Nuclear and Particle Physics

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H Evaluation of some rotation matrix elements $d^{(j)}_{m'm}$

An angular momentum state $|jm\rangle$ is transformed under a finite rotation θ about the y or 2 axis into a linear combination of the $2j+1$ states $|jm'\rangle$ with $m' = j, j-1, \ldots, -j+1, -j$. The transformation is conventionally written

$$\exp(-\mathrm{i}\theta J_2)|jm\rangle = \sum_{m'} d^{(j)}_{m'm}(\theta)|jm'\rangle.$$

Then,

$$d^{(j)}_{m'm} = \langle jm'|\exp(-\mathrm{i}\theta J_2)|jm\rangle.$$

The coefficients $d^{(j)}_{m'm}$ are the (m', m)th elements of the matrix $\exp(-\mathrm{i}\theta J_2)$, where J_2 is the operator for the 2-component of angular momentum.

Case 1: $j = \frac{1}{2}$

Written in terms of the Pauli spin matrix,

$$\sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

we have

$$\exp(-\mathrm{i}\theta J_2) = \exp\left[-\mathrm{i}\left(\frac{\theta}{2}\right)\sigma_2\right] = 1 - \mathrm{i}\left(\frac{\theta}{2}\right)\sigma_2 - \left(\frac{\theta}{2}\right)^2\frac{\sigma_2^2}{2!} + \mathrm{i}\left(\frac{\theta}{2}\right)^3\frac{\sigma_2^3}{3!}$$
$$+ \left(\frac{\theta}{2}\right)^4\frac{\sigma_2^4}{4!} - \mathrm{i}\left(\frac{\theta}{2}\right)^5\frac{\sigma_2^5}{5!} - \left(\frac{\theta}{2}\right)^6\frac{\sigma_2^6}{6!} + \cdots.$$

Now $\sigma_2^2 = I$, the unit matrix and, since

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

we have

$$\exp\left[-\mathrm{i}\left(\frac{\theta}{2}\right)\sigma_2\right] = I\cos\left(\frac{\theta}{2}\right) - \mathrm{i}\sigma_2\sin\left(\frac{\theta}{2}\right)$$

$$= \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}.$$

Hence,

$$d^{(\frac{1}{2})}_{-\frac{1}{2}\frac{1}{2}} = -d^{(\frac{1}{2})}_{\frac{1}{2}-\frac{1}{2}} = \sin\left(\frac{\theta}{2}\right)$$

$$d^{(\frac{1}{2})}_{\frac{1}{2}\frac{1}{2}} = d^{(\frac{1}{2})}_{-\frac{1}{2}-\frac{1}{2}} = \cos\left(\frac{\theta}{2}\right).$$

Case 2: $j = 1$

From equation (8.21) with $j = 1$ we have

$$J_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \qquad J_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

and

$$J_2^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

It is then a simple matter to show that $J_2^{2n+1} = J_2$ and $J_2^{2n} = J_2^2$ $(n = 1, 2, \ldots)$.

On expanding the exponential we have

$$\begin{aligned}
\exp(-\mathrm{i}\theta J_2) &= 1 - \mathrm{i}\theta J_2 - \frac{\theta^2}{2!}J_2^2 + \mathrm{i}\frac{\theta^3}{3!}J_2^3 + \frac{\theta^4}{4!}J_2^4 - \mathrm{i}\frac{\theta^5}{5!}J_2^5 - \frac{\theta^6}{6!}J_2^6 + \cdots \\
&= 1 - \mathrm{i}J_2\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) - J_2^2\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right) \\
&= 1 - \mathrm{i}J_2 \sin\theta - J_2^2(1 - \cos\theta) \\
&= \begin{pmatrix} \frac{1}{2}(1+\cos\theta) & -\frac{\sin\theta}{\sqrt{2}} & \frac{1}{2}(1-\cos\theta) \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ \frac{1}{2}(1-\cos\theta) & \frac{\sin\theta}{\sqrt{2}} & \frac{1}{2}(1+\cos\theta) \end{pmatrix}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&d^{(1)}_{01} = -d^{(1)}_{10} = -d^{(1)}_{0-1} = d^{(1)}_{-10} = \frac{\sin\theta}{\sqrt{2}} \\
&d^{(1)}_{11} = d^{(1)}_{-1-1} = \tfrac{1}{2}(1 + \cos\theta) \\
&d^{(1)}_{-11} = d^{(1)}_{1-1} = \tfrac{1}{2}(1 - \cos\theta) \\
&d^{(1)}_{00} = \cos\theta.
\end{aligned}$$