Problem 1

We'll start by calculating $\nabla^2\left(e^{ikr}/r\right)$. This is most conveniently tackled in spherical coordinates for which there is only dependence on a single coordinate, r. Consulting standard references,

$$\nabla^2\left(\frac{e^{ikr}}{r}\right)=\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\left(\frac{e^{ikr}}{r}\right)\right]=$$

$$=\frac{1}{r^2}\frac{d}{dr}\left[r^2\left(ik\frac{e^{ikr}}{r}+e^{ikr}\frac{d}{dr}\left(\frac{1}{r}\right)\right)\right]=$$

Note that I have left the last term in its differential form because eventually I want to make use of the relation

$$\nabla^2\frac{1}{r}=-4\pi\delta^3(\vec{r})$$

Continuing, then,

$$\nabla^2\left(\frac{e^{ikr}}{r}\right)=\frac{1}{r^2}\frac{d}{dr}\left[ikre^{ikr}+r^2e^{ikr}\frac{d}{dr}\left(\frac{1}{r}\right)\right]=$$

$$=\frac{1}{r^2}\left[ike^{ikr}-rk^2e^{ikr}+ike^{ikr}r^2\frac{d}{dr}\left(\frac{1}{r}\right)+e^{ikr}\frac{d}{dr}r^2\frac{d}{dr}\left(\frac{1}{r}\right)\right]$$

$$= \frac{1}{r^2}\left[\cancel{ik e^{ikr}} - r k^2 e^{ikr} - \cancel{ik e^{ikr}} + e^{ikr} \frac{d}{dr} r^2 \frac{d}{dr}\left(\frac{1}{r}\right)\right]$$

$$= -k^2 \frac{e^{ikr}}{r} + e^{ikr}\left[\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr}\left(\frac{1}{r}\right)\right]$$

$$= -k^2 \frac{e^{ikr}}{r} - 4\pi e^{ikr} \delta^3(\vec{r}) = -k^2 \frac{e^{ikr}}{r} - 4\pi \delta^3(\vec{r})$$

since $e^{ikr} = 1$ at $r = 0$. So, now we can write

$$\left(+\frac{\hbar^2}{2m}\nabla^2 + E\right) G(\vec{r}) = \left(+\frac{\hbar^2}{2m}\nabla^2 + E\right)\left(-\frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\right)$$

$$= -\frac{1}{4\pi}\nabla^2 \frac{e^{ikr}}{r} - \frac{Em}{2\pi\hbar^2}\frac{e^{ikr}}{r}$$

$$= +\delta^3(\vec{r}) + \frac{k^2}{4\pi}\frac{e^{ikr}}{r} - \frac{\cancel{\hbar^2} k^2 \cancel{m}}{4\pi \cancel{m} \cancel{\hbar^2}} = \delta^3(\vec{r})$$

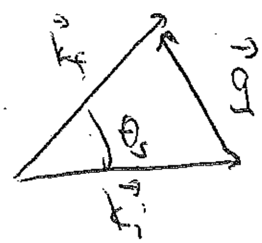
It checks.

HW 5

~~Bruce Schumm 2005 Quantum Problems I~~

Problem

(a) For elastic scattering, $|\vec{k}_{final}| = |\vec{k}_{initial}| = k$ and



by the law of cosines

$$q^2 = |\vec{k}_i|^2 + |\vec{k}_f|^2 - 2|\vec{k}_i||\vec{k}_f|\cos\theta_s$$

$$= 2k^2 - 2k^2\cos\theta_s = 2k^2[1-\cos\theta_s] = 4k^2\sin^2\theta_s/2$$

$$\boxed{q = 2k\sin\theta_s/2}$$

(b) According to the leading-order Born approximation,

$$f(q) = -\frac{m}{2\pi\hbar^2}\langle\chi_f|V|\chi_i\rangle.$$ Let's calculate

$$\langle\chi_f|V|\chi_i\rangle = \int e^{-i\vec{k}_f\cdot\vec{r}}\left[V_0\frac{e^{-\lambda r}}{r}\right]e^{i\vec{k}_i\cdot\vec{r}}\,d^3r =$$

$$= \int e^{i(\vec{k}_i - \vec{k}_f)\cdot\vec{r}} \left[V_0 \frac{e^{-\lambda r}}{r} \right] d^3r$$

$$= \iiint e^{-i\vec{q}\cdot\vec{r}} \left[V_0 \frac{e^{-\lambda r}}{r} \right] r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= 2\pi \iint \left[V_0 \frac{e^{-\lambda r}}{r} \right] r^2 dr \; e^{-i\vec{q}\cdot\vec{r}} \sin\theta \, d\theta$$

Without loss of generality, assume $\vec{q}$ is directed along the polar axis

$$= 2\pi \int_0^{\infty} V_0 e^{-\lambda r} r \, dr \int_0^{\pi} e^{-iqr\cos\theta} \sin\theta \, d\theta$$

$$= 2\pi \int_0^{\infty} V_0 e^{-\lambda r} r \, dr \int_{-1}^{1} e^{-iqr\cos\theta} \, d(\cos\theta)$$

$$= 2\pi \int_0^{\infty} V_0 e^{-\lambda r} r \, dr \left[\frac{e^{-iqrx}}{-iqr} \right]_{-1}^{1}$$

$$= -\frac{2\pi}{iq} \int_0^{\infty} V_0 e^{-\lambda r} dr \left[\cos qrx - i \sin qrx \right]_{-1}^{1}$$

and by the even/oddness of cos/sin, then

$$\langle \psi_f | V | \psi_i \rangle = -\frac{2\pi}{iq} \int_0^\infty V_0 e^{-\lambda r} dr \left[-2i \sin qr \right]$$

$$= \frac{4\pi}{q} \int_0^\infty V_0 e^{-\lambda r} \sin(qr) dr =$$

$$= \frac{4\pi V_0}{q} \int_0^\infty e^{-\lambda r} \sin(qr) dr = \frac{4\pi V_0}{q} \left[\frac{q}{\lambda^2 + q^2} \right] = \frac{4\pi V_0}{\lambda^2 + q^2}$$

Thus,

$$f(q) = -\frac{m}{2\pi\hbar^2} \frac{4\pi V_0}{\lambda^2 + q^2} = -\frac{2mV_0}{\hbar^2} \frac{1}{\lambda^2 + q^2}$$

and so

$$\frac{d\sigma}{d\Omega}(\theta) = \left| f(\theta) \right|^2 = \frac{4m^2 V_0^2}{\hbar^4} \frac{1}{\lambda^2 + 4k^2 \sin^2(\theta_s/2)}$$

$$\boxed{\frac{d\sigma}{d\Omega}(\theta) = \frac{4m^2 V_0^2}{\lambda^2 \hbar^4} \frac{1}{1 + 4(k/\lambda)^2 \sin^2(\theta_s/2)}}$$

Problem 3

(1) In the partial wave expansion, the total cross section is given by a sum over partial-wave contributions:

$$\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2 \delta_l$$

The case of the hard sphere is handled in Shankar 19.5, which provides us with the following beneficial relation:

$$\delta_l = \tan^{-1}\left[\frac{j_l(kr_0)}{\eta_l(kr_0)}\right]$$

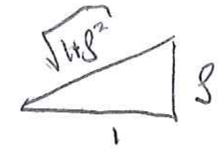
where r_0 is the radius of the sphere. Setting $\rho = kr_0$ and noting that $\rho \to 0$ as $k \to 0$, we find for the Bessel and Neumann functions (c.f. 12.6.33-34 in Shankar)

$$j_l(\rho) \xrightarrow[\rho\to 0]{} \frac{\rho^l}{(2l+1)!!} \qquad \eta_l(\rho) \xrightarrow[\rho\to 0]{} -\frac{(2l-1)!!}{\rho^{l+1}}$$

Thus, in the small-l limit, we find that

$$\frac{j_l(\rho)}{\eta_l(\rho)} = \frac{\rho^l}{(2l+1)!!} \frac{-\rho^{l+1}}{(2l-1)!!} = -\frac{\rho^{2l+1}}{(2l+1)!!\,(2l-1)!!}$$

which, for $\rho \to 0$, is dominated by the $l=0$ term:



$$\sin^2\delta_0 \to \sin^2\left\{\tan^{-1}\left[\frac{-\rho}{(1)!!\,(-1)!!}\right]\right\} = \sin^2\tan^{-1}(-\rho) = \frac{\rho^2}{1+\rho^2}$$

Thus, as $k \to 0$, we have that

$$\sigma \to \frac{4\pi}{k^2}\left(\frac{\rho^2}{1+\rho^2}\right) = \frac{4\pi r_0^2}{(k^2 \cancel{r_0^2})}\,\frac{\cancel{(k^2 r_0^2)}}{1+k^2 r_0^2} \to 4\pi r_0^2$$

as expected.

(2) The maximum angular momentum that can be achieved when bouncing off the hard sphere is

$$l_{max} = k r_0$$

Thus, we have

$$\sigma = \sum_{l=0}^{kr_0} \sigma_l = \frac{4\pi}{k^2}\sum_{0}^{kr_0}(2l+1)\sin^2\delta_l$$

But

$$\delta_l = \tan^{-1}\left[\frac{j_l(kr_0)}{n_l(kr_0)}\right] \xrightarrow[kr_0\to\infty]{} \tan^{-1}\left[\frac{\frac{1}{\rho}\sin(\rho - l\pi/2)}{-\frac{1}{\rho}\cos(\rho - l\pi/2)}\right]$$

$$= \tan^{-1}\left[-\tan(\rho - l\pi/2)\right] = -(\rho - l\pi/2)$$

where we've again defined $\rho \equiv kr_0$. Thus,

$\sin^2\delta_\ell = \sin^2(kr_0 - \ell\pi/2)$ as suggested. Then,

$$\sigma = \frac{4\pi}{k^2}\sum_0^{kr_0}(2\ell+1)\sin^2(kr_0 - \ell\pi/2)$$

It's noted that for $kr_0 >> 1$, the sume can be taken over into an integral

$$\sigma \cong \frac{4\pi}{k^2}\int_0^{kr_0}(2\ell+1)\sin^2(kr_0 - \ell\pi/2)\,d\ell$$

$$\cong \frac{4\pi}{k^2}\int_0^{kr_0}(2\ell)\sin^2(kr_0 - \ell\pi/2)\,d\ell$$

Now since kr_0 is a big number, $\sin^2(kr_0 - \ell\pi/2)$ will oscillate rapidly as ℓ goes from 0 to $\pi/2$ so we can fairly set it to its mean value, which is $\frac{1}{2}$, so

$$\sigma \to \frac{4\pi}{k^2}\left(\frac{1}{2}\right)\int_0^{kr_0}2\ell\,d\ell = \frac{2\pi}{k^2}\left[\ell^2\right]_0^{kr_0} = \frac{2\pi k^2 r_0^2}{k^2} = 2\pi r_0^2$$

as, again, suggested!

Problem 4

~~In light of eq. (14), the Born approximation is valid at all energies, which of course includes the low energy regime, when~~

$$\frac{2mg}{\hbar^2\mu} \ll 1 .$$

~~This condition guarantees that the small phase shift approximation is valid as long as $k \lesssim \mu$.~~

2. Consider the case of low-energy scattering from a spherical delta-function shell,

$$V(r) = V_0\delta(r-a),$$

where V_0 and a are constants. Calculate the scattering amplitude, $f(\theta)$, the differential cross-section and the total cross-section, under the assumption that $ka \ll 1$, so that only s-wave scattering is important.

First, we solve the radial Schrodinger equation. Following the standard steps, we write the solution as:

$$\psi(\vec{x}) = \frac{u_\ell(r)}{r} Y_\ell^m(\theta,\phi).$$

where $u_\ell(r) = rR_\ell(r)$ is related to the radial wave function $R(r)$, and satisfies the reduced radial equation,

$$-\frac{\hbar^2}{2m}\frac{d^2u_\ell}{dr^2} + \left[V(r) + \frac{\hbar^2\ell(\ell+1)}{2mr^2}\right]u_\ell(r) = Eu_\ell(r),$$

subject to the boundary condition that $u_\ell(r=0) = 0$. The phase shift is defined by the asymptotic behavior of $R_\ell(r)$:

$$R_\ell(r) \xrightarrow[r\to\infty]{} \frac{1}{kr}\sin(kr - \tfrac{1}{2}\ell\pi + \delta_\ell). \tag{18}$$

In the low-energy limit, we may assume that $ka \ll 1$ so that only s-wave scattering is important. Thus, we can simply take $\ell = m = 0$. Hence, for the delta-function potential, we must solve:

$$-\frac{d^2u}{dr^2} + \frac{2mV_0}{\hbar^2}\delta(r-a)u(r) = k^2u(r), \tag{19}$$

where $u(r) \equiv u_{\ell=0}(r)$ and $E = \hbar^2k^2/(2m)$ defines k as usual.

For $r \neq a$, the delta function vanishes, and we can solve the simple equation:

$$\frac{d^2u}{dr^2} = -k^2u(r). \tag{20}$$

The solution is thus given by:

$$u(r) = \begin{cases} A\sin kr, & \text{for } r < a, \\ B\sin(kr+\delta_0), & \text{for } r > a, \end{cases} \tag{21}$$

where we have imposed the boundary condition at the origin, $u(r=0)=0$. We have also identified the s-wave phase shift δ_0 according to the asymptotic behavior of $u(r)$ as $r \to \infty$ [cf. eq. (18)].

To make further progress, we must impose the correct boundary conditions at $r = a$. These are:

$$\text{(i)} \quad u(r) \text{ is continuous at } r = a\,,$$

$$\text{(ii)} \quad \left[\left.\frac{du}{dr}\right|_{a+\epsilon} - \left.\frac{du}{dr}\right|_{a-\epsilon}\right] = \frac{2mV_0}{\hbar^2}u(a)\,.$$

Boundary condition (ii) arises after integrating eq. (19) from $r = a - \epsilon$ to $a + \epsilon$ (where $0 < \epsilon \ll 1$). In particular, in the limit of $\epsilon \to 0$,

$$-\int_{a-\epsilon}^{a+\epsilon} \frac{d^2u}{dr^2}\,dr + \frac{2mV_0}{\hbar^2}u(a) = 0\,.$$

The right hand side is zero as a consequence of boundary condition (i) above. I have also used the fact that for any $\epsilon \neq 0$ (no matter how small),

$$\int_{a-\epsilon}^{a+\epsilon} \delta(r-a)f(r)dr = f(a)\,,$$

for any well-behaved function $f(r)$. The remaining integral is:

$$\int_{a-\epsilon}^{a+\epsilon} \frac{d^2u}{dr^2}\,dr = \left.\frac{du}{dr}\right|_{a+\epsilon} - \left.\frac{du}{dr}\right|_{a-\epsilon}\,,$$

which establishes condition (ii) above.

From eq. (21), we obtain:

$$\frac{du}{dr} = \begin{cases} kA\cos kr\,, & \text{for } r < a\,, \\ kB'\cos(kr+\delta_0)\,, & \text{for } r > a\,. \end{cases}$$

Applying conditions (i) and (ii) then yield:

$$B'\sin(ka+\delta_0) = A\sin ka\,,$$
$$B'\cos(ka+\delta_0) = A\cos ka - \frac{2mV_0}{\hbar^2 k}A\sin ka\,.$$

Diving these two equation, one obtains,

$$\cot(ka+\delta_0) = \cot ka + \frac{2mV_0}{\hbar^2 k}\,, \tag{22}$$

which is an implicit equation for the s-wave phase shift. To obtain δ_0, we make use of the trigonometric identity:

$$\cot(ka+\delta_0) = \frac{\cot ka \cot\delta_0 - 1}{\cot ka + \cot\delta_0}\,.$$

We can then rewrite eq. (22) as:

$$\frac{\cot ka \cot \delta_0 - 1}{\cot ka + \cot \delta_0} = \cot ka + \frac{2mV_0}{\hbar^2 k}\,.$$

Cross-multiplying and solving for $\cot \delta_0$, we find:

$$\cot \delta_0 = -\frac{\hbar^2 k}{2mV_0}\left[1 + \cot ka\left(\cot ka + \frac{2mV_0}{\hbar^2 k}\right)\right].$$

It is slightly more convenient to rewrite the above in terms of $\tan \delta_0 = 1/\cot \delta_0$,

$$\tan \delta_0 = -\frac{2mV_0}{\hbar^2 k}\left[\frac{1}{\sin^2 ka} + \frac{2mV_0}{\hbar^2 k}\frac{\cos ka}{\sin ka}\right]^{-1},$$

where we have used the identity $1 + \cot^2 ka = 1/\sin^2 ka$ and the definition of the cotangent. A simple rearrangement yields the desired result,

$$\tan \delta_0 = \frac{-\sin^2 ka}{\dfrac{\hbar^2 k}{2mV_0} + \sin ka \cos ka}\,. \tag{23}$$

The partial wave expansion of the scattering amplitude is given by:

$$f_k(\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)\left(\frac{e^{i\delta_\ell}\sin \delta_\ell}{k}\right)P_\ell(\cos\theta)\,.$$

In the limit of low-energies where $ka \ll 1$, only s-wave scattering is important. Thus, we can approximate

$$f_k(\theta) \simeq \frac{e^{i\delta_0}\sin \delta_0}{k}\,.$$

Applying the low-energy limit, $ka \ll 1$, to eq. (23) yields,

$$\tan \delta_0 \simeq -ka\left[1 + \frac{\hbar^2}{2mV_0 a}\right]^{-1}.$$

Indeed, $|\tan \delta_0| \ll 1$ (as expected), in which case,

$$e^{i\delta_0}\sin \delta_0 \simeq \sin \delta_0 \simeq \tan \delta_0 \simeq \delta_0\,.$$

Hence,

$$f_k(\theta) \simeq -a\left[1 + \frac{\hbar^2}{2mV_0 a}\right]^{-1}.$$

Finally, the differential and total cross sections are respectively given by:

$$\frac{d\sigma}{d\Omega} = |f_k(\theta)|^2 = a^2\left[1 + \frac{\hbar^2}{2mV_0 a}\right]^{-2},$$

$$\sigma = 4\pi a^2\left[1 + \frac{\hbar^2}{2mV_0 a}\right]^{-2}.$$

Homework Problem 5

We'll begin by finding the relation between $d\sigma/d\Omega\,(\theta_d)$ and the event rate observed in the little square detector patch.

Consider a particle in one of the beams. In passing through the other beam, it will encounter N particles w/ cross section σ squeezed into an area πa^2. Thus, the probability that particle will interact will be given by $N\sigma/\pi a^2$. Since there are N particles in the beam, not one, the total number of interactions per beam crossing will be $N^2\sigma/\pi a^2$.

Now, the detector subtends some solid angle $d\Omega_d$ at the angle θ_d. The number of particles per crossing scattering into the detector is thus

$$n_{scat} = \frac{N^2}{\pi a^2}\frac{d\sigma}{d\Omega}(\theta_d)\, d\Omega_d$$

and multiplying this by the frequency f will give the number per second, N_{scat}. But $d\Omega_d$ is just the fraction of 4π steradians subtended by the detector, ie,

$$d\Omega_d = \frac{A}{4\pi R^2}$$

Putting this together,

$$N_{scat} = \frac{N^2 A}{4\pi^2 a^2 R^2}\frac{d\sigma}{d\Omega}(\theta_d)$$

We now need to determine $\frac{d\sigma}{d\Omega}(\theta_d)$ for these various scenarios.

From class, the partial wave expansion is given by

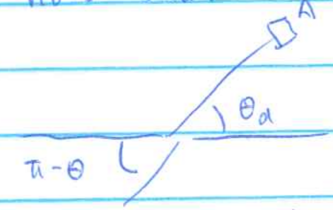
$$f(\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\,T_\ell(E)\,P_\ell(\cos\theta)$$

where $d\sigma/d\Omega = |f(\theta)|^2$. Keeping just the first two terms,

$$f(\theta) \to \frac{1}{k}(1)\,T_0 + \frac{1}{k}(3)\,T_1\cos\theta$$

$$= \sqrt{\sigma_0} + \sqrt{\sigma_0}\,e^{i\delta}\cos\theta$$

We now calculate $d\sigma/d\Omega = |f(\theta)|^2$ for several cases. We first note that since we didn't specify the color of the ball that hits the detector,



it can be either a right-moving ball scattering through θ_d or a left-moving ball scattering through $\pi-\theta_d$.

Distinguishable Case

In this case, $d\sigma/d\Omega$ is given by the sum of squared amplitudes:

$$\frac{d\sigma}{d\Omega_D} = \sigma_0 \left[|1 + e^{i\delta}\cos\theta|^2 + |1 + e^{i\delta}\cos(\pi - \theta)|^2 \right]$$

$$= \sigma_0 \left[|1 + e^{i\delta}\cos\theta|^2 + |1 - e^{i\delta}\cos\theta|^2 \right]$$

$$= \sigma_0 \left[1 + (e^{i\delta} + e^{-i\delta})\cos\theta + \cos^2\theta + 1 - (e^{i\delta} + e^{-i\delta})\cos\theta + \cos^2\theta \right]$$

$$\frac{d\sigma}{d\Omega_D} = 2\sigma_0 (1 + \cos^2\theta)$$

<u>Indistinguishable Symmetric Case</u>

We symmetrize by adding together the amplitudes for scattering through θ and $\pi - \theta$:

$$f_s(\theta) = \sqrt{\sigma_0} \left[1 + e^{i\delta}\cos\theta + 1 + e^{i\delta}\cos(\pi - \theta) \right]$$

$$= \sqrt{\sigma_0} \left[1 + e^{i\delta}\cos\theta + 1 - e^{i\delta}\cos(\theta) \right] = 2\sqrt{\sigma_0}$$

$$\frac{d\sigma}{d\Omega_s} = |f_s(\theta)|^2 = 4\sigma_0$$

Indistinguishable Antisymmetric Case

We antisymmetrize by subtracting the θ and $\pi-\theta$ amplitudes:

$$f_A(\theta) = \sqrt{\sigma_0}\left[1+e^{i\delta}\cos\theta - 1 + e^{i\delta}\cos\theta\right] = 2\sqrt{\sigma_0}\cos\theta\, e^{i\delta}$$

$$\underline{\frac{d\sigma}{d\Omega_A} = |f_A(\theta)|^2 = 4\sigma_0\cos^2\theta}$$

Now we need to put this all together to answer the four questions asked about the event rate into the little detector mounted at θ_d.

a) These are distinguishable bosons, so

$$N_{scat} = \frac{N^2 A}{4\pi^2 a^2 R^2}\frac{d\sigma}{d\Omega_D}(\theta_d) = \frac{N^2 A\,\sigma_0}{2\pi^2 a^2 R^2}(1+\cos^2\theta_d)$$

b) Half the collisions will be distinguishable (red/green or green/red) and the other half indistinguishable (red/red or green/green).

$$N_{scat} = \frac{N^2 A}{4\pi^2 a^2 R^2}\frac{1}{2}\left[\frac{d\sigma}{d\Omega_D}(\theta_d) + \frac{d\sigma}{d\Omega_S}(\theta_d)\right] = \frac{N^2 A\sigma_0}{4\pi^2 a^2 R^2}\left[3+\cos^2\theta_d\right]$$

$$= \frac{N^2 A\,\sigma_0}{2\pi^2 a^2 R^2}\left[\frac{3}{2}+\frac{1}{2}\cos^2\theta_d\right]$$

c) This is just the distinguishable case again

$$N_{scat} = \frac{N^2 A \sigma_0}{2\pi^2 a^2 R^2} \left(1 + \cos^2\theta_d\right)$$

d) As we've seen, this will be ½ distinguishable, and ½ indistinguishable. However, the half that's indistinguishable will be (see Shankar p. 561) 3/4 asymmetric and 1/4 symmetric. Thus,

$$N_{scat} = \frac{N^2 A}{4\pi^2 a^2 R^2} \left[\frac{1}{2} \frac{d\sigma}{d\Omega_D}(\theta_d) + \frac{1}{8} \frac{d\sigma}{d\Omega_S}(\theta_d) + \frac{3}{8} \frac{d\sigma}{d\Omega_A}(\theta_d) \right]$$

$$= \frac{N^2 A \sigma_0}{4\pi^2 a^2 R^2} \left[1 + 1\cos^2\theta + \frac{1}{2} + \frac{3}{2}\cos^2\theta \right]$$

$$= \frac{N^2 A \sigma_0}{2\pi^2 a^2 R^2} \left[\frac{3}{4} + \frac{5}{4}\cos^2\theta \right]$$

e) It's easy to see that $\int_0^{\pi/2} \cos^n\theta \, d\Omega = \frac{2\pi}{n}$, where the four cases give

$$\sigma_D = 2\sigma_0\left(4\pi + \tfrac{4}{3}\pi\right) = \left(\tfrac{32}{3}\right)\pi\sigma_0$$

$$\sigma_S = 4\sigma_0(4\pi) = 16\pi\sigma_0 = 1.5\,\sigma_D$$

$$\sigma_A = 4\sigma_0\left(\tfrac{4}{3}\pi\right) = \left(\tfrac{16}{3}\right)\pi\sigma_0 = 0.5\,\sigma_D$$

Note that we integrate only to $\pi/2$, and not π, since this is the cross section for either ball to be scattered at angle θ, so to integrate all the way to π would double-count.