HW IV

Problem 1 (Courtesy Edward Groth, Princeton)

~~[illegible]~~ Preliminaries

The single-particle wavefunctions are of the form

$$|n_a, n_b\rangle = \sqrt{\frac{4}{ab}} \sin\left(\frac{\pi n_a x}{a}\right) \sin\left(\frac{\pi n_b y}{b}\right) \qquad n_a, n_b > 0$$

and the energies are given by

$$E_{n_a n_b} = \frac{\hbar^2 \pi^2}{2m}\left[\left(\frac{n_a}{a}\right)^2 + \left(\frac{n_b}{b}\right)^2\right]$$

The lowest-energy single particle state is $|1,1\rangle$ with energy

$$E_{11} = \frac{\hbar^2 \pi^2}{2m}\left[\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2\right]$$

Since b is the largest side of the box, the next-lowest-energy state is $|1,2\rangle$ with energy

$$E_{12} = \frac{\hbar^2 \pi^2}{2m}\left[\left(\frac{1}{a}\right)^2 + \left(\frac{2}{b}\right)^2\right]$$

We write two-particle states as

$$|n_a^1, n_b^1\rangle |n_a^2, n_b^2\rangle$$

(c) Two distinguishable particles

Each particle can be placed in the lowest-energy single particle state, which is

$$\psi = |1,1\rangle |1,1\rangle$$

with energy

$$E_{1111} = 2\frac{\hbar^2\pi^2}{2m}\left[\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2\right]$$

Now, to take the "contact interaction" $V = ab\, V_0\, \delta^2(\vec{r}_1 - \vec{r}_2)$ into account, we need the product of two integrals $\Delta_x \Delta_y$ of the form

$$\Delta_x = \frac{4}{a^2}\int_0^a dx_1 \int_0^a dx_2 \sin^2\left(\frac{\pi n_a^1 x_1}{a}\right)\sin^2\left(\frac{\pi n_a^2 x_2}{a}\right) a\,\delta(x_1 - x_2)$$

$$= \frac{4}{a}\int_0^a dx \sin^2\left(\frac{\pi n_a^1 x}{a}\right)\sin^2\left(\frac{\pi n_a^2 x}{a}\right)$$

$$= 1 + \frac{1}{2}\delta_{n_a^1, n_a^2}$$

For the distinguishable case, $n_a^1 = n_a^2 = 1$ and the integral is $3/2$. Putting this all together, we find

$$\Delta E = V_0(\Delta_x \Delta_y) = V_0(\Delta_x)^2 = \frac{9}{4}V_0$$

(b) Two identical bosons

In this case, we have the added constraint that the wavefunction must be symmetric under the exchange of the two particles. But the state $|1,1\rangle|1,1\rangle$ is just that, so no difference in the state, energy, or contact interaction energy relative to that of part (a).

(c) Identical spin-$\frac{1}{2}$ fermions in the singlet state

In this case the total wavefunction

$$\psi = \psi_{space}\,\psi_{spin}$$

must be antisymmetric. But the spin singlet wavefunction

$$\psi^1_{spin} = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right]$$

is itself antisymmetric, so the total wavefunction

$$\psi_{tot} = \psi_{space}\,\psi_{spin} = \left[|1,1\rangle|1,1\rangle\right]\frac{1}{\sqrt{2}}\left[|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right]$$

is indeed antisymmetric under particle exchange. So again, no difference relative to (a).

(d) Identical spin-½ fermions in the triplet state.

In this case ψ_{spin} is symmetric so ψ_{space} must be antisymmetric. Our lowest-energy state is thus

$$\psi = \frac{1}{\sqrt{2}}\left[|1,1\rangle|1,2\rangle - |1,2\rangle|1,1\rangle\right]$$

with energy

$$E_{1112} = E_{11} + E_{12} = \frac{\hbar^2\pi^2}{2m}\left[2\left(\frac{1}{a}\right)^2 + 5\left(\frac{1}{b}\right)^2\right]$$

Finally, we need the matrix element of the perturbation for this state. Note that

$$\Delta_y = \frac{4}{b^2}\frac{1}{2}\int_0^b dy_1\int_0^b dy_2\left[\sin\left(\frac{\pi y_1}{b}\right)\sin\left(\frac{2\pi y_2}{b}\right) - \sin\left(\frac{2\pi y_1}{b}\right)\sin\left(\frac{\pi y_2}{b}\right)\right]^2 b\delta(y_1 - y_2)$$

$$= \frac{2}{\pi}\int_0^\pi dy\left[\sin y\sin 2y - \sin 2y\sin y\right] = 0$$

Not surprisingly, the particles in the triplet state are prevented from being at the same place by the Pauli exclusion principle, so the contact interaction is 0.

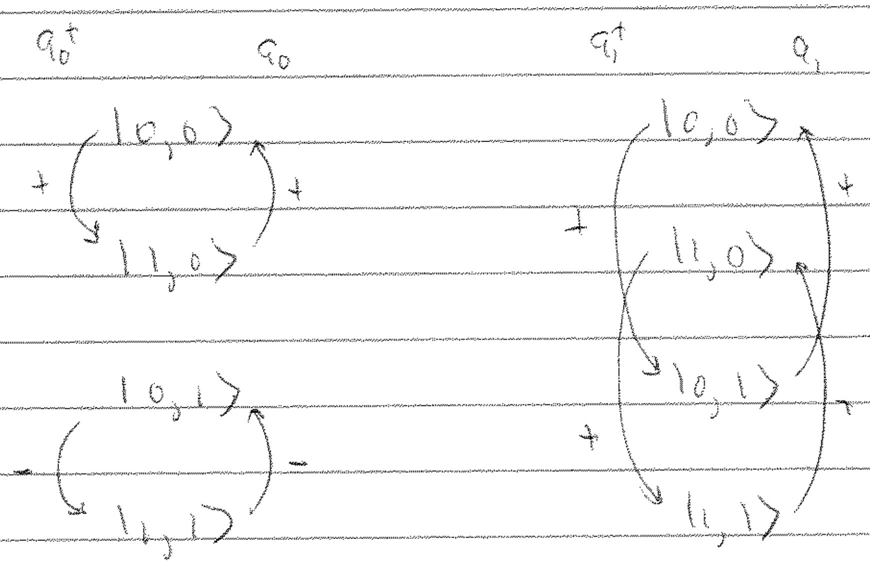
Problem 2 (Baym 19.1)

In the nomenclature $|n_0, n_1\rangle$, let's choose the basis

$$|0,0\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad |1,0\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad |0,1\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad |1,1\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

We need to use a bit of caution with respect to signs, c.f. p 415 of Baym, which are set to establish the proper exchange symmetry for fermions.

Below, we indicate the actions of $a_0^\dagger, a_0, a_1^\dagger, a_1$ on the basis states, showing the correct relative signs. Note that we arbitrarily associate the "−" signs associated with filling and depleting the $|1,1\rangle$ state w/ the first level; we presumably could have chosen the second level instead and gotten the same result.



This then gives us the following four matrices (you can easily verify!)

$$a_0 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad a_0^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$a_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad a_1^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

We are asked to check the anticommutation relations

$$\{a_0, a_0^+\} \qquad \{a_1, a_1^+\} \qquad \{a_0, a_1\} \qquad \{a_0, a_1^+\}$$

$$a_0 a_0^+ + a_0^+ a_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{1}_4 \checkmark$$

$$a_1 a_1^+ + a_1^+ a_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{1}_4 \checkmark$$

$$a_0 a_1 + a_1 a_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0 \quad \checkmark$$

$$a_0 a_1^\dagger + a_1^\dagger a_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0 \quad \checkmark$$

and so on...

Problem 3

We have, from the second expression,

$$N = \sum_{\lambda} \int d^3r\, \Psi^{\dagger}_{\lambda}(\vec{r})\, \Psi_{\lambda}(\vec{r})$$, making use of the definitions of the field operators

$$= \sum_{\lambda} \int d^3r \sum_{p} \frac{e^{-i\vec{p}\cdot\vec{r}}}{\sqrt{V}}\, a^{\dagger}_{\vec{p},\lambda} \sum_{p'} \frac{e^{i\vec{p}'\cdot\vec{r}}}{\sqrt{V}}\, a_{\vec{p}',\lambda}$$

$$= \sum_{\lambda \vec{p}\vec{p}'} a^{\dagger}_{\vec{p}',\lambda}\, a^{\dagger}_{\vec{p},\lambda} \int d^3r\, \frac{e^{i(\vec{p}'-\vec{p})\cdot\vec{r}}}{(\sqrt{V})^2}$$

The integral over r is just $\delta(\vec{p}-\vec{p}')$, whence

$$N = \sum_{\vec{p},\lambda} a^{\dagger}_{\vec{p},\lambda}\, a^{\dagger}_{\vec{p},\lambda}$$

and so it's shown.

Problem 3

The guidance we seek is provided on pp. 425-426 of Baym. In one dimension, we have for particles of spin s

$$G_s(x-x') = \langle \Phi_0 | \psi_s^+(x) \psi_s(x') | \Phi_0 \rangle$$

where Φ_0 is the $T=0$ ground state of the Fermi gas, which is fully occupied, for each spin, for all states up to the Fermi level p_f.

Now, $\psi_s(x) = \frac{1}{\sqrt{L}} \sum_p e^{ipx} a_{ps}$, where we find that

$$G_s(x-x') = \frac{1}{L} \sum_{pp'} e^{-ipx} e^{ip'x'} \langle \Phi_0 | a_{ps}^+ a_{p's} | \Phi_0 \rangle$$

$$= \frac{1}{L} \sum_{p,p'} e^{-ipx} e^{ip'x'} \delta_{pp'} n_{ps}$$

$$= \frac{1}{L} \sum_p e^{-ip(x-x')} n_{ps}$$

But, $n_{ps} = \begin{cases} 1 & -p_f \le p \le p_f \\ 0 & |p| > p_f \end{cases}$

for a $T=0$ Fermi gas, so for periodic boundary conditions in a 1-D box of length L,

$$G_s(x-x') = \frac{1}{L}\sum_{-p_f}^{p_f} e^{-ip(x-x')} = \frac{1}{L}\sum_{n=-n_f}^{+n_f} e^{-ip(x-x')}$$

where $n_f = \frac{Lp_f}{2\pi\hbar} = \frac{N_s}{2}$ with N_s the total number of fermions of spin s in the gas

Note that, due to this relation by the way, the density

$$\rho = \frac{N}{L} = \frac{2N_s}{L} = \frac{4Lp_f}{2\pi\hbar L} = \frac{2p_f}{\pi\hbar} \qquad (A)$$

Now, we note that the spacing between states in the fermi gas (box) is

$$\Delta p = \frac{2\pi\hbar}{L}$$

so converting the sum over p into an integral over dp,

$$G_s(x-x') \to \frac{1}{L}\left(\frac{L}{2\pi\hbar}\right)\int_{-p_f}^{p_f} e^{-ip(x-x')}\,dp$$

$$= \frac{1}{2\pi\hbar}\left\{\int_{-p_f}^{0} e^{-ip(x-x')}\,dp + \int_{0}^{p_f} e^{-ip(x-x')}\,dp\right\}$$

$$= -\frac{1}{2\pi\hbar}\frac{1}{i\Delta}\left\{1 - e^{ip_f\Delta} + e^{-ip_f\Delta} - 1\right\}$$

where we have defined $\Delta = x - x'$. Following through,

$$G_s(x-x') = \frac{1}{2\pi\hbar\Delta}\,\frac{1}{i}\left\{e^{ip_f\Delta} - e^{-ip_f\Delta}\right\}$$

$$= \frac{\sin p_f\Delta}{\pi\hbar\Delta}$$

If we now let $z \equiv p_f\Delta = p_f(x-x')$, we find

$$G_s(x-x') = \frac{p_f}{\pi\hbar}\,\frac{\sin z}{z} = \frac{\rho}{2}\,\frac{\sin z}{z}$$

where ρ is the gas density, by (2) above. Substituting in the explicit form for z,

$$G_s(x-x') = \frac{\rho}{2}\,\frac{\sin[p_f(x-x')]}{p_f(x-x')}$$

This function seems to me to be an effective "coherence length" of the influence of a particle of spin s at a position x' on a particle of the same spin at a distance $x-x'$ away due to the influence of the Pauli exclusion principle.

Problem 5 (Baym Problem 19.4)

The box needs to be multidimensional or the answer diverges (I found this out the hard way). So let's let it be a 3D box. Also, let the states be labelled $\vec{p}_1 s_1$ and $\vec{p}_2 s_2$. We'll only consider the case $\vec{p}_1 \neq \vec{p}_2$ since otherwise the parallel (equal) spin configuration is not allowed and the answer is indeterminate.

So, our state is

$$|\Xi\rangle = |\vec{p}_1 s_1, \vec{p}_2 s_2\rangle$$

with $\vec{p}_1 \neq \vec{p}_2$, and we'll explore the differences between $s_1 = s_2$ and $s_1 \neq s_2$.

From p434 of Baym, or class notes, we know that the interaction energy operator is given by

$$V_{op} = \frac{1}{2} \sum_{ss'} \int d^3r_1 d^3r_2 \, V(\vec{r}_1 - \vec{r}_2) \, \psi_s^+(\vec{r}_1) \psi_{s'}^+(\vec{r}_2) \psi_{s'}(\vec{r}_2) \psi_s(\vec{r}_1)$$

In the mean time, the field annihilation (creation) operator is given by

$$\psi_s^{(+)}(\vec{r}) = \sum_{\vec{p}} \frac{e^{(-)i\vec{p}\cdot\vec{r}}}{\sqrt{V}} a^{(+)}_{\vec{p},s}$$

Taking the expectation value of V for our state $|\Xi\rangle$, and using this expression for the field operators, we find that (writing for shorthand $\vec{p}\cdot\vec{r} = pr$)

$$\bar{V} = \langle \Phi | V_{op} | \Phi \rangle =$$

$$= \frac{1}{2}\sum_{ss'} \int d^3r_1 d^3r_2 \, V(\vec{r}_1 - \vec{r}_2) \frac{1}{V^2} \sum_{pp'qq'} \Big[\langle \Phi | e^{-ipr_1} e^{-iqr_2} e^{iq'r_2} e^{ip'r_1} \times$$

$$\times \, a^\dagger_{ps} a^\dagger_{qs'} a_{q's'} a_{p's} | \Phi \rangle \Big]$$

$$= \frac{1}{2V^2} \int d^3r_1 d^3r_2 \, V(\vec{r}_1 - \vec{r}_2) \sum_{ss'pp'qq'} e^{-i(p-p')r_1} e^{-i(q-q')r_2} \langle \Phi | a^\dagger_{ps} a^\dagger_{qs'} a_{q's'} a_{p's} | \Phi \rangle$$

$$\boxed{\boxed{s_1 \neq s_2}}$$

We have two cases to consider. First, we consider $s_1 \neq s_2$. The two possibilities are $(s_1, s_2) = (+,-)$ and $(-,+)$, ~~and~~ we'll want the average of those two. But the interaction energy will be the same for both, so we just pick one or the other, say $(+,-)$. ~~Then~~ Keeping this in mind we then note that ~~when~~ if we remove a particle of momentum $\vec{p}'$ and spin s, when we put the particle with spin s back into the state it will also need to have momentum $\vec{p}'$. ~~So~~, Similarly, for the oppositely spinning s' will need to be put back with ~~whatever~~ whatever momentum it had when it was removed. Thus, when $s \neq s'$

$$p = p' \qquad q = q'$$

is required.

Thus,

$$\bar{V}_{+-} = \frac{1}{2V^2}\int d^3r_1 d^3r_2 V(\vec{r}_1-\vec{r}_2) \sum_{ss'pp'qq'} \delta_{pp'}\delta_{qq'} e^{-i(p-p')r_1 - i(q-q')r_2} \langle \Phi | a^\dagger_{p's} a^\dagger_{q's'} a_{qs'} a_{ps} | \Phi \rangle$$

$$= \frac{1}{2V^2}\int d^3r_1 d^3r_2 V(\vec{r}_1-\vec{r}_2) \sum_{ss'pq} \langle \Phi | a^\dagger_{ps} a^\dagger_{qs'} a_{qs'} a_{ps} | \Phi \rangle$$

Now recalling that $|\Phi\rangle = |\vec{p}_1 s_1, \vec{p}_2 s_2\rangle$, then there are only two non-zero contributions from the sum over s, s', p, q

$$\left.\begin{array}{llll} \vec{p} = \vec{p}_1 & s = + & q = \vec{p}_2 & s' = - \\ p = \vec{p}_2 & s = - & q = \vec{p}_1 & s' = + \end{array}\right\} \text{ so}$$

$$\bar{V}_{+-} = \frac{1}{2V^2}\int d^3r_1 d^3r_2 V(\vec{r}_1-\vec{r}_2)\Big[\langle \Phi | a^\dagger_{p_1s_1} a^\dagger_{p_2s_2} a_{p_2s_2} a_{p_1s_1} | \Phi \rangle + \langle \Phi | a^\dagger_{p_2s_2} a^\dagger_{p_1s_1} a_{p_1s_1} a_{p_2s_2} | \Phi \rangle\Big]$$

But since $\{a_{ps}, a^\dagger_{p's'}\} = 0$ for $ps \neq p's'$, and $\{a_{ps}, a_{ps}\} = 0$ for any ps, then we can anticommute $a^\dagger_{p_1s_1}$ two places to the right in the first term, and $a_{p_2s_2}$ two places to the left in the second term, to get

$$\bar{V}_{+-} = \frac{1}{2V^2}\int d^3r_1 d^3r_2 V(\vec{r}_1-\vec{r}_2) \cdot 2\Big[\langle \Phi | a^\dagger_{p_2s_2} a_{p_2s_2} a^\dagger_{p_1s_1} a_{p_1s_1} | \Phi \rangle\Big] =$$

$$\frac{1}{V^2}\int d^3r_1\, d^3r_2\, V(\vec{r}_1-\vec{r}_2)\left[\langle\Phi| n_{p_2 s_2}\, n_{p_1 s_1}|\Phi\rangle\right]$$

But since there is exactly one particle in each of the states $p_1 s_1$ and $p_2 s_2$, the term in brackets is just 1, so

$$V_{+-} = \frac{1}{V^2}\int d^3r_1\, d^3r_2\, V(\vec{r}_1-\vec{r}_2) \qquad \text{(A)}$$

We hold off on evaluating this integral until later.

$$\boxed{s_1 = s_2}$$

For this we want an average of ++ and --, but again they should be the same, so we choose ++ just to be definite.
We recall our general expression

$$\overline{V} = \frac{1}{2V^2}\int d^3r_1\, d^3r_2\, V(\vec{r}_1-\vec{r}_2) \sum_{ss'pp'qq'} e^{-i(p-p')r_1}\, e^{-i(q-q')r_2}\, \langle\Phi| a^\dagger_{ps}\, a^\dagger_{qs'}\, a_{q's'}\, a_{p's}|\Phi\rangle$$

First, we note that since $s=s'$, we now have

$$\overline{V}_{++} = \frac{1}{2V^2}\int d^3r_1\, d^3r_2\, V(\vec{r}_1-\vec{r}_2) \sum_{spp'qq'} e^{-i(p-p')r_1}\, e^{-i(q-q')r_2}\, \langle\Phi| a^\dagger_{ps}\, a^\dagger_{qs}\, a_{q's}\, a_{p's}|\Phi\rangle$$

So if we remove a particle w/ momentum p' and replace it, again with p', we can have either $p=p'$ or $q=p'$. So we get two possibilities rather than one:

$\vec{p}' = \vec{p}$ $\quad \vec{q}' = \vec{q}$ $\quad$ as before

$\vec{p}' = \vec{q}$ $\quad \vec{q}' = \vec{p}$ $\quad$ new term!

Let's work on the inner product factor:

$$\langle \Phi | a^+_{p's} a^+_{q's} a_{qs} a_{ps} | \Phi \rangle \longrightarrow \delta_{pp'}\delta_{qq'} \langle \Phi | a^+_{ps} a^+_{qs} a_{qs} a_{ps} | \Phi \rangle +$$

$$+ \delta_{pq'}\delta_{qp'} \langle \Phi | a^+_{ps} a^+_{qs} a_{ps} a_{qs} | \Phi \rangle$$

Since by hypothesis $p \neq q$ (remember we argued for that at the very beginning of the problem), then we can anticommute the creation/annihilation operators to manipulate them into the form $a^+_{ps} a_{ps} a^+_{qs} a_{qs} = n_{ps} n_{qs}$. This involves two anticommutes (net "+" sign) for the first term, but only 1 anticommute (net "−" sign) for the second term, so

$$\langle \Phi | a^+_{p's} a^+_{q's} a_{qs} a_{ps} | \Phi \rangle \to (\delta_{pp'}\delta_{qq'} - \delta_{pq'}\delta_{qp'}) \langle \Phi | n_{ps} n_{qs} | \Phi \rangle \text{ and so}$$

$$V_{++} = \frac{1}{2V^2} \int d^3r_1 d^3r_2 V(\vec{r}_1 - \vec{r}_2) \sum_{spp'qq'} \Big[\delta_{pp'}\delta_{qq'} \langle \Phi | n_{ps} n_{qs} | \Phi \rangle -$$

$$- \delta_{pq'}\delta_{qp'} e^{-i(p-q)r_1} e^{-i(q-p)r_2} \langle \Phi | n_{ps} n_{qs} | \Phi \rangle \Big]$$

→ being very pedantic here to avoid confusion

$$= \frac{1}{2V^2} \int d^3r_1 d^3r_2 V(\vec{r}_1 - \vec{r}_2) \sum_{spq} \Big[\langle \Phi | n_{ps} n_{qs} | \Phi \rangle - e^{-i(p-q)(r_1 - r_2)} \langle \Phi | n_{ps} n_{qs} | \Phi \rangle \Big]$$

Now, our possibilities for p, q and s are

$p = p_1 \; s=+ \quad q = p_2 \; s=+$

$p = p_2 \; s=+ \quad q = p_1 \; s=+$

so using these two possibilities, we resolve the sums over s, p, q and

$$\sum_{spq} \langle \Phi | n_{ps} n_{qs} | \Phi \rangle \rightarrow \langle \Phi | n_{p_1+} n_{p_2+} | \Phi \rangle + \langle \Phi | n_{p_2+} n_{p_1+} | \Phi \rangle = 2$$

$$\sum_{spq} e^{-i(p-q)(r_1-r_2)} \langle \Phi | n_{ps} n_{qs} | \Phi \rangle \rightarrow$$

$$\rightarrow e^{-i(p_1-p_2)(r_1-r_2)} \langle \Phi | n_{p_1 s} n_{p_2 s} | \Phi \rangle + e^{-i(p_2-p_1)(r_1-r_2)} \langle \Phi | n_{p_2 s} n_{p_1 s} | \Phi \rangle$$

$$= e^{i(p_1-p_2)(r_1-r_2)} + e^{-i(p_1-p_2)(r_1-r_2)} = 2\cos\left[(\vec{p}_1-\vec{p}_2)\cdot(\vec{r}_1-\vec{r}_2)\right]$$

putting the vector notation back into the dot product. So,

$$V_{++} = \frac{1}{2V^2}\int d^3r_1 d^3r_2 \, V(\vec{r}_1-\vec{r}_2)\left[2 - 2\cos\left[(\vec{p}_1-\vec{p}_2)\cdot(\vec{r}_1-\vec{r}_2)\right]\right]$$

$$= \underbrace{\frac{1}{V^2}\int d^3r_1 d^3r_2 \, V(\vec{r}_1-\vec{r}_2)}_{A} - \underbrace{\frac{1}{V^2}\int d^3r_1 d^3r_2 \, V(\vec{r}_1-\vec{r}_2)\cos\left[(\vec{p}_1-\vec{p}_2)\cdot(\vec{r}_1-\vec{r}_2)\right]}_{B}$$

But, the A term on this page is identical to that on p. 4, i.e.,

$$V_{++} = V_{+-} - \frac{1}{V^2}\int d^3r_1 d^3r_2 \, V(\vec{r}_1-\vec{r}_2)\cos\left[(\vec{p}_1-\vec{p}_2)\cdot(\vec{r}_1-\vec{r}_2)\right]$$

This integral, for $V = \dfrac{e^2}{|\vec{r}_1-\vec{r}_2|}$, is thus the answer we seek!

Pursuing this, we are simply asking to compute the "exchange energy"

$$V_X = \frac{1}{V^2}\int d^3r_1 d^3r_2\, V(\vec{r}_1-\vec{r}_2)\cos\left[(\vec{p}_1-\vec{p}_2)\cdot(\vec{r}_1-\vec{r}_2)\right]$$

$$= \frac{1}{V^2}\int d^3R\, d^3r\, V(\vec{r})\cos(\vec{\Delta}\cdot\vec{r})$$

where we have defined $\vec{\Delta} \equiv \vec{p}_1-\vec{p}_2$ and made the transformation

$\vec{R} = \frac{1}{2}(\vec{r}_1+\vec{r}_2) \quad \vec{r} = \vec{r}_1-\vec{r}_2$

Performing the R integral and substituting $V(\vec{r}) = e^2/r$,

$$V_X = \frac{e^2}{V}\int d^3r\, \frac{\cos(\vec{\Delta}\cdot\vec{r})}{r}$$

Now, wolog, we align the z axis with $\vec{\Delta} = \vec{p}_1-\vec{p}_2$ (energy can't depend on orientation of the axis), so $\vec{\Delta}\cdot\vec{r} = \Delta r\cos\theta$. Performing the ϕ integration (does that every do anything other than bring out a factor of 2π, I ask you!?)

$$V_X = \frac{2\pi e^2}{V}\int_0^{\rho}\int_{-1}^{1}\frac{r^2dr\, d(\cos\theta)\cos(\Delta r\cos\theta)}{r} = \frac{2\pi e^2}{V}\int_0^{\rho}\int_{-1}^{1} dr\,dz\, r\cos(\Delta r z)$$

where we have let "ρ" be the radius of the sphere of volume V. Continuing with the integration

$$V_x = \frac{2\pi e^2}{V}\int_0^{\rho} dr\,dz\,r\left[\frac{\sin(\Delta r z)}{\Delta r}\right]_{-1}^{1}$$

$$= \frac{4\pi e^2}{\Delta V}\int_0^{\rho} dr\,\sin(\Delta r)$$ Letting $y = \Delta r$, so $dr = dy/\Delta$
$r < \rho \Rightarrow y = \Delta\rho$

$$= \frac{4\pi e^2}{\Delta^2 V}\int_0^{\Delta\rho}\sin y\,dy = \frac{4\pi e^2}{\Delta^2 V}\left[1 - \cos(\Delta\rho)\right]$$

Finally, recognizing that $V = 4/3\,\pi\rho^3$ is the volume of the sphere, and using $1 - \cos\theta = 2\sin^2\theta/2$, we find the exchange energy

$$V_x = \frac{6e^2}{\Delta^2\rho^3}\sin^2\left(\frac{\Delta\rho}{2}\right)$$

Finally, we make use of Bohm's x variable $X = \Delta\rho$ to write

$$\boxed{V_x = \frac{6e^2}{\rho x^2}\sin^2\left(\frac{x}{2}\right)}$$