

Nuclear and Particle Physics

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H Evaluation of some rotation matrix elements $d_{m'm}^{(j)}$

An angular momentum state $|jm\rangle$ is transformed under a finite rotation θ about the y or 2 axis into a linear combination of the $2j + 1$ states $|jm'\rangle$ with $m' = j, j - 1, \dots, -j + 1, -j$. The transformation is conventionally written

$$\exp(-i\theta J_2)|jm\rangle = \sum_{m'} d_{m'm}^{(j)}(\theta)|jm'\rangle.$$

Then,

$$d_{m'm}^{(j)} = \langle jm'|\exp(-i\theta J_2)|jm\rangle.$$

The coefficients $d_{m'm}^{(j)}$ are the (m', m) th elements of the matrix $\exp(-i\theta J_2)$, where J_2 is the operator for the 2-component of angular momentum.

Case 1: $j = \frac{1}{2}$

Written in terms of the Pauli spin matrix,

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

we have

$$\begin{aligned} \exp(-i\theta J_2) = \exp\left[-i\left(\frac{\theta}{2}\right)\sigma_2\right] &= 1 - i\left(\frac{\theta}{2}\right)\sigma_2 - \left(\frac{\theta}{2}\right)^2 \frac{\sigma_2^2}{2!} + i\left(\frac{\theta}{2}\right)^3 \frac{\sigma_2^3}{3!} \\ &+ \left(\frac{\theta}{2}\right)^4 \frac{\sigma_2^4}{4!} - i\left(\frac{\theta}{2}\right)^5 \frac{\sigma_2^5}{5!} - \left(\frac{\theta}{2}\right)^6 \frac{\sigma_2^6}{6!} + \dots \end{aligned}$$

Now $\sigma_2^2 = I$, the unit matrix and, since

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

we have

$$\begin{aligned} \exp\left[-i\left(\frac{\theta}{2}\right)\sigma_2\right] &= I \cos\left(\frac{\theta}{2}\right) - i\sigma_2 \sin\left(\frac{\theta}{2}\right) \\ &= \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}. \end{aligned}$$

Hence,

$$d_{\frac{1}{2}\frac{1}{2}}^{(\pm)} = -d_{\frac{1}{2}-\frac{1}{2}}^{(\pm)} = \sin\left(\frac{\theta}{2}\right)$$

$$d_{\frac{3}{2}\frac{3}{2}}^{(\pm)} = d_{-\frac{3}{2}-\frac{3}{2}}^{(\pm)} = \cos\left(\frac{\theta}{2}\right).$$

Case 2: $j = 1$

From equation (8.21) with $j = 1$ we have

$$J_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \quad J_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

and

$$J_2^2 = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

It is then a simple matter to show that $J_2^{2n+1} = J_2$ and $J_2^{2n} = J_2^2$ ($n = 1, 2, \dots$).

On expanding the exponential we have

$$\begin{aligned}
 \exp(-i\theta J_2) &= 1 - i\theta J_2 - \frac{\theta^2}{2!} J_2^2 + i \frac{\theta^3}{3!} J_2^3 + \frac{\theta^4}{4!} J_2^4 - i \frac{\theta^5}{5!} J_2^5 - \frac{\theta^6}{6!} J_2^6 + \dots \\
 &= 1 - iJ_2 \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \right) - J_2^2 \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \dots \right) \\
 &= 1 - iJ_2 \sin \theta - J_2^2 (1 - \cos \theta) \\
 &= \begin{pmatrix} \frac{1}{2}(1 + \cos \theta) & -\frac{\sin \theta}{\sqrt{2}} & \frac{1}{2}(1 - \cos \theta) \\ \frac{\sin \theta}{\sqrt{2}} & \cos \theta & -\frac{\sin \theta}{\sqrt{2}} \\ \frac{1}{2}(1 - \cos \theta) & \frac{\sin \theta}{\sqrt{2}} & \frac{1}{2}(1 + \cos \theta) \end{pmatrix}
 \end{aligned}$$

Hence,

$$d_{01}^{(1)} = -d_{10}^{(1)} = -d_{0-1}^{(1)} = d_{-10}^{(1)} = \frac{\sin \theta}{\sqrt{2}}$$

$$d_{11}^{(1)} = d_{-1-1}^{(1)} = \frac{1}{2}(1 + \cos \theta)$$

$$d_{-11}^{(1)} = d_{1-1}^{(1)} = \frac{1}{2}(1 - \cos \theta)$$

$$d_{00}^{(1)} = \cos \theta.$$