

THE STATIC QUARK MODEL (1964- Gell-Mann, Ne'eman)

Symmetry Groups and Their Representations

Recap: angular momentum + rotations in 3-space

$$J_z = -i(\vec{r} \times \vec{\sigma})_z = -i\left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right) \quad (1)$$

is QM angular momentum operator (plus cyclic permutations $\underbrace{x \rightarrow y \rightarrow z}$)

As we discussed, J_i generates rotations about the i^{th} axis

$$R_i(\theta) = \exp(-i\theta J_i)$$

From (1), it's easy to show that the angular momentum operators satisfy an algebra

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$

where $\epsilon_{ijk} = +1 (-1)$ if i, j, k are cyclic (anticyclic) permutations of x, y, z , and 0 otherwise.

$[\epsilon_{ijk}]$ known as structure constants of the (Lie) Group

• anticyclic = cyclic, but with one additional exchange of two indices

Additionally, the Casimir operator $J^2 = J_x^2 + J_y^2 + J_z^2$ commutes w/ all 3 angular momentum operators

$$[J^2, J_i] = 0 \quad i = x, y, z$$

and so can be simultaneously diagonalized w/ ons of the operators, say J_z . Thus, states are labelled according to total angular momentum j and its z projection M_j , since all physics invt. under rotation in real space. $\Rightarrow |j, M_j\rangle$

Recall from QM that it is exactly the algebra that permits us to define raising and lowering operators (via the relation $[J_z, J_{\pm}] = \pm J_{\pm}$)

$$J_{\pm} |j, M_j\rangle \quad J_{\pm} = J_x \pm i J_y$$

with the property

$$J_{\pm} |j, M_j\rangle = [j(j+1) - m(m \pm 1)]^{1/2} |j, M_j \pm 1\rangle \quad (\textcircled{A})$$

which, from any state $|j, M_j\rangle$ allows us to construct $2j+1$ states, $-j \leq M_j \leq j$, which all transform similarly under rotations, via the same $3(2j+1) \times (2j+1)$ rotation matrixes, as elucidated in homework problems 1.2 and 1.3 (We explicitly constructed ~~exactly~~ for the case of $j=1/2$ and $j=1$).

$$R_y = e^{-i\omega_j t}$$

(Qn-27)

This set of similarly transforming states is said to form an irreducible representation, of dimension $2j+1$, of the rotation group.

Note ALSO that the algebra of the rotation group is identical to that of the Pauli Spin Matrices

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2} \right] = i \epsilon_{ijk} \frac{\sigma_k}{2}$$

and so the ^{irreducible} representations of $SU(2)$ w/ integer spin are identical to those of $SO(3)$, but in addition, we gain half-integer representations.

Direct Products of Irreducible Representations

The algebra, via the relations (A), also tell us how to combine representations

$$\phi_1(j_1, m_1) \phi_2(j_2, m_2) = \sum_{\substack{j_1=j_2, \\ j_1+j_2 \\ m=m_1+m_2}} C_{j_1, m_1; j_2, m_2}^{jm} \psi(j, m)$$

↑ Clebsch-Gordan Coef

For example, for $j_1=1 \quad j_2=\frac{1}{2}$

$\phi_1(1, 1)$	$\phi_2(\frac{1}{2}, \frac{1}{2})$
$\phi_1(1, 0)$	$\phi_2(\frac{1}{2}, -\frac{1}{2})$
$\phi_1(1, -1)$	

$$\phi_1(1,1) \phi_2(\frac{1}{2},\frac{1}{2}) = \chi(3\frac{1}{2}, 3\frac{1}{2})$$

$$\phi_1(1,1) \phi_2(\frac{1}{2}, -\frac{1}{2}) = \frac{1}{\sqrt{3}} \chi(3\frac{1}{2}, \frac{1}{2}) + \frac{\sqrt{2}}{\sqrt{3}} \chi(\frac{1}{2}, \frac{1}{2})$$

$$\phi_1(1, -\frac{1}{2}) \phi_2(\frac{1}{2}, \frac{1}{2}) = \frac{\sqrt{1}}{\sqrt{3}} \chi(\frac{1}{2}, \frac{1}{2}) \mp \frac{\sqrt{2}}{\sqrt{3}} \chi(\frac{1}{2}, -\frac{1}{2})$$

$$\phi_1(1, -\frac{1}{2}) \phi_2(\frac{1}{2}, -\frac{1}{2}) = \chi(3\frac{1}{2}, -\frac{1}{2})$$

In other words, the product of the (3-d) $j=1$ and (2-d) $j=\frac{1}{2}$ representations is an ~~combi~~
additive combination (sum) of the (4-d) $j=\frac{3}{2}$
and (2-d) $j=\frac{1}{2}$ representations

In short-hand, we write the cardinality relation

$$3 \otimes 2 = 4 \oplus 2$$

$$\begin{matrix} j = & 1 & \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \end{matrix}$$

w/ weights (suppressed in this notation) given by Clebsch-Gordan coefficients, which exhibits the decomposition of the product representation into irreducible representations.

$SU(2)$

Finally, it's pretty obvious that any ~~rotation-group~~
representation can be obtained by multiplying enough
2-d ($j=\frac{1}{2}$) representations together. Thus, the $j=\frac{1}{2}$
representation is known as the fundamental
representation.

Another way to view it: Consider all states you can make ~~without~~ and out of combining

~~J=1~~ and $J_z = \frac{1}{2}$. Catalog them via a weight diagram - one "x" for every obtainable value of m_J .

$$m_{J_1} = j \quad m_{J_2} = \frac{1}{2} \quad m_J =$$

$$1 \quad \frac{1}{2} \quad \frac{3}{2}$$

$$-\frac{1}{2} \quad \frac{1}{2}$$

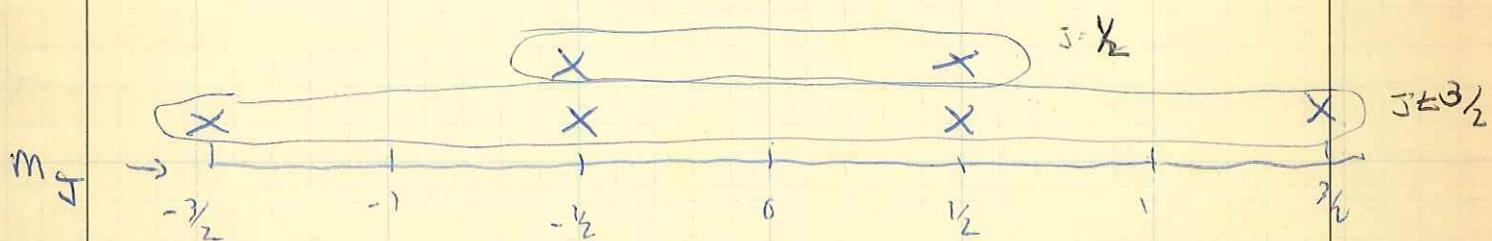
$$0 \quad \frac{1}{2} \quad \frac{1}{2}$$

$$-\frac{1}{2}$$

$$-1 \quad \frac{1}{2} \quad -\frac{1}{2}$$

$$-\frac{1}{2} \quad -\frac{3}{2}$$

Take such states of representation ① and "add" representation ② to it.



This is clearly the sum of the two irreducible rep's

$$\text{S.B.L} \quad J = \frac{1}{2}$$

PARTICLE CLASSIFICATION ACCORDING TO IRREDUCIBLE REPRESENTATIONS

If the strong force is invariant under rotations in isospin space [SU(2)], then particles formed from two members of the fundamental $I = \frac{1}{2}$ representation should fall into the product representation

$$2 \otimes 2 = 3 \oplus 1$$

$$\begin{matrix} J = & \frac{1}{2} & \frac{1}{2} & 1 & 0 \end{matrix}$$

i.e., for $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we expect 3 $I=1$ $q\bar{q}$ combos, and the 1's, and 1 $I=0$.

As mentioned, the latter was found 1961 A Pais et al

$$\pi^+ + d \rightarrow p + \pi^+ \pi^- \pi^0$$

$$m = 550 \text{ MeV}/c^2$$

w/ Dalitz analysis yielding $I(JP) = 0(0^-) \Rightarrow \eta$ candidate.
 [For pions $I(JP) = 1(0^-)$]
 η is $I=0$ partner of π^0 .

But, a little later (higher energy) on $\eta \pi^+ \pi^-$ at $M = 958 \text{ MeV}/c^2$ resonance w/ $I(JP) = 0(0^-)$ was found - what is this?
 It's the η' , but how does it fit in?

$$SU(3)_{\text{color}}$$

THE MOTIVATION FOR SU(3)

A look @ hadron spectrum, for a given J^P , did not provide a really satisfying description w/ $SU(2)$. In particular, consider the pseudoscalar mesons ($J^P = 0^-$), which were all known by ~ 1960

$$\pi^+ \rightarrow 0^+ \quad \pi^+ \quad S=1 \quad s=0 \quad \sim 135 \text{ MeV}$$

$$K^0 \quad K^+ \quad J=\frac{1}{2} \quad s=1 \quad \sim 495 \text{ MeV}$$

$$\bar{K}^0 \quad K^- \quad J=\frac{1}{2} \quad s=-1 \quad \sim 493 \text{ MeV}$$

$$\begin{matrix} \eta, \eta' \\ \nearrow \quad \searrow \\ \Sigma_B \quad M=958 \end{matrix} \quad J=0 \quad S=0$$

Isospin symmetry explained the near mass-equality of the multiplet members, and some interaction + decay properties, but you still have a zoo. On the other hand, in addition to I, I_3 , we see here the existence of another quantum number, strangeness, which is conserved by the strong interaction. This further label of strong eigenstates suggest that perhaps a larger symmetry group is at play, governing transformations not just w/in I multiplets ($I_3 \rightarrow I_3 \pm 1$), but also between multiplets of different strangeness ($s \rightarrow s \pm 1$). For the sake of development, we'll label this new quantum number "Y" (hypercharge), and start by seeing that Y is related to, but not equal to, s .

Note: w/ $SU(3)$ connecting all states,
 you would now expect π, η to have same mass if
 $SU(3)$ is unbroken

Finally, it should be remarked that we expect at the outset
that the symmetry is rather badly broken

$$m_\pi \sim 135 \text{ MeV/c}^2$$

$$m_K \sim 495 \text{ MeV/c}^2$$

~~Under $SU(3)$ symmetry~~

We'll look into this more carefully when we get to our discussion
of hadron masses; for now, we'll assume the symmetry
is unbroken, in order to derive our particle mass multiplets.

The most obvious extension of isospin ($SU(2)$) symmetry that
incorporates a third conserved quantum no. is also the
apparently correct one - $SU(3)$ symmetry.

Note that this is a static symmetry which merely establishes
relationships between strong interaction effects, but does not further
the cause of a dynamic theory (in the sense of QED) of the S.I.
This static symmetry is known as $SU(3)_{flavor}$ (u,d,s and
q, b flavors), as opposed to the dynamical symmetry
 $SU(3)_{color}$ associated w/ strong interaction dynamics. More
on this when we discuss gauge theories in general, and then
QCD in specific.

Dynamical: Int. strength, cross sections, etc., vs particle
spectrum.

WHAT IS SU(3)

Recall $SU(2)$ fundamental representation is a doublet $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

transforming as

$$\psi \rightarrow \psi' = U\psi$$

2.2

w/ U any unitary transformation of determinant $+1$. We found that in general

$$U = \exp(-\frac{1}{2}i(\vec{\Theta} \cdot \vec{\tau}))$$

for ordinary $\vec{\Theta} = (\theta_1, \theta_2, \theta_3)$, and $\vec{\tau}_i$ the Pauli spin matrices.

Note: $\vec{\tau}_i$ is not pauli spin matrices for non-fundamental rep.

Clearly, the fundamental representation of $SU(3)$ operates on a triplet

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \rightarrow U \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

In this case $U = \exp(-\frac{1}{2}i(\vec{\Theta} \cdot \vec{\tau}))$

where $\vec{\Theta}$ is an 8-dimensional vector parameter specifying the "rotation" in the eight different "directions" labelled by the generators λ_i $i=1, \dots, 8$

($SU(n)$ has $n^2 - 1$ free parameters, or generators)

$(Qm=3)$

Note: Strictly speaking, these are the generators of transformations of the fundamental representations. Note also that these look like

subscribing the commutation relations (algebra)

defn for the antifundamental

$$[\frac{1}{2}\lambda_i, \frac{1}{2}\lambda_j] = \sum_k i f_{ijk} \frac{1}{2}\lambda_k$$

$$\left\{ \begin{array}{l} f_{123} = 1 \\ f_{147} = f_{246} = f_{257} = f_{345} = f_{376} = f_{637} = \frac{1}{2} \\ f_{458} = f_{678} = \sqrt{3}/2 \\ \text{All others } 0! \end{array} \right. \quad \begin{array}{l} + \text{ antisymmetric in exchange} \\ \text{index of any two indices} \end{array}$$

A set of ^{Sp}_n matrices satisfying

A set of ^{true}_{spacetime} unitary matrices satisfying the relationship

(corresponding to a particularly useful choice of basis) is

$$I \quad \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$v \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$v \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Thinking in terms of 1 being generator of $U(1)$, or $SO(2)$

$$U = e^{i\theta} I$$

In the extension of $SU(2)$ to $SU(3)$ the basic doublet of $SU(2)$ is replaced by a triplet

$$\varphi \equiv \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$$

and this basic triplet is assumed to transform as

$$\varphi \rightarrow \varphi' = U\varphi \quad (10.1)$$

where the matrices U are arbitrary, unitary, unimodular 3×3 matrices, a canonical representation of which is

$$U \equiv \exp(-\frac{1}{2}i\theta\hat{n}\cdot\lambda). \quad (10.2)$$

The eight generators* $\frac{1}{2}\lambda_j$ play an analogous role to the three Pauli matrices in $SU(2)$ and the standard form, which was introduced by Gell-Mann,¹ is

$$\begin{aligned} \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{aligned} \quad (10.3)$$

* In $SU(n)$ there are $n^2 - 1$ generators of the group.

note

We note that $\lambda_1, \lambda_2, \lambda_3$ form an $SO(2)$ subgroup

$$\lambda_i = \begin{pmatrix} \tau_i & 0 \\ 0 & 0 \end{pmatrix} \quad i=1,3$$

which we associate w/ isospin without assigning

$$I_{\pm} = \frac{1}{2} (\lambda_1 \pm i\lambda_2)$$

$$I_3 = \frac{1}{2} \lambda_3$$

We also see that the elements λ_4, λ_5 and λ_6, λ_7 are associated w/ $SO(2)$ subgroups, although they are ~~not~~ ^{is} diagonal in this basis ^{of associated w/ these subgroups} (not simultaneously diagonalizable), and we define \Rightarrow no conserved quantity

$$V_{\pm} = \partial V \mp \frac{1}{2} (\lambda_4 \pm i\lambda_5)$$

$$V_3 = \frac{1}{2} (\lambda_6 \pm i\lambda_7)$$

The final generator is diagonal, providing a good quantum no, which we can associate w/ hypercharge Y

$$Y = \frac{1}{\sqrt{3}} \lambda_8$$

As f. $SO(2)$, they have mutually-commuting Casimir operator $I^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ that commute w/ all λ_i .

$\Rightarrow I, I_3, Y$ are mutually observable QM
that we can use to label states!

$(Qm=3^q)$

Representations of $SU(3)$

Finding the representations of $SU(3)$

For $SU(2)$, irreducible representations were classified by their dimension, characterized by j or I , and eigenstates within the representation by the single quantum number m_j or I_3 . The transformation properties of the multiplet were given by

$$\phi \rightarrow \phi' = \exp(-i\theta \hat{n} \cdot \vec{\tau}) \phi$$

$(2j+1)(2j+1) \times (2j+1)$

where T_1, T_2, T_3 are (y-axis) matrices

satisfying the $SU(2)$ algebra. (To work down T_2 for jets, similarly
 $(T_j \propto \sigma_j^z \text{ for } j=1)$)

actually labelled
 also by number,
 but j or I
 is redundant w/
 that is this case.

Similarly, the $SU(3)$ irreducible representations are labelled by dimension, and correspond to the set of states which mutually transform under the operation

$$\exp(-i\theta \cdot \vec{F}) \phi$$

not the same before 10, $\vec{F} = \vec{\lambda}$ for first 12 rep.

$$\phi \rightarrow \phi' = \exp(-i\theta \hat{n} \cdot \vec{F}) \phi$$

where F are 8×8 matrices satisfying the $SU(3)$ algebra. It is the job of mathematicians to find all such dimensions d for which a representation exists.

Ans

(Qm-40)

doesn't this in fact commute? I think it does $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$!
 I think it does commute, and in a perfect world, would be used to label representations.

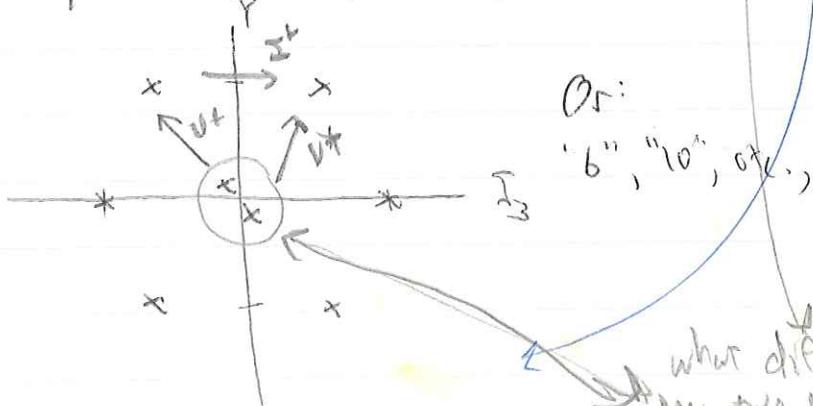
To label states within a representation, note that two generators, F_3 and F_8 ($I_3 + Y$) are diagonal. A third linearly independent operator $I^2 = F_1^2 + F_2^2 + F_3^2$ can be constructed. [although

~~in this case it does not commute w/ all generators, and so is not a good candidate for labeling states (necessary in fact).~~

in this case it does not commute w/ all generators, and so is not a good candidate for labeling states (necessary in fact). This, states within a representation are labelled by their values of I, I_3, Y .
 (the casimir op. of $SO(3)$)

Typically, a representation is depicted by a two dimensional weight diagram in I_3 and Y , w/ the I -degeneracy at each point specified by a number

"8" =



General rule (not for us to prove): for irreducible reps:

Degeneracy at each (I_3, Y) point increases by 1 w/ every step inside the boundary, until you touch a triangular boundary, each point of which has an identical weight.

(as you move in:



Qn 2-4)

See B&J,
p 329; I

is good for
labeling
states (necessary
in fact).

Note: thus
no casimir operator
labeling of the

tops as they
are for $SO(2)$,
which can't
be labelled by "

U and V Spin

The $SU(3)$ generators λ_8, λ_5 and λ_6, λ_7 are associated w/ two other $SU(2)$ subgroups of $SU(3)$, inspiring the assignments

$$\left. \begin{array}{l} U_{\pm} = \frac{1}{2} (\lambda_6 \pm i \lambda_7) \\ V_{\pm} = \frac{1}{2} (\lambda_8 \pm i \lambda_5) \end{array} \right\} \begin{array}{l} \text{all one needs to find raising} \\ \text{lowering op's of fm} \\ U_{\pm} |U, m_a\rangle = \sqrt{U(U+1)-m(m\pm 1)} |U, m\pm 1\rangle \end{array}$$

These are not simultaneously diagonalizable with χ, I_3 . In fact,

$$[\chi, U_{\pm}] = \pm U_{\pm} \quad [I_3, V_{\pm}] = \pm 1/2 V_{\pm}$$

$$[\chi, V_{\pm}] = \pm V_{\pm} \quad [I_3, U_{\pm}] = \mp \frac{1}{2} U_{\pm}$$

From these relations, it can be shown

<u>Operator</u>	<u>$\Delta \chi$</u>	<u>ΔI_3</u>	<u>ΔU_{\pm}</u>	<u>ΔV_{\pm}</u>
I_{\pm}	0	± 1	0	$\pm \frac{1}{2}$
U_{\pm}	± 1	$\mp \frac{1}{2}$	± 1	$\pm \frac{1}{2}$
V_{\pm}	$\pm i$	$\pm \frac{1}{2}$	$\mp \frac{1}{2}$	$\pm i$

These op's move you around w/in a rep. as I_{\pm} did for $SU(2)$.

(Qm-42)

The Fundamental Representation

Study explicitly, since all reps can be built from this one.

Recall our fundamental representation eigenstates

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\phi_1 \quad \phi_2 \quad \phi_3$

for which the generators are just the 3×3 λ -matrices.

Operating w/

$$I_3 = \frac{1}{2} \lambda_3 \quad \text{and} \quad Y = \frac{1}{\sqrt{3}} \lambda_8 = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}$$

= $\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$

hit-sight

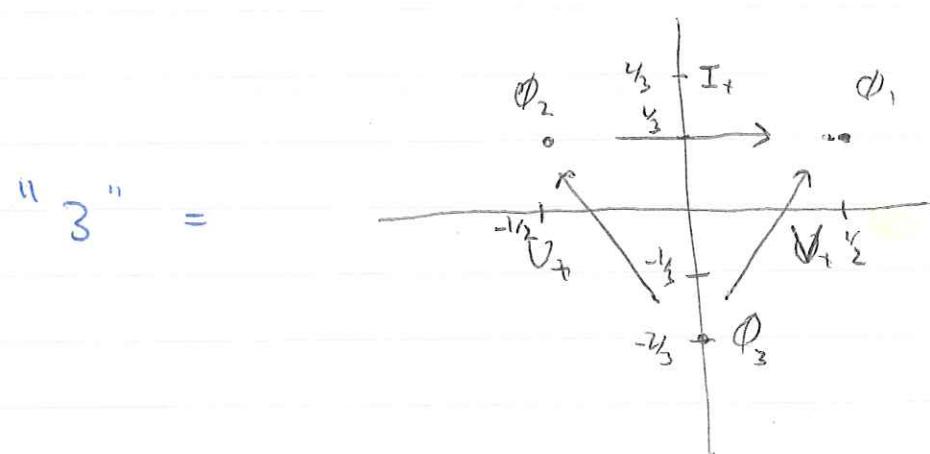
Convenience factors: $\frac{1}{\sqrt{3}}$ gets Y as close as possible to stringy "S".

We see that

$$\phi_1 : |I_3, Y\rangle = \left(\frac{1}{2}, \frac{1}{3}\right) \quad \text{connected by } I_3, Y, V$$

$$\phi_2 : \quad = \left(-\frac{1}{2}, \frac{1}{3}\right) \quad \text{spin operators}$$

$$\phi_3 : \quad = \left(0, -\frac{2}{3}\right)$$



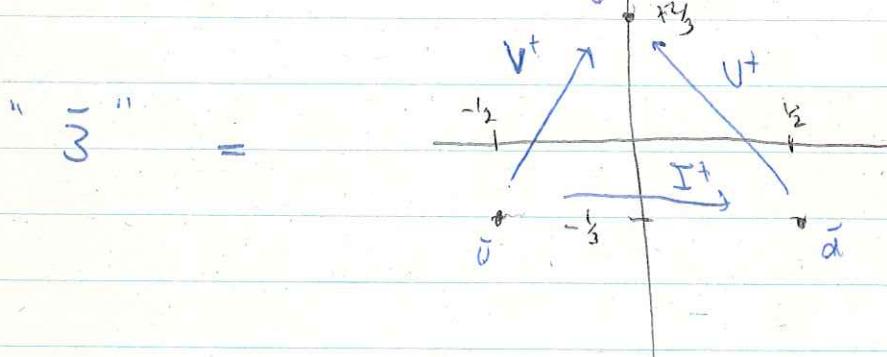
LEAVE ON
BOARD

(in progress)

The Charge Conjugate Representation

The property of charge conjugation, which takes particles to antiparticles, clearly takes $\bar{s}_3 \rightarrow -s_3$.

Also since $\psi \rightarrow \overline{\psi}$, that γ^5 also plays a role in charge or "antifundamental" assignment, $\gamma^5 \rightarrow -\gamma^5$. Thus, the charge conjugate (anti-matter) fundamental representation is given by $\bar{C} = \text{op. swap sign of all intrinsic numbers.}$



LEAVE ON
BOARD

Product of Representations

The fundamental rep. represents new fermi systems (as np did for isospin) \Rightarrow we must look at higher dimensional Rep's. On the other hand, we know that all these can be obtained by multiplying together fundmtl representations. This is how we'll build up the meson + baryon multiplets.

(Qm-4u)