rho)^2 \, d\rho = \int \rho e^{-2} (4 - 4\rho + \rho^2) \, d\rho$$

$$= 4 \int \rho e^{-2\rho} \, d\rho - 4 \int \rho^2 e^{-2\rho} \, d\rho + \int \rho^3 e^{-2\rho} \, d\rho$$

$$= \int x e^{-x} \, dx - \frac{1}{2} \int x^2 e^{-x} \, dx + \frac{1}{16} \int x^3 e^{-x} \, dx$$

$$= 1 - 1 + \frac{6}{16} = \frac{3}{8}$$

$$DEN_{31} = 4(27a_0^3) \int \rho^4 e^{-2\rho} \, d\rho - 4(27a_0^3) \int \rho^5 e^{-2\rho} \, d\rho + (27a_0)^3 \int \rho^6 e^{-2\rho} \, d\rho$$

$$= 27a_0^3 \left[\frac{1}{8} \int x^4 e^{-x} \, dx - \frac{1}{16} \int x^5 e^{-x} \, dx + \frac{1}{128} \int x^6 e^{-x} \, dx \right]$$

$$= 27a_0^3 \left[\frac{24}{8} - \frac{120}{16} + \frac{720}{128} \right] = 27a_0^3 \left[\frac{24 - 60 + 45}{8} \right] = \frac{243}{8} a_0^3$$

$$\text{Ratio} = \boxed{\langle r^{-3} \rangle_{31} = \frac{3}{8} \frac{8}{243} \frac{1}{a_0^3} = \frac{1}{81 a_0^3}}$$

And, with what feeble strength we still maintain,

$$\mathrm{NUM}_{32} = \int \left|\frac{1}{3a_0\rho}\rho^2 e^{-\rho}\right|^2 (3a_0 d\rho) = \int \rho^3 e^{-2\rho}\, d\rho$$

$$= \frac{1}{16}\int x^3 e^{-x}\, dx = \frac{6}{16} = \frac{3}{8}$$

$$\mathrm{DEN}_{32} = (27a_0^3)\int \rho^6 e^{-2\rho}\, d\rho = \frac{27a_0^3}{128}\int x^6 e^{-x}\, dx = \frac{(720)(27)}{128} a_0^3$$

$$= \frac{1215}{8} a_0^3$$

$$\boxed{\langle r^{-3}\rangle_{32} = \frac{3}{8}\cdot\frac{8}{1215}\frac{1}{a_0^3} = \frac{1}{405 a_0^3}}$$

We can now calculate the spin-orbit strengths.

$$\beta^{31} = \beta^{21}\cdot\frac{\langle r^{-3}\rangle_{31}}{\langle r^{-3}\rangle_{21}} = \frac{1}{96}\lambda\cdot\frac{1/81a_0^3}{1/24a_0^3} = \frac{1}{96}\lambda\cdot\frac{24}{81} = \frac{1}{324}\lambda$$

$$\beta^{32} = \beta^{21}\cdot\frac{\langle r^{-3}\rangle_{32}}{\langle r^{-3}\rangle_{21}} = \frac{1}{96}\lambda\cdot\frac{1/405a_0^3}{1/24a_0^3} = \frac{1}{96}\lambda\cdot\frac{8}{135} = \frac{1}{1620}\lambda$$

$$\boxed{\beta^{31} = \frac{1}{324}\lambda}$$

$$\boxed{\beta^{32} = \frac{1}{1620}\lambda}$$

$$\boxed{\beta^{30} = 0}$$

We now need to calculate the spin-orbit energy shift for the $n=3$ states. We have $l = 0, 1, 2$ and $s = \frac{1}{2}$. We know we get nothing for $l=0$ (S states), so we need only consider the 3P and 3D states. For each of these, we have the possibilities $j = l \pm \frac{1}{2}$, so we make the following table:

	J^2	L^2	S^2	$J^2-L^2-S^2$	β	ΔE_{SO}
					$\underbrace{}$ in units of λ	
$3P_{1/2}$	$3/4$	2	$3/4$	-2	$\frac{1}{324}$	$-\frac{1}{162}$
$3P_{3/2}$	$15/4$	2	$3/4$	$+1$	$\frac{1}{324}$	$+\frac{1}{324}$
$3D_{3/2}$	$15/4$	6	$3/4$	-3	$\frac{1}{1620}$	$-\frac{1}{540}$
$3D_{5/2}$	$35/4$	6	$3/4$	$+2$	$\frac{1}{1620}$	$+\frac{1}{810}$
reminder! $2P_{1/2}$	$3/4$	2	$3/4$	-2	$\frac{1}{96}$	$-\frac{1}{48}$
$2P_{3/2}$	$15/4$	2	$3/4$	$+1$	$\frac{1}{96}$	$+\frac{1}{96}$

The $n=2$ states were added in here just for comparison; since everything is decaying from $n=3$ to $n=1$, we don't need the $n=2$ states anymore, really.

These shifts perturb coulomb-interaction levels of

$$E_n^{(0)} = -\frac{e^2}{2a_0 n^2} = -13.6/n^2 \text{ eV}$$

so the unperturbed transition energy is $E_{13} = -\frac{e^2}{2a_0}\left[\frac{1}{9}-1\right] = \frac{4}{9}\frac{e^2}{a_0}$
~~where~~

Furthermore, the 1s levels are not split, or even displaced, by the spin-orbit interaction. Thus, the difference with respect to the unperturbed transition is just the SO energy shift. The degeneracy is just $2(2j+1)$, so we have the following transitions! (the factor of 2 arises because the ground state is doubly degenerate)

Transition	Energy	Degeneracy	Forbidden?
$3S_{1/2} \to 1S_{1/2}$	$\frac{4}{9}\frac{e^2}{a_0}$	2×2	$l=0 \to l=0$ (Yes)
$3P_{1/2} \to 1S_{1/2}$	$\frac{4}{9}\frac{e^2}{a_0} - \frac{\lambda}{162}$	2×2	No
$3P_{3/2} \to 1S_{1/2}$	$\frac{4}{9}\frac{e^2}{a_0} + \frac{\lambda}{324}$	2×4	No
$3D_{3/2} \to 1S_{1/2}$	$\frac{4}{9}\frac{e^2}{a_0} - \frac{\lambda}{540}$	2×4	No
$3D_{5/2} \to 1S_{1/2}$	$\frac{4}{9}\frac{e^2}{a_0} + \frac{\lambda}{810}$	2×6	$\Delta j = 2$ (Yes)

The splittings are small - for example, $\lambda/324 \cong 4.5\times10^{-6}$ eV compared to an overall transition energy of ~ 10 eV!

Problem 4

The interaction energy from the coupling of the spin into the external magnetic field is given by

$$\Delta E = -\vec{\mu}\cdot\vec{B} = g_L \mu_B m_j B \hbar$$

where g_L is the Landé g-factor for the state, and m_j the projection (in units of $\hbar$) of the total angular momentum into the direction of the B-field. This will remove the remaining degeneracy in the n=3 levels that remained after we "turned on" the spin-orbit coupling. It will also remove the two-fold degeneracy in the n=1 state.

First, we need to calculate the g_L's, given in the notes as $g_L = \left[1 + \frac{j(j+1) + s(s+1) - l(l+1)}{2j(j+1)}\right]$

Here's a little table:

STATE	j	s	l	g_L
$s_{1/2}$	$1/2$	$1/2$	0	2
$p_{1/2}$	$1/2$	$1/2$	1	$2/3$
$p_{3/2}$	$3/2$	$1/2$	1	$4/3$
$d_{3/2}$	$3/2$	$1/2$	2	$4/5$
$d_{5/2}$	$5/2$	$1/2$	2	$6/5$

There are no remaining occupied-state degeneracies. Allowed transitions have $\Delta j = -1, 0, +1$ and $\Delta m_j = -1, 0, +1$, except $\ell=0 \to \ell=0$ is not allowed, so only the 3p states have allowed transitions. Here's a little table that shows all the allowed transitions. The energy shifts are all relative to that of the $4/9\ e^2/a_0$ energy of the purely coulombic $n=3 \to n=1$ transition.

	$\lvert \ell, j, m_j \rangle$ n=3 state	$\Rightarrow$ $\lvert \ell, j, m_j \rangle$ n=1 state	In units of $\frac{1}{2}\mu_B h_B$: $\Delta E_{n=3}$	$\Delta E_{n=1}$	ΔE_Z	In units of λ: ΔE_{so}
$3p_{1/2}$	$\lvert 1, \frac{1}{2}, \frac{1}{2} \rangle$	$\to \lvert 0, \frac{1}{2}, \frac{1}{2} \rangle$	$2/3$	$+2$	$-4/3$	$-\frac{1}{162}$
	$\lvert 1, \frac{1}{2}, \frac{1}{2} \rangle$	$\to \lvert 0, \frac{1}{2}, -\frac{1}{2} \rangle$	$2/3$	-2	$+8/3$	$-1/162$
$3p_{1/2}$	$\lvert 1, \frac{1}{2}, -\frac{1}{2} \rangle$	$\to \lvert 0, \frac{1}{2}, \frac{1}{2} \rangle$	$-2/3$	$+2$	$-8/3$	$-1/162$
	$\lvert 1, \frac{1}{2}, -\frac{1}{2} \rangle$	$\to \lvert 0, \frac{1}{2}, -\frac{1}{2} \rangle$	$-2/3$	-2	$+4/3$	$-1/162$
$3p_{3/2}$	$\lvert 1, \frac{3}{2}, \frac{3}{2} \rangle$	$\to \lvert 0, \frac{1}{2}, \frac{1}{2} \rangle$	4	2	$+2$	$+1/324$
	$\lvert 1, \frac{3}{2}, \frac{1}{2} \rangle$	$\to \lvert 0, \frac{1}{2}, \frac{1}{2} \rangle$	$4/3$	2	$-2/3$	$+1/324$
	$\lvert 1, \frac{3}{2}, \frac{1}{2} \rangle$	$\to \lvert 0, \frac{1}{2}, -\frac{1}{2} \rangle$	$4/3$	-2	$+10/3$	$+1/324$
	$\lvert 1, \frac{3}{2}, -\frac{1}{2} \rangle$	$\to \lvert 0, \frac{1}{2}, \frac{1}{2} \rangle$	$-4/3$	2	$-10/3$	$+1/324$
	$\lvert 1, \frac{3}{2}, -\frac{1}{2} \rangle$	$\to \lvert 0, \frac{1}{2}, -\frac{1}{2} \rangle$	$-4/3$	-2	$+2/3$	$+1/324$
	$\lvert 1, \frac{3}{2}, -\frac{3}{2} \rangle$	$\to \lvert 0, \frac{1}{2}, -\frac{1}{2} \rangle$	-4	-2	-2	$+1/324$

Problem 5: ~~QM~~ (~~[illegible]~~)

A

(a) $$-\frac{\hbar^2}{2m}\left[\frac{d^2}{dx_1^2}+\frac{d^2}{dx_2^2}\right]\psi(x) + \left\{\frac{1}{2}C\left[x_1^2+x_2^2\right]+\frac{1}{2}k(x_1-x_2)^2\right\}\psi(x) = E\psi$$

(b) We write the Hamiltonian as

$$-\frac{\hbar^2}{m}\left[\frac{d^2}{d(x_1+x_2)^2}+\frac{d^2}{d(x_1-x_2)^2}\right]+\frac{1}{2}\left[Cx_1^2+Cx_2^2+kx_1^2+kx_2^2-2kx_1x_2\right]$$

We need to write the potential term in terms of the system variables R^2 and r^2

$$4A\left[\frac{1}{2}(x_1+x_2)\right]^2+B(x_1-x_2)^2=\frac{1}{2}\left[(C+k)(x_1^2+x_2^2)-2kx_1x_2\right]$$

$$\Rightarrow (A+B)(x_1^2+x_2^2)+2(A-B)x_1x_2=\frac{1}{2}(C+k)(x_1^2+x_2^2)-kx_1x_2$$

$$\therefore (A+B)=\frac{C+k}{2} \qquad A-B=-\frac{k}{2}$$

or

$$A=\frac{C}{4} \qquad B=\frac{1}{2}\left(\frac{C}{2}+k\right)$$

and so the potential term becomes

$$CR^2 + \frac{1}{2}\left(\frac{c}{2}+k\right)r^2 = \frac{1}{2}(2c)R^2 + \frac{1}{2}\left(\frac{c}{2}+k\right)r^2$$

and in total the Hamiltonian becomes

$$-\frac{\hbar^2}{4m}\frac{d^2}{dR^2} - \frac{\hbar^2}{m}\frac{d^2}{dr^2} + \frac{1}{2}(2c)R^2 + \frac{1}{2}\left(\frac{c}{2}+k\right)r^2$$

$$= \left\{-\frac{\hbar^2}{4m}\frac{d^2}{dR^2} + \frac{1}{2}(2c)R^2\right\} + \left\{-\frac{\hbar^2}{m}\frac{d^2}{dr^2} + \frac{1}{2}\left(\frac{c}{2}+k\right)r^2\right\}$$

$$= H_1(R) + H_2(r)$$

(c) Examining the Hamiltonian, we see that the problem has separated into that of two individual effective oscillators. Recalling the HO hamiltonian

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx} + \frac{1}{2}Kx^2$$

We see that

Variable	Effective Mass M	Effective Strength K
R	$2m$	$2c$
r	$m/2$	$(c/2 + k)$

Since the energy levels of the HO are given by

$E_n = (n+\frac{1}{2})\hbar\omega \qquad \omega = \sqrt{K/M}$ we have levels

$$R \qquad \hbar\sqrt{c/m}\,\left(\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots\right)$$

$$r \qquad \hbar\sqrt{\tfrac{c+2k}{m}}\,\left(\tfrac{1}{2}, \cancel{\tfrac{3}{2}}, \tfrac{5}{2}, \ldots\right)$$

However, we must worry about exchange symmetry for the identical bosons. Under $x_1 \leftrightarrow x_2$,

$$R \rightleftarrows R \qquad r \leftrightarrow -r$$

so only even functions of r are allowable! This eliminates the 1st, 3rd etc. excited states. Hence we cross out "3/2" for the r-dependent wave function.

Finally, letting $k = C/2$, the levels become

$R \Rightarrow \hbar\omega_0 \left(\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \tfrac{7}{2}, \ldots\right)$

$\omega_0 = \sqrt{C/m}$

$r \Rightarrow \sqrt{2}\,\hbar\omega_0 \left(\tfrac{1}{2}, \tfrac{5}{2}, \tfrac{9}{2}, \ldots\right)$

The four lowest energies are thus

$$E_0 = \hbar\omega_0 \left(\tfrac{1}{2} + \tfrac{\sqrt{2}}{2}\right) \doteq 1.21\,\hbar\omega_0$$

$$E_1 = \hbar\omega_0 \left(\tfrac{3}{2} + \tfrac{\sqrt{2}}{2}\right) \doteq 2.21\,\hbar\omega_0$$

$$E_2 = \hbar\omega_0 \left(\tfrac{5}{2} + \tfrac{\sqrt{2}}{2}\right) \doteq 3.21\,\hbar\omega_0$$

$$E_3 = \hbar\omega_0 \left(\tfrac{1}{2} + \tfrac{5\sqrt{2}}{2}\right) \doteq 4.04\,\hbar\omega$$

$$E_4 = \hbar\omega_0 \left(\tfrac{7}{2} + \tfrac{\sqrt{2}}{2}\right) \doteq 4.21\,\hbar\omega$$