Nuclear and Particle Physics

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10 The quark model

SU(3) · Quarks · Mesons in the quark model · Baryons in the quark model · Hadron masses in the quark model · Baryon magnetic moments · Heavy-meson spectroscopy

As we saw in chapter 8 the isospin symmetry of the strong interactions leads naturally to the observation that all hadrons belong to I spin multiplets. In the absence of symmetry-breaking effects there is a degeneracy in the mass of the members of a multiplet. The electromagnetic interaction does not respect isospin symmetry and through it the degeneracy is removed with resulting mass differences of a per cent or so. The isospin operators commute with the strong interaction Hamiltonian H and therefore with other operators which also commute with H, in particular the angular momentum and parity operators. Consequently, all members of an isospin multiplet have the same spin-parity.

In the last chapter we discovered that there are larger groups of mesons and baryons with the same spin-parity, these groups containing I spin multiplets with different values of the strangeness quantum number. The observed regularities in the hadron spectrum led to the search for a higher symmetry, i.e. a symmetry higher than the SU(2) group of which isospin is an example, which would explain the existence of these larger groups of particles. A higher symmetry will involve a further additive quantum number, in addition to I and I_3, which is conserved in the strong interations but not necessarily in the weak and electromagnetic interactions. The strangeness S is such a quantum number but the hypercharge* Y, the sum of strangeness and baryon number, is found to be more convenient. The appropriate mathematical group is then SU(3).

* The name hypercharge arises because it is twice the average charge of an I spin multiplet, as can be seen from the Gell-Mann–Nishijima relation $Q = I_3 + Y/2$.

In this chapter we show how, in spite of the fact that the symmetry is quite badly broken, SU(3) and the associated quark model give a good description of the observed hadron spectrum.

10.1 SU(3)

10.1.1 The SU(3) generators

In the extension of SU(2) to SU(3) the basic doublet of SU(2) is replaced by a triplet

$$\varphi \equiv \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix}$$

and this basic triplet is assumed to transform as

$$\varphi \to \varphi' = U\varphi \tag{10.1}$$

where the matrices U are arbitrary, unitary, unimodular 3×3 matrices, a canonical representation of which is

$$U \equiv \exp(-\tfrac{1}{2}\mathrm{i}\theta\hat{\boldsymbol{n}}\cdot\boldsymbol{\lambda}). \tag{10.2}$$

The eight generators* $\frac{1}{2}\lambda_j$ play an analogous role to the three Pauli matrices in SU(2) and the standard form, which was introduced by Gell-Mann,[1] is

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{10.3}$$

* In SU(n) there are $n^2 - 1$ generators of the group.

Table 10.1
The structure constants of SU(3)

$f_{123} = 1$
$f_{147} = f_{246} = f_{257} = f_{345} = f_{516} = f_{637} = \frac{1}{2}$
$f_{458} = f_{678} = \sqrt{3}/2$

As in SU(2), the generators satisfy the commutation relations

$$[\tfrac{1}{2}\lambda_i, \tfrac{1}{2}\lambda_j] = \mathrm{i} f_{ijk} \tfrac{1}{2}\lambda_k \tag{10.4}$$

where the structure constants f_{ijk}, which are easily obtained by explicit calculation, have the values given in table 10.1. The f_{ijk} are antisymmetric under the interchange of any two indices. These matrices form a *three-dimensional* representation of SU(3). Among the eight generators of the SU(3) group we note that only λ_3 and λ_8 are diagonal. In SU(2) the states in a given I spin multiplet were labelled with the eigenvalues of the diagonal matrix $\frac{1}{2}\tau_3$. In a search for a higher dimensional representation of the hadronic states it is appropriate to label the states by quantum numbers associated with the eigenvalues of λ_3 and λ_8. Apart from an extra row and column of zeros the matrices λ_i, $i = 1, 2, 3$, are just the Pauli matrices, i.e.

$$\lambda_i = \left(\begin{array}{cc:c} \multicolumn{2}{c:}{\multirow{2}{*}{τ_i}} & 0 \\ & & 0 \\ \hdashline 0 & 0 & 0 \end{array}\right) \qquad (i = 1, 2, 3)$$

and therefore λ_1, λ_2 and λ_3 are associated with the I spin operators which form an SU(2) subgroup of SU(3): in particular λ_3 is associated with I_3 and linear combinations of λ_1 and λ_2 are formed to produce I spin step operators. The eigenvalues of λ_8 are related to the hypercharge Y. In SU(3) the states in a multiplet are labelled with the eigenvalues of I_3 and Y. Accordingly, the states in SU(3) multiplets will occupy sites on a two-dimensional grid in I_3–Y space, in contrast to the one-dimensional SU(2) multiplets.

As in the case of SU(2) we generalize (10.4) by defining the generators as $F_i \equiv \frac{1}{2}\lambda_i$; the F_i will then satisfy the commutation relations

$$[F_i, F_j] = \mathrm{i} f_{ijk} F_k. \tag{10.5}$$

The study of SU(3) amounts essentially to finding higher dimensional $N \times N$ matrices F_i which satisfy (10.5) and which transform N-dimensional states according to

$$\varphi \to \varphi' = (1 - \mathrm{i}\theta\hat{\boldsymbol{n}}\cdot\boldsymbol{F})\varphi. \tag{10.6}$$

These states belong to N-dimensional multiplets of SU(3).

To facilitate the search for these multiplets we form linear combinations of the non-commuting operators to produces step operators. Specifically, we define

$$\left.\begin{aligned} I_\pm &= F_1 \pm \mathrm{i}F_2 \\ I_3 &= F_3 \\ V_\pm &= F_4 \pm \mathrm{i}F_5 \\ U_\pm &= F_6 \pm \mathrm{i}F_7 \\ Y &= \frac{2}{\sqrt{3}} F_8 . \end{aligned}\right\} \qquad (10.7)$$

10.1.2 The representations of SU(3)

The operators $U_\pm$ and $V_\pm$ are called 'U spin' and 'V spin' operators. We shall have particular use for the U spin operators later in this chapter.

We begin with the fundamental triplet. An examination of the matrix representations of the commuting operators

$$I_3 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}$$

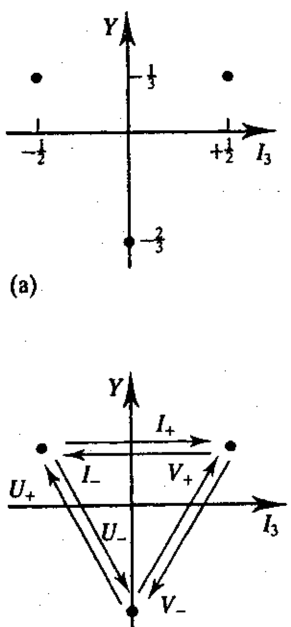

Figure 10.1
(a) The fundamental triplet of SU(3); (b) the action of the step operators $I_\pm$, $U_\pm$ and $V_\pm$ on the fundamental triplet.

shows that the fundamental triplet is described by the three eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

These are simultaneously eigenstates of I_3 and Y with eigenvalues (I_3, Y) of $(\frac{1}{2}, \frac{1}{3})$, $(-\frac{1}{2}, \frac{1}{3})$ and $(0, -\frac{2}{3})$, i.e. the fundamental triplet consists of an I spin doublet with $Y = \frac{1}{3}$ and an I spin singlet with $Y = -\frac{2}{3}$. The weight diagram – a plot of Y against I_3 – for the fundamental triplet is shown in figure 10.1(a).

In order to proceed further we require the commutation relations between the step operators defined in (10.7) and I_3 and Y. They are

$$[I_3, I_\pm] = \pm I_\pm \qquad [I_3, U_\pm] = \mp\tfrac{1}{2}U_\pm \qquad [I_3, V_\pm] = \pm\tfrac{1}{2}V_\pm$$

and (10.8)

$$[Y, I_\pm] = 0 \qquad [Y, U_\pm] = \pm U_\pm \qquad [Y, V_\pm] = \pm V_\pm .$$

We know from our study of I spin that the state $I_+|I, I_3\rangle$, for example, is an eigenstate of I^2 and I_3 with eigenvalue $I_3' = I_3 + 1$, i.e. I_+ is the raising operator for I spin. From the commutation relation $[I_3, U_\pm] = \mp\frac{1}{2}U_\pm$ we have

$$\begin{aligned} I_3U_\pm|I, I_3\rangle &= (U_\pm I_3 \mp \tfrac{1}{2}U_\pm)|I, I_3\rangle \\ &= U_\pm(I_3 \mp \tfrac{1}{2})|I, I_3\rangle \\ &= (I_3 \mp \tfrac{1}{2})U_\pm|I, I_3\rangle \end{aligned}$$

and therefore the operator $U_+(U_-)$ lowers (raises) the eigenvalue of I_3 by $\frac{1}{2}$. In a similar fashion, using $[Y, U_\pm] = \pm U_\pm$, it can be shown that $U_+(U_-)$ raises (lowers) the eigenvalue of Y by one. The overall effect of the commutation relations (10.8) can be summarized by the statements

$I_\pm$ induces the changes $\Delta Y = 0$, $\Delta I_3 = \pm 1$

$U_\pm$ induces the changes $\Delta Y = \pm 1$, $\Delta I_3 = \mp\frac{1}{2}$

$V_\pm$ induces the changes $\Delta Y = \pm 1$, $\Delta I_3 = \pm\frac{1}{2}$

and the action of these step operators on the fundamental triplet is shown in figure 10.1(b).

A particular representation, or multiplet, of SU(3) is completely specified when it is known which sites in the Y–I_3 plane are occupied and the multiplicity at each site is known, i.e. the number of states with a particular weight. A weight diagram contains precisely this information. As in SU(2) all states in an irreducible representation can be generated from a particular state by repeated application of the step operators. Before discussing the actual multiplets which occur in SU(3) we list some of their general properties.[2]

(a) The multiplets have hexagonal symmetry so that in general the boundary is a six-sided non-reentrant figure; in some cases it may be triangular.
(b) Every possible site on and inside the boundary is occupied by at least one state.
(c) The multiplicity of weights on layer 1, the boundary (see figure 10.2), is one, that on layer 2 is two in general, and so on until all sites have been accounted for. If a triangular layer is reached the multiplicity ceases to increase thereafter so that all sites on and inside a triangular layer have the same multiplicity.

It is convenient to have a shorthand notation which will succinctly describe the character of a particular SU(3) representation. If, starting from the state with the highest weight (maximum I_3 value), p applications of V_- (U_+ and I_+) and q applications of I_- (V_+ and U_-) are required to generate the boundary the multiplet is simply denoted as (p, q). Thus,

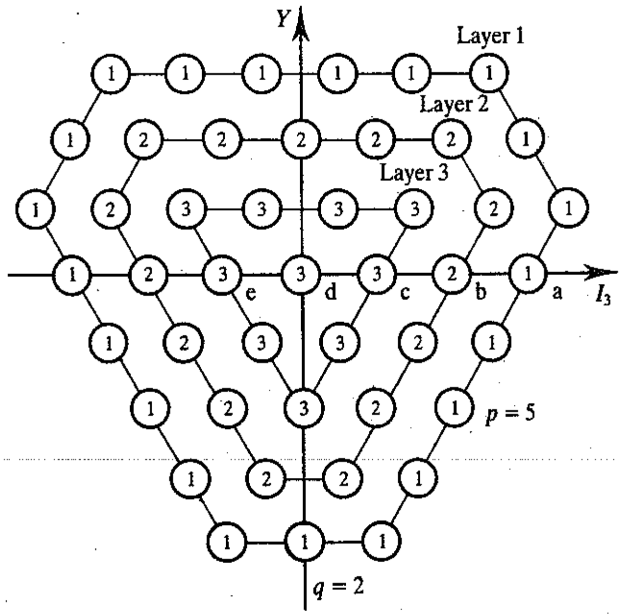

Figure 10.2
Weight diagram of a general SU(3) multiplet showing the multiplicity at each site.

for example, the fundamental triplet shown in figure 10.1 is denoted by (1, 0) and the multiplet in figure 10.2 by (5, 2). It can be shown[2] that the total number of states or the dimensionality of a multiplet $n(p, q)$ is

$$n(p, q) = \tfrac{1}{2}(1 + p)(1 + q)(2 + p + q)$$

so that the dimensionality of the (5, 2) representation is **81**. Sometimes multiplets are labelled simply by their dimensionality, e.g. **3** for the fundamental triplet and **81** for the (5, 2) representation.

It is clear that in the irreducible representations of SU(3) there can, in general, be a degeneracy of states with a particular weight (I_3, Y) so that an additional quantum number is required to distinguish between them. To be a good quantum number it must commute with both I_3 and Y. The square of the total I spin, $I^2 = \tfrac{1}{2}(I_+I_- + I_-I_+) + I_3^2$, has this property so that its eigenvalues can be used in conjunction with I_3 and Y to specify the states uniquely. Effectively, the degeneracy arises because for a given weight (I_3, Y) states from different I spin multiplets can contribute to the multiplicity of the weight. By way of illustration, consider the states with $Y = 0$ in figure 10.2, i.e. the states that sit on the I_3 axis. In essence these states form a *reducible* I spin multiplet. Seven of these states belong to an $I = 3$ multiplet, five to a multiplet with $I = 2$ and the remainder form an I spin triplet.

Before we attempt to combine multiplets in SU(3) we would like to make a remark concerning the fundamental representation. Apart from the SU(3) singlet (0, 0), the fundamental representation (1, 0) with dimensionality **3** is the simplest. Unlike the situation in SU(2), where the fundamental nucleon doublet $\binom{p}{n}$ with dimensionality **2** and its conjugate

$\bar{2}$, $\left(\begin{smallmatrix}-\bar{n}\\ \bar{p}\end{smallmatrix}\right)$, transform in the same way (see example 10.1), the conjugate representation $\bar{\mathbf{3}}$ in SU(3), denoted by (0, 1), does not transform in the same way as the **3** representation. The weight diagrams of these *inequivalent* representations of SU(3) are shown in figure 10.3.

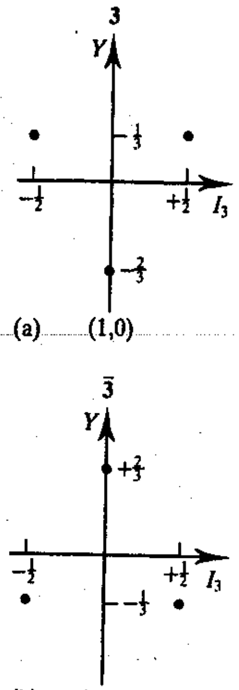

Figure 10.3
The fundamental triplet **3** and the conjugate triplet $\bar{\mathbf{3}}$ in SU(3).

10.1.3 Products of representations

The simplest way to obtain products of representations in SU(3) is to use a graphical technique. Any required reduction can be carried out by making use of the general properties of irreducible representations introduced in the last section. To illustrate the procedure we consider the product $\mathbf{3} \otimes \bar{\mathbf{3}}$. A superposition of the weight diagram of figure 10.3(b) on each weight in figure 10.3(a) yields the result shown on the left in figure 10.4. We note that the weights on the boundary of the product representation have unit multiplicity in accord with the properties of irreducible representations. The multiplicity at the centre of the representation, layer 2 in the notation of the last section, is, however, three and for an irreducible representation it ought to be two. To conform with the properties of irreducible representations the nonet must reduce to an octet and a singlet as shown in figure 10.4. The octet (1, 1) consists of two I spin doublets with $Y = +1$ and $Y = -1$ and an I spin triplet and a singlet both with $Y = 0$: the SU(3) singlet is also an I spin singlet with $Y = 0$. In the notation of group theory the product is written

$$\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}. \tag{10.9}$$

As a second example we derive the result $\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}$. In this case the superposition technique yields the nonet of states with the configuration shown on the left in figure 10.5. In this case there are three sites on the boundary which are doubly occupied. Again, to conform with the requirement of unit multiplicity on the boundary the nonet must reduce to a sextet (2, 0) and the conjugate triplet (0, 1).

Finally, we consider the product of three SU(3) triplets, $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$. We

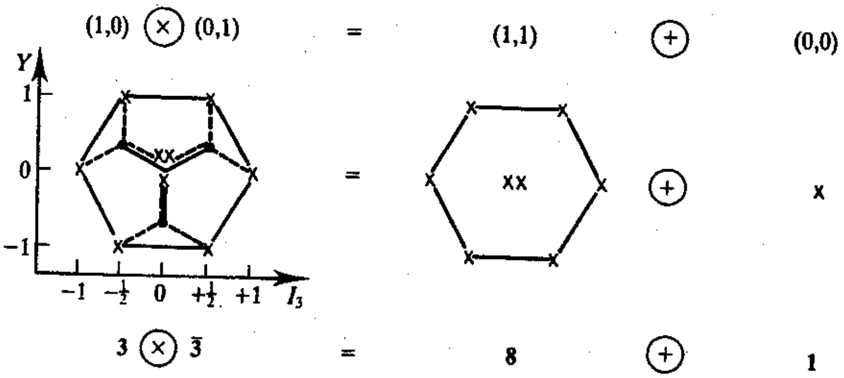

Figure 10.4
Graphical technique illustrating the reduction of the product $\mathbf{3} \otimes \bar{\mathbf{3}}$.

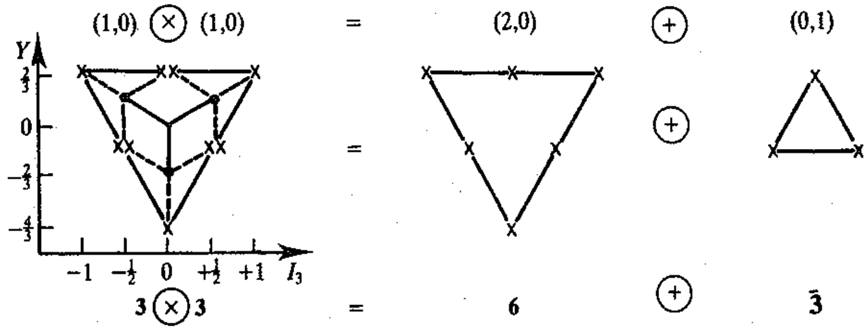

Figure 10.5
Reduction of the product $3 \otimes 3$.

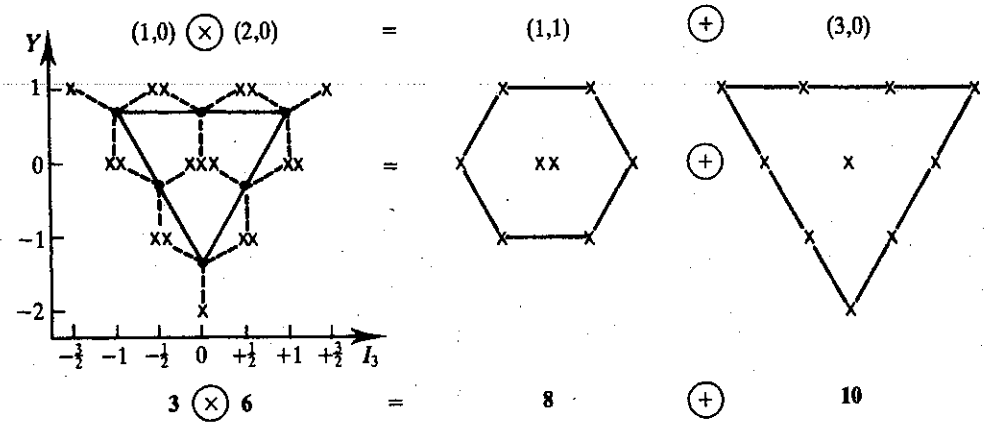

Figure 10.6
Reduction of the product $3 \otimes 6$.

have, from the last example,

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{3} \otimes (\mathbf{6} \oplus \bar{\mathbf{3}}) = \mathbf{3} \otimes \mathbf{6} \oplus \mathbf{3} \otimes \bar{\mathbf{3}}. \tag{10.10}$$

We already know that $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$, therefore we consider the product $\mathbf{3} \otimes \mathbf{6}$. Again, the superposition technique yields the weight diagram on the left of figure 10.6 which is clearly reducible. The reduction is effected by removing one of the states from each site on the boundary with multiplicity two. By the rule that all inner sites of an irreducible representation must be occupied, and that successive inner layers have a multiplicity which increases by one, we see that we must also remove two of the states from the site which has a multiplicity three. This procedure produces an octet (1, 1) and a decuplet (3, 0) both of which are irreducible. Thus $\mathbf{3} \otimes \mathbf{6} = \mathbf{8} \oplus \mathbf{10}$ and, finally,

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10}. \tag{10.11}$$

We note that the decuplet consists of an $I = \frac{3}{2}$ quartet with $Y = 1$, a triplet with $I = 1$ and $Y = 0$, a $Y = -1$ I spin doublet and an I spin singlet with $Y = -2$.

10.2 Quarks

The simplest and most elegant SU(3) scheme which successfully describes the observed hadron spectrum is the quark model proposed by Gell-Mann[3] and independently by Zweig.[4] They introduced a triplet of quarks with baryon number $B = \frac{1}{3}$ and proposed that the fundamental SU(3) triplet consists of a doublet of strangeness 0 quarks with $I = \frac{1}{2}$, the *up* quark u ($I_3 = +\frac{1}{2}$) and the *down* quark d ($I_3 = -\frac{1}{2}$), and an I spin singlet, the *strange* quark s with strangeness -1. The conjugate triplet $\bar{\mathbf{3}}$ consists of antiquarks with opposite sign of the additive quantum numbers I_3, B and S. The quantum numbers of the quarks and antiquarks are summarized in table 10.2. Note in particular the fractional charges $+\frac{2}{3}$ and $-\frac{1}{3}$ of the quarks. Furthermore, the eigenvalues of I_3 and Y of the quark triplet are precisely those of the diagonal generators given in section 10.1.2.

The weight diagrams for the fundamental quark and antiquark triplets are shown in figure 10.7.

10.3 Mesons in the quark model

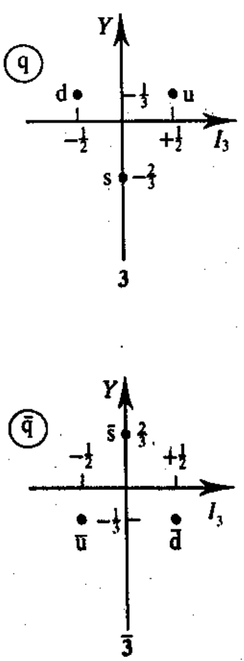

Figure 10.7
Weight diagrams for the fundamental quark and antiquark triplets.

The most economical way to construct mesons in the quark model is to form $q\bar{q}$ combinations by taking the direct product of the fundamental $\mathbf{3}$ and $\bar{\mathbf{3}}$ representations. The nonet of states is obtained in the usual way by superimposing the antiquark weight diagram on each site of the quark weight diagram and, in accordance with the properties of irreducible representations, the nonet reduces to an octet and a singlet as shown in figure 10.8. The quark content of the states on the boundary of the octet is unambiguous and is indicated in the figure. The unitary singlet with $Y = 0$ and $I = 0$ contains all the quarks on an equal footing and the normalized singlet state is therefore

$$\{\mathbf{1}, |0, 0\rangle\} = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}). \tag{10.12}$$

The notation on the left-hand side of (10.12) is $\{n, |I, I_3\rangle\}$ where n is the dimensionality of the representation. Equation (10.12) is seen as a natural extension from SU(2) to SU(3) if we make the substitution $\mathrm{p} \to \mathrm{u}$, $\mathrm{n} \to \mathrm{d}$, $\bar{\mathrm{p}} \to \bar{\mathrm{u}}$ and $\bar{\mathrm{n}} \to \bar{\mathrm{d}}$ (see example 10.2). Of the two states at the centre of the octet, one belongs to an I spin triplet and the other is an I spin singlet: both have $I_3 = 0$. We can write down the quark wavefunction of the $I_3 = 0$ triplet state immediately,

$$\{\mathbf{8}, |1, 0\rangle\} = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}). \tag{10.13}$$

Table 10.2
Quantum numbers of the light quarks and antiquarks

Flavour	Spin	Charge	I_3	Baryon number	Strangeness	Hypercharge
u	$\frac{1}{2}$	$+\frac{2}{3}$	$+\frac{1}{2}$	$\frac{1}{3}$	0	$\frac{1}{3}$
d	$\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$\frac{1}{3}$	0	$\frac{1}{3}$
s	$\frac{1}{2}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	-1	$-\frac{2}{3}$
$\bar{u}$	$\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{1}{2}$	$-\frac{1}{3}$	0	$-\frac{1}{3}$
$\bar{d}$	$\frac{1}{2}$	$+\frac{1}{3}$	$+\frac{1}{2}$	$-\frac{1}{3}$	0	$-\frac{1}{3}$
$\bar{s}$	$\frac{1}{2}$	$+\frac{1}{3}$	0	$-\frac{1}{3}$	$+1$	$+\frac{2}{3}$

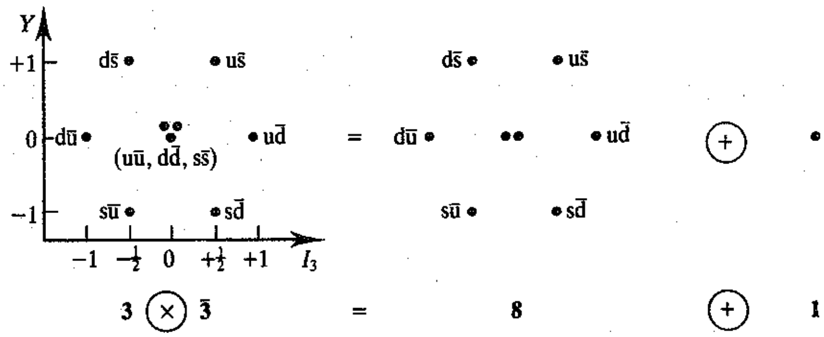

Figure 10.8
The nonet of $q\bar{q}$ states reduces to an octet and a singlet.

Alternatively, the state (10.13) may be obtained by using the I spin lowering operator on the $|1, 1\rangle$ state, i.e. $|-u\bar{d}\rangle$. We note that since the s and $\bar{s}$ quarks are I spin singlets they cannot couple to give an $I = 1$ state. They can, however, couple to give an $I = 0$ state so that the $I = 0$ state at the centre of the octet will be a linear combination of $u\bar{u}$, $d\bar{d}$ and $s\bar{s}$. The properly normalized state, which is orthogonal to both (10.12) and (10.13), is

$$\{\mathbf{8}, |0, 0\rangle\} = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}). \tag{10.14}$$

The quark model therefore predicts that mesons should belong to SU(3) octets and singlets. In the octets there are two I spin doublets with $Y = +1$ and $Y = -1$ which are particle and antiparticle, and an I spin triplet and an I spin singlet both with $Y = 0$. The unitary singlet is of course an I spin singlet. This is precisely the hypercharge $-I$ spin structure which is observed amongst the known mesons.

Let us now check that the other quantum numbers such as spin-parity and C-parity, where appropriate, agree with the experimentally observed values. Recall that the quarks are spin $\frac{1}{2}$ fermions so that in a $q\bar{q}$ state the spins may couple to give a total spin $S = 0$ or 1. If the quark–antiquark pair have relative orbital angular momentum L the total angular momentum will be the vector sum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$. The parity of the

Table 10.3
The possible J^P values of meson states arising from the coupling of quark–antiquark spin and relative orbital angular momentum

Orbital angular momentum L	Quark spins	
	Singlet $S = 0$	Triplet $S = 1$
0	0^-	1^-
1	1^+	$0^+\ 1^+\ 2^+$
2	2^-	$1^-\ 2^-\ 3^-$

states will be $(-1)^{L+1}$ where the factor $(-1)^L$ arises from the orbital motion and the factor -1 is due to the opposite intrinsic parities of quark and antiquark. The possible J^P values of the meson states formed from the coupling of quark–antiquark spin and orbital angular momentum are shown in table 10.3. The pseudoscalar and vector mesons appear as $q\bar{q}$ states with $L = 0$ and total quark spin 0 and 1 respectively, and the 2^+ tensor nonet results from the orbital excitation ($L = 1$) of the spin-triplet $q\bar{q}$ state. In figure 10.9 we show the SU(3) multiplets for the 0^-, 1^- and 2^+ mesons.

C-parity is a good quantum number only for states with $Q = B = S = 0$ where here S is the strangeness quantum number, and is therefore applicable only to the unitary singlets and to the two states at the centres of the octets. The argument which leads to the determination of C for

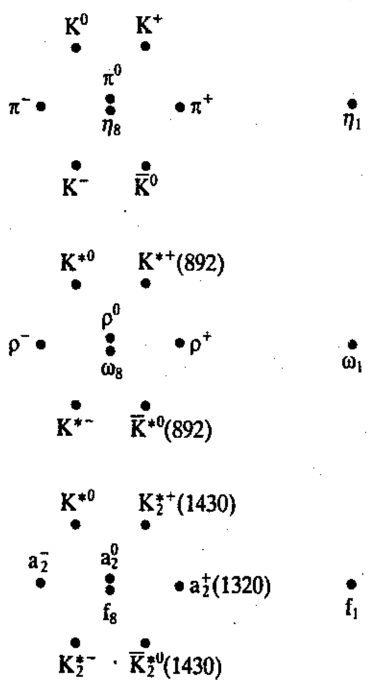

Figure 10.9 The $J^P = 0^-$, 1^- and 2^+ mesons arranged in SU(3) octets and singlets. The states shown as η_8, η_1, ω_8, ω_1 and f_8, f_1 are mixtures of the physical states η and $\eta'(958)$, $\omega(783)$ and $\phi(1020)$, and $f_2(1270)$ and $f_2'(1525)$. Mixing in the meson nonets is discussed in section 10.5.

Table 10.4
Quark model assignments of the pseudoscalar, vector and tensor mesons

$^{2S+1}L_J$	J^{PC}	$u\bar{u}, d\bar{d}, s\bar{s}$ $I = 0$	$u\bar{d}, u\bar{u}, d\bar{d}$ $I = 1$	$u\bar{s}, d\bar{s}$ $I = \frac{1}{2}$
$^1S_0$	0^{-+}	$\eta, \eta'(958)$	π	K
$^3S_1$	1^{--}	$\phi(1020), \omega(783)$	$\rho(770)$	$K^*(892)$
$^3P_2$	2^{++}	$f_2'(1525), f_2(1270)$	$a_2(1320)$	$K_2^*(1430)$

a $p\bar{p}$ state has already been given in section 8.8: for a $q\bar{q}$ state the argument is identical so that $C = (-1)^{L+S}$ where S is the total spin of the $q\bar{q}$ system and L the relative orbital angular momentum. We are thus led to the quark model assignments of the pseudoscalar, vector and tensor mesons shown in table 10.4.

10.4 Baryons in the quark model

The quark model description of baryons is more complicated than for mesons. All hadrons must be colour singlets and in section 10.3 it was tacitly assumed that in each $q\bar{q}$ pair the colour–anti-colour combinations yielded colour-singlet mesons. Since quarks have $B = \frac{1}{3}$ the simplest way to construct baryons from the basic quark triplet is to form qqq states. The quark *content* of these states is unambiguous but in order to explain the observed baryon spectrum we need to consider the *symmetry* of the quark *wavefunctions*. The overall wavefunctions

$$\Psi = \psi(\text{space})\phi(\text{flavour})\chi(\text{spin})\xi(\text{colour})$$

must be antisymmetric. Each quark flavour comes in three colours, red, green and blue (RGB), which form a fundamental triplet of the SU(3) colour group, $SU(3)_C$, which, unlike SU(3) flavour symmetry, is assumed to be exact. The $SU(3)_C$ singlet wavefunction for baryons

$$\xi = \frac{1}{\sqrt{6}}\{|RGB\rangle + |GBR\rangle + |BRG\rangle - |GRB\rangle - |BGR\rangle - |RBG\rangle\} \quad (10.15)$$

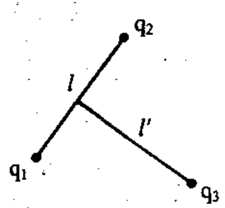

Figure 10.10
Relative orbital angular momenta in a three-quark system.

is antisymmetric in the exchange of any two quark colours. Its inclusion in the overall wavefunction Ψ guarantees antisymmetry provided $\psi(\text{space})\phi(\text{flavour})\chi(\text{spin})$ is symmetric.

Let us focus on the lowest-lying baryon multiplets, the $J^P = \frac{1}{2}^+$ octet and the $\frac{3}{2}^+$ decuplet. The relative orbital angular momenta l and l' in these three-quark states (figure 10.10) are assumed to be zero and therefore $\psi(\text{space})$ is symmetric. In the direct product of the SU(3) flavour and the SU(2) multiplets we therefore require symmetric combinations. In SU(3) we have

$$\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10}_S \oplus \mathbf{8}_{M_S} \oplus \mathbf{8}_{M_A} \oplus \mathbf{1}_A \qquad (10.16)$$

and in SU(2) the direct product of three spin doublets is

$$\mathbf{2} \otimes \mathbf{2} \otimes \mathbf{2} = \mathbf{4}_S \oplus \mathbf{2}_{M_S} \oplus \mathbf{2}_{M_A}. \qquad (10.17)$$

The subscripts denote the symmetry of the wavefunctions. For example, the direct product of three flavour triplets leads to a symmetric decuplet, two octets of mixed symmetry M_S and M_A and an antisymmetric singlet. In the mixed symmetry octets S (A) implies that the wavefunctions are symmetric (antisymmetric) with respect to interchange of the first two quark flavours. The symmetry properties of the multiplets become apparent only on examination of the wavefunctions of the states (see examples 10.4–10.7).

In order to determine the nature of the baryon multiplets predicted by the quark model with spin, SU(6), we must combine the SU(3) flavour multiplets with the SU(2) spin multiplets. In the direct product $(\mathbf{10}_S \oplus \mathbf{8}_{M_S} \oplus \mathbf{8}_{M_A} \oplus \mathbf{1}_A) \otimes (\mathbf{4}_S \oplus \mathbf{2}_{M_S} \oplus \mathbf{2}_{M_A})$ the only symmetric combination is the **56** representation $(\mathbf{10}, \mathbf{4}) \oplus (\mathbf{8}, \mathbf{2})$. The notation here is $(n_{SU(3)}, n_{SU(2)})$ where n is the dimensionality. The quark model with spin therefore successfully predicts a decuplet of $\frac{3}{2}^+$ and an octet of $\frac{1}{2}^+$ baryons. These multiplets and their quark content are shown in figure 10.11.

Having successfully assigned the low-lying mesons and baryons to

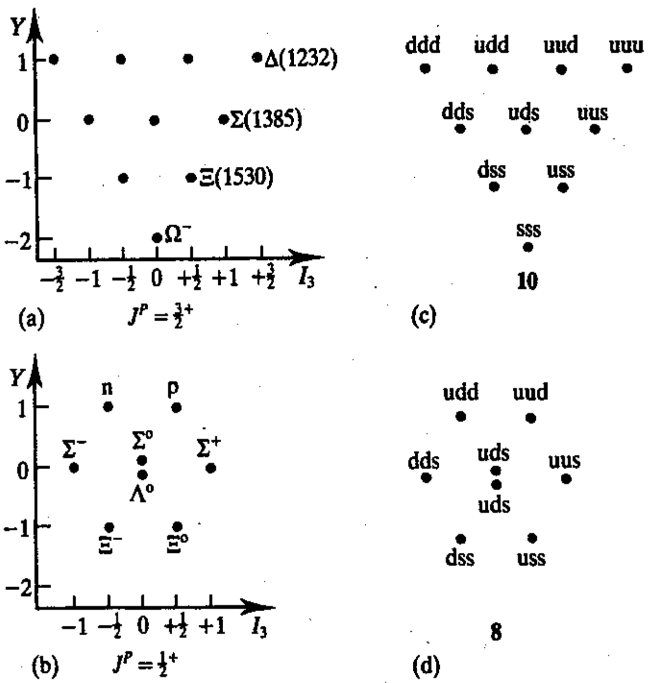

Figure 10.11
(a) The $\frac{3}{2}^+$ baryon decuplet and (b) the $\frac{1}{2}^+$ baryon octet. (c), (d) The quark content of the decuplet and octet states respectively.

multiplets of SU(3) it is natural to investigate the quark model further by examining its predictive power and comparing with experiment.

10.5 Hadron masses in the quark model

It is clear from the mass splittings in the multiplets that although flavour SU(3) describes the hadron spectrum very well, it is not an exact symmetry. If it were, the states in a given multiplet would be degenerate.

In an attempt to explain the mass splitting Gell-Mann proposed that the strong interaction Hamiltonian, H, consists of two parts – a 'very strong' interaction which is assumed to be exactly SU(3) invariant and described by a Hamiltonian H_0, plus a 'medium strong' interaction described by a Hamiltonian H', which is assumed to give deviations from exact symmetry. H_0 will therefore commute with all the SU(3) generators,

$$[H_0, F_i] = 0 \qquad (i = 1, 2, \ldots, 8) \tag{10.18}$$

while the symmetry-breaking part of the Hamiltonian will have non-zero commutation relations with some of the F_i,

$$[H', F_i] \neq 0. \tag{10.19}$$

Since isospin and hypercharge are conserved in the strong interaction, it follows that

$$[H', Y] = [H', I_3] = [H', I_\pm] = 0. \tag{10.20}$$

Gell-Mann proposed that H' should transform like one of the SU(3) generators and, since F_8 – the hypercharge – is the only generator that commutes with I_3, I_+ and I_- and, of course, with itself, he suggested that H' should transform like the hypercharge.

In order to make quantitative statements concerning the mass splitting it is convenient to work with U spin. In section 10.1.2 we saw that the U spin shift operators relate multiplet members with different hypercharge and I spin, i.e. members with different masses. From inspection of any of the multiplets it is apparent that

$$Y = U_3 + \tfrac{1}{2}Q. \tag{10.21}$$

Since all members of a U spin multiplet have the same electric charge it follows that Q commutes with all the U spin operators:

$$[U, Q] = 0. \tag{10.22}$$

In other words, the hypercharge, and consequently the symmetry-breaking part of the Hamiltonian H', transforms as a linear superposition of the third component of a vector in U spin space and a U spin scalar. Thus we may write

$$H' = H'_v + H'_s \tag{10.23}$$

where v and s stand for vector and scalar respectively. The mass of a particular U spin state $|U, U_3\rangle$ is then given by

$$\begin{aligned}\langle U, U_3|H|U, U_3\rangle &= \langle U, U_3|H_0 + H'_v + H'_s|U, U_3\rangle \\ &= m_0 + m_v + m_s.\end{aligned} \tag{10.24}$$

The contribution m_0 to the mass arising from the 'very strong' part of the interaction is the same for all members of a multiplet. In a given U spin multiplet m_s is the same for all members while m_v is proportional to U_3.

When equation (10.24) is applied to the U spin quartet of negatively charged $\frac{3}{2}^+$ baryons (see appendix J) we obtain the mass relation (example 10.8)

$$m_\Sigma - m_\Delta \approx m_\Xi - m_\Sigma \approx m_\Omega - m_\Xi \approx 150 \text{ MeV}.$$

Since the quark content of the Δ, Σ, Ξ and Ω is ddd, dds, dss and sss respectively, the near equality of the mass differences suggests that in some sense the mass of the strange quark is about 150 MeV greater than the mass of the d and u quarks. The much smaller mass difference between members of I spin multiplets suggests that $m_u \approx m_d$.

For the neutral members of the $\frac{1}{2}^+$ baryon octet the relation

$$\tfrac{1}{2}m_n + \tfrac{1}{2}m_{\Xi^0} = \tfrac{1}{4}m_{\Sigma^0} + \tfrac{3}{4}m_{\Lambda^0} \tag{10.25}$$

holds (see example 10.9). This is an example of the Gell-Mann–Okubo mass formula and, on substituting the measured mass values, is found to be accurate to about 1 per cent.

Since the SU(3) couplings are the same for any octet one might expect the results for the baryon octet to apply without change to the meson octets. Thus, for example, for the pseudoscalar mesons one might predict

$$\tfrac{1}{2}(m_{K^0} + m_{\bar{K}^0}) = \tfrac{1}{4}m_{\pi^0} + \tfrac{3}{4}m_{\eta^0}. \tag{10.26}$$

The masses of the K^0 and $\bar{K}^0$ are equal by the CPT theorem so that one expects

$$m_{\eta^0} = \tfrac{1}{3}(4m_{K^0} - m_{\pi^0}). \tag{10.27}$$

On substituting $m_\eta = 549$ MeV, $m_K = 498$ MeV and $m_\pi = 135$ MeV into equation (10.27) one finds a discrepancy of about 12 per cent. It is found that the mass formula works better for mesons if the square of the particle mass is used rather than the mass.* The Gell-Mann–Okubo mass formula for the pseudoscalar mesons then becomes

$$m_{\eta^0}^2 = \tfrac{1}{3}(4m_{K^0}^2 - m_{\pi^0}^2). \tag{10.28}$$

The discrepancy in this case is still about 7 per cent. However, it is found that the η^0 is not a 'wholly octet' state: because of SU(3)-breaking the physical η^0 and η' are mixtures of the SU(3) octet and singlet states.

The Gell-Mann–Okubo formula (10.28) assumes no mixing between the SU(3) octet and singlet states. In the presence of mixing we can write

$$\begin{aligned} \eta' &= \eta_1 \cos\theta + \eta_8 \sin\theta \\ \eta &= -\eta_1 \sin\theta + \eta_8 \cos\theta \end{aligned} \tag{10.29}$$

where η' and η denote the physical states, η_1 and η_8 the singlet and octet states and θ the mixing angle in the pseudoscalar nonet. The physical states η' and η are related to the SU(3) singlet and octet states by a rotation through the angle θ. For small θ the parametrization (10.29) implies that the η' is largely a singlet state and the η largely an octet state. We assume that the matrix elements of the Hamiltonian, or the mass-matrix elements, are quadratic in the mass rather than linear. With respect to η_1 and η_8 base states we have

$$H\begin{pmatrix}\eta_1\\ \eta_8\end{pmatrix} = \begin{pmatrix} M_{11}^2 & M_{18}^2 \\ M_{18}^2 & M_{88}^2 \end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_8\end{pmatrix} \tag{10.30}$$

with $M_{88}^2 = \frac{1}{3}(m_K^2 - m_\pi^2)$, in analogy with equation (10.28). Diagonalization of the mass matrix (see example 10.13) leads to

$$\tan^2\theta = \frac{M_{88}^2 - m_\eta^2}{m_{\eta'}^2 - M_{88}^2}. \tag{10.31}$$

Similar expressions hold for the vector- and tensor-meson nonets in which there is ϕ–ω and f'_2–f_2 mixing respectively. The mixing angles are $\theta_P \approx -11°$, $\theta_V \approx 40°$ and $\theta_T \approx 32°$. Equation (10.31) does not determine the sign of the mixing angle: it is negative (positive) according to whether the mass of the mainly octet member is less than (greater than) that of the mainly singlet member.

* Some justification for this derives from the Dirac equation (section 11.4.3) which describes the relativistic motion of spin $\frac{1}{2}$ fermions and leads to fermion propagators which depend on the fermion *mass*. Mesons on the other hand are described by the Klein–Gordon equation (section 11.4.1) which leads to meson propagators dependent on *mass squared*.

By introducing the mixing angle θ we have obtained consistency between the observed masses of the mesons, but this would be only of passing interest if no other consequences resulted from the concept of mixing. Important predictions concerning the dominant decay modes of the isoscalar states result from the observation that the 1^- and 2^+ nonets are very nearly 'ideally mixed'. The singlet and octet wavefunctions for the isoscalar states are

$$\{\mathbf{1}, |0, 0\rangle\} \equiv \psi_1 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

$$\{\mathbf{8}, |0, 0\rangle\} \equiv \psi_8 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}).$$

In general, the octet–singlet mixing is parametrized by the equations

$$m_1 = \psi_1 \cos\theta + \psi_8 \sin\theta$$

$$m_8 = -\psi_1 \sin\theta + \psi_8 \cos\theta$$

where m_1 denotes the physical, mainly singlet meson and m_8 the physical, mainly octet meson. If $\sin\theta = 1/\sqrt{3}$, we have

$$m_1 \approx u\bar{u} + d\bar{d}$$

$$m_8 \approx s\bar{s}$$

and the nonet is said to be ideally mixed in the sense that the singlet state consists only of $u\bar{u}$ and $d\bar{d}$ quarks and the octet state of $s\bar{s}$ quarks. Ideal mixing occurs for $\theta \approx 35°$: this is approximately the case for the 1^- and 2^+ nonets but not for the pseudoscalar nonet. We would therefore expect that the mainly singlet members of these nonets should decay predominantly to pseudoscalar mesons consisting of u and d quarks (pions) and the mainly octet members to strange pseudoscalar mesons (kaons). This is borne out by the observed branching fractions, $B(\varphi \to \mathrm{K\bar{K}}) \approx 84$ per cent and $B(\omega \to \pi^+\pi^-\pi^0) \approx 89$ per cent, for the 1^- isoscalars, and $B(\mathrm{f}_2' \to \mathrm{K\bar{K}}) \approx 71$ per cent and $B(\mathrm{f}_2 \to \pi\pi) \approx 85$ per cent for the 2^+ isoscalars. In contrast, the branching fraction for the decay $\varphi \to \pi^+\pi^-\pi^0$, which is favoured over the $\mathrm{K\bar{K}}$ decay mode by phase space considerations, is only about 2 per cent. We return to the suppression of this decay mode in section 10.7.2.

So far in this section we have obtained mass *relations* between members of various $\mathrm{SU(3)_{flavour}}$ multiplets, but have said nothing concerning the value of m_0 in equation (10.24) – the common mass which all members of a specific multiplet would have if $\mathrm{SU(3)_{flavour}}$ were an exact symmetry. Why is m_0 different for different multiplets? Why is it, for example, that the mass of the K^+, with quark content $\mathrm{u\bar{s}}$, is less than that of the

$K^{*+}(892)$ which has the same quark content? Similarly, why is the Δ^+ (uud) heavier than the proton (uud)? The pseudoscalar and vector meson octets differ in the relative orientation of the quark spins: they are antiparallel in the 0^- octet and parallel in the 1^- octet. The spins of the three quarks in the baryon octet couple to give $J^P = \frac{1}{2}^+$ while in the $\frac{3}{2}^+$ decuplet they are parallel. It seems likely, therefore, that if we view hadrons as bound states of *interacting* quarks, the mass differences between multiplets and, indeed, between members of the same multiplet, can be attributed to a spin–spin interaction.

The currently accepted theory of interacting quarks is quantum chromodynamics (QCD) in which coloured quarks interact via the exchange of coloured gluons. Like the photon in QED the gluons are massless and at short distances the QCD potential has the form of the QED Coloumb potential,

$$V(r) \approx -\frac{\alpha_s}{r} \tag{10.32}$$

where α_s is the strong coupling constant. In QED, the spin–spin interaction gives rise to hyperfine splitting[5] in which

$$\Delta E_{\mathrm{hfs}} = \tfrac{2}{3}\boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2 |\psi(0)|^2. \tag{10.33}$$

In units in which $\hbar = c = 1$, the magnetic moment μ_i of a particle with electric charge e_i, spin s_i and mass m_i is given by

$$\mu_i = \frac{e_i}{m_i} s_i. \tag{10.34}$$

Hence,

$$\Delta E_{\mathrm{hfs}} = \frac{8\pi\alpha}{3} |\psi(0)|^2 \frac{s_1 \cdot s_2}{m_1 m_2} \tag{10.35}$$

where we have used the relation $e_1 e_2 = e^2 = 4\pi\alpha$, α being the fine-structure constant. In equations (10.33) and (10.35) $\psi(0)$ is the value of the wavefunction $\psi(r_1, r_2)$ at zero separation. To obtain the analogous result for QCD we have to replace the electric charges e_1 and e_2 by the appropriate colour charges (see appendix M). For mesons and baryons this amounts to the substitutions

$$\alpha \to \begin{cases} \tfrac{4}{3}\alpha_s & (q\bar{q}) \\ \tfrac{2}{3}\alpha_s & (qq) \end{cases}$$

so,

$$\Delta E_{\mathrm{hfs}} = \frac{32}{9} \pi\alpha_s |\psi(0)|^2 \frac{s_1 \cdot s_2}{m_1 m_2} \tag{10.36}$$

for mesons, and

$$\Delta E_{\rm hfs} = \frac{16}{9}\pi\alpha_s|\psi(0)|^2 \sum_{i<j} \frac{\boldsymbol{s}_i\cdot\boldsymbol{s}_j}{m_i m_j} \tag{10.37}$$

for baryons. We therefore construct a simple model in which hadron masses are supposed to arise from a sum of constituent quark masses and hyperfine interactions. Thus, for mesons

$$m(q_1\bar{q}_2) = m_1 + m_2 + a\frac{\boldsymbol{s}_1\cdot\boldsymbol{s}_2}{m_1 m_2} \tag{10.38}$$

and for baryons

$$m(q_1q_2q_3) = m_1 + m_2 + m_3 + a' \sum_{i<j} \frac{\boldsymbol{s}_i\cdot\boldsymbol{s}_j}{m_i m_j}. \tag{10.39}$$

We will regard the constants a and a' and the quark masses as free parameters and attempt to explain the hadron mass spectrum with a consistent set of values.

By way of illustration we calculate baryon masses and leave meson masses as an exercise (example 10.14). In terms of the spins of the constituent quarks the baryon spin $\boldsymbol{J}$ is given by

$$J^2 = (\boldsymbol{s}_1 + \boldsymbol{s}_2 + \boldsymbol{s}_3)^2 = s_1^2 + s_2^2 + s_3^2 + 2(\boldsymbol{s}_1\cdot\boldsymbol{s}_2 + \boldsymbol{s}_1\cdot\boldsymbol{s}_3 + \boldsymbol{s}_2\cdot\boldsymbol{s}_3)$$

hence,

$$\sum_{i<j} \boldsymbol{s}_i\cdot\boldsymbol{s}_j = \tfrac{1}{2}[j(j+1) - \tfrac{9}{4}] = \begin{cases} +\frac{3}{4} & \text{for } j = \frac{3}{2} \text{ (decuplet)} \\ -\frac{3}{4} & \text{for } j = \frac{1}{2} \text{ (octet).} \end{cases} \tag{10.40}$$

Consider first the octet baryons. For the nucleon we have (equating the masses of the u and d quarks)

$$m_{\rm N} = 3m_{\rm u} - \frac{3}{4}\frac{a'}{m_{\rm u}^2}. \tag{10.41}$$

In the evaluation of the Σ and Λ masses we have to take into account the fact that $m_s > m_u, m_d$ in the calculation of the hyperfine splitting term. The Σ and Λ have isospin $I = 1$ and 0 respectively. Since the strange quark has $I = 0$, the u and d quarks must be in an $I = 1$ combination (symmetric) in the Σ and $I = 0$ (antisymmetric) in the Λ. Therefore, in the Σ the spins of the u and d quarks must couple to give spin 1 (symmetric) and in the Λ they must give spin 0 (antisymmetric) in order that the spin/flavour wavefunctions be symmetric: the antisymmetric colour wavefunction guarantees overall antisymmetry. Thus, for the Σ we have

$$J_{ud}^2 = (\boldsymbol{s}_u + \boldsymbol{s}_d)^2 = s_u^2 + s_d^2 + 2\boldsymbol{s}_u \cdot \boldsymbol{s}_d = 2$$

and

$$\boldsymbol{s}_u \cdot \boldsymbol{s}_d = \tfrac{1}{4}.$$

For the Λ, $J_{ud} = 0$ and

$$\boldsymbol{s}_u \cdot \boldsymbol{s}_d = -\tfrac{3}{4}.$$

Hence,

$$\begin{aligned} m_\Lambda &= m_u + m_d + m_s + a'\left(\frac{\boldsymbol{s}_u \cdot \boldsymbol{s}_d}{m_u m_d} + \frac{\boldsymbol{s}_u \cdot \boldsymbol{s}_s}{m_u m_s} + \frac{\boldsymbol{s}_d \cdot \boldsymbol{s}_s}{m_d m_s}\right) \\ &= 2m_u + m_s + a'\left[\frac{\boldsymbol{s}_u \cdot \boldsymbol{s}_d}{m_u^2} + \frac{1}{m_u m_s}(\boldsymbol{s}_u \cdot \boldsymbol{s}_s + \boldsymbol{s}_d \cdot \boldsymbol{s}_s)\right]. \end{aligned}$$

Noting that $\boldsymbol{s}_u \cdot \boldsymbol{s}_s + \boldsymbol{s}_d \cdot \boldsymbol{s}_s = \boldsymbol{s}_1 \cdot \boldsymbol{s}_2 + \boldsymbol{s}_1 \cdot \boldsymbol{s}_3 + \boldsymbol{s}_2 \cdot \boldsymbol{s}_3 - \boldsymbol{s}_u \cdot \boldsymbol{s}_d = -\tfrac{3}{4} + \tfrac{3}{4} = 0$, we have, finally,

$$m_\Lambda = 2m_u + m_s - \frac{3}{4}\frac{a'}{m_u^2}. \tag{10.42}$$

A similar calculation yields

$$m_\Sigma = 2m_u + m_s + \frac{a'}{4}\left(\frac{1}{m_u^2} - \frac{4}{m_u m_s}\right). \tag{10.43}$$

The Ξ mass is obtained from equation (10.43) by the interchange u $\leftrightarrow$ s, hence

$$m_\Xi = m_u + 2m_s + \frac{a'}{4}\left(\frac{1}{m_s^2} - \frac{4}{m_u m_s}\right). \tag{10.44}$$

In the case of the $\frac{3}{2}^+$ decuplet, the quark spins are aligned such that each pair combines to give spin 1. Thus, for example,

$$J_{12}^2 = (\boldsymbol{s}_1 + \boldsymbol{s}_2)^2 = s_1^2 + s_2^2 + 2\boldsymbol{s}_1 \cdot \boldsymbol{s}_2$$

and

$$\boldsymbol{s}_1 \cdot \boldsymbol{s}_2 = \tfrac{1}{2}[j(j+1) - s_1(s_1+1) - s_2(s_2+1)] = \tfrac{1}{4}.$$

Hence, for all states in the decuplet,

$$\boldsymbol{s}_1 \cdot \boldsymbol{s}_2 = \boldsymbol{s}_1 \cdot \boldsymbol{s}_3 = \boldsymbol{s}_2 \cdot \boldsymbol{s}_3 = \tfrac{1}{4}.$$

Table 10.5
Quark model predictions for the masses of the $\frac{3}{2}^+$ baryons

Baryon	Coefficient of m_u or m_d	Coefficient of m_s	ΔE_{hfs}	Predicted mass/ (GeV/c^2)	Observed mass/ (GeV/c^2)
N	3	0	$-3a'/4m_u^2$	0.939	0.939
Λ	2	1	$-3a'/4m_u^2$	1.114	1.116
Σ	2	1	$(3a'/4m_u^2) - (a'/m_u m_s)$	1.179	1.192
Ξ	1	2	$(a'/4m_s^2) - (a'/m_u m_s)$	1.327	1.318
$\Delta(1232)$	3	0	$3a'/4m_u^2$	1.239	1.232
$\Sigma(1385)$	2	1	$(a'/4m_u^2) + (a'/2m_u m_s)$	1.381	1.385
$\Xi(1530)$	1	2	$(a'/4m_s^2) + (a'/2m_u m_s)$	1.529	1.533
Ω^-	0	3	$3a'/4m_s^2$	1.682	1.672

(After Gasiorowicz S and Rosner J L 1981 *Am J Phys* **49** (954).)

Table 10.6
Current and constituent masses of the u, d and s quarks

Quark	Current mass/(MeV/c^2)	Constituent mass /(MeV/c^2)	
		Mesons	Baryons
u	5.6 ± 1.1	310	363
d	9.9 ± 1.1		
s	199 ± 33	483	538

The current masses are evaluated at a scale of 1 GeV/c^2.

It is then a simple matter to calculate the masses of the decuplet members using equation (10.39). The results, for constituent masses $m_u = m_d = 0.363\ GeV/c^2$, $m_s = 0.538\ GeV/c^2$ and $a'/m_u^2 = 0.2\ GeV/c^2$, are given in table 10.5.

In spite of the simplicity of the model and the fact that effects such as variation in $|\psi(0)|^2$, different binding energies and different kinetic energies have been neglected, the agreement between the predicted and observed masses is impressive. The effective or constituent masses of the light quarks, as they appear in mesons and baryons, are summarized in table 10.6. That the effective masses in baryons appear to be about 50 MeV/c^2 greater than in mesons may be attributable to small differences in binding effects.

The constituent masses of the quarks are quite distinct from the 'current' quark masses which appear in the QCD Lagrangian describing the interactions between quarks and gluons. These are quark masses free of the dynamical effects experienced in hadrons. The $SU(3)_{flavour}$ symmetry of strong interactions arises because the current quark masses are small compared with typical hadronic mass scales. To the extent that quark masses may be neglected, the strong interactions are flavour independent: the u, d, s, . . . , quarks experience the same strong interactions. The symmetry breaking appears in the Lagrangian through terms of the form

$$L = m_u u\bar{u} + m_d d\bar{d} + m_s s\bar{s} + \cdots \tag{10.45}$$

where m_f is the current mass of the quark with flavour f. In section 10.7 we shall discover that, in addition to the light quarks u, d and s, much heavier quarks (charm c and bottom b) exist. The specific pattern of quark masses appearing in nature is a mystery: the standard model of Glashow, Weinberg and Salam (chapter 13) has nothing to say on this matter. In the study of 'ordinary' hadrons the effect of c and b quarks may safely be neglected. Consistent values for the light quark masses have been obtained from consideration of the pseudoscalar-meson masses, baryon masses and the decay $\eta \to 3\pi$. For example, the symmetry-breaking term in the Lagrangian (10.45) has been used to compute ratios of masses of the pseudoscalar mesons:[6]

$$\frac{m_d}{m_u} = \frac{m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2}{2m_{\pi^0}^2 + m_{K^+}^2 - m_{K^0}^2 - m_{\pi^+}^2} \approx 1.8$$

$$\frac{m_s}{m_d} = \frac{m_{K^0}^2 + m_{K^+}^2 - m_{\pi^+}^2}{m_{K^0}^2 - m_{K^+}^2 + m_{\pi^+}^2} \approx 20.$$

When combined with an estimate of the strange quark mass from the Λ–N mass difference, $m_s \approx 150$ MeV/c^2, one obtains $m_u \approx 4.2$ MeV/c^2, and $m_d \approx 7.5$ MeV/c^2. The current quark masses shown in table 10.6 are taken from the 'Review of particle properties'.[7]

We shall see in chapter 14 that the mass scale characteristic of the strong interactions has a value $\Lambda_{QCD} \approx 0.2$ GeV. The masses of the u and d quarks are negligible in comparison and it is this which results in the (accidental) isospin symmetry of the strong interactions discussed in section 8.7. The mass of the s quark is comparable with Λ_{QCD} and as a result $SU(3)_{flavour}$ is only an approximate symmetry.

10.6 Baryon magnetic moments

As a further example of the predictive power of the quark model we consider the magnetic dipole moments of the $\frac{1}{2}^+$ octet baryons. In this ground-state octet the quarks have zero relative orbital angular momentum so the net magnetic moment of a baryon is simply the vector sum of the dipole moments of the constituent quarks, $\boldsymbol{\mu} = \boldsymbol{\mu}_1 + \boldsymbol{\mu}_2 + \boldsymbol{\mu}_3$. It is a matter of convention that when one speaks of the magnetic moment of a particle, rather than the vector $\boldsymbol{\mu}$, one means the *maximum observable component* of the magnetic moment μ_z, which, for a positively charged particle, corresponds to a spin orientation 'along' the positive z axis. For a point-like spin $\frac{1}{2}$ particle of charge e and mass m the magnetic moment is

$$\mu = \frac{e}{2m}. \tag{10.46}$$

Thus, for the structureless spin $\frac{1}{2}$ quarks, the magnetic moments are

$$\mu_u = \frac{2}{3}\frac{e}{2m_u} \qquad \mu_d = -\frac{1}{3}\frac{e}{2m_d} \qquad \mu_s = -\frac{1}{3}\frac{e}{2m_s}. \tag{10.47}$$

In order to calculate the magnetic moments of the ground-state baryons we need the quark wavefunctions for the baryons with spin component $s_z = +\frac{1}{2}$. In section 10.4 we saw that the ground-state baryons belong to a **56** of SU(6) and that the totally symmetric octet arises from the linear combination

$$\sqrt{\tfrac{1}{2}}[(\mathbf{8}_{M_S}, \mathbf{2}_{M_S}) + (\mathbf{8}_{M_A}, \mathbf{2}_{M_A})]. \tag{10.48}$$

The explicit wavefunction for a spin-up proton, for example, is

$$|p\uparrow\rangle = \sqrt{\tfrac{1}{2}}(p_{M_S}\chi_{M_S} + p_{M_A}\chi_{M_A}) \tag{10.49}$$

where p_{M_S}, p_{M_A} are the flavour wavefunctions. The spin wavefunctions χ_{M_S} and χ_{M_A} have the same form as p_{M_S} and p_{M_A} with the replacement u $\rightarrow \uparrow$ and d $\rightarrow \downarrow$. Explicitly

$$|p\uparrow\rangle = \sqrt{\tfrac{1}{2}}\{\sqrt{\tfrac{1}{6}}[(ud + du)u - 2uud]\sqrt{\tfrac{1}{6}}[(\uparrow\downarrow + \downarrow\uparrow)\uparrow - 2\uparrow\uparrow\downarrow] + \sqrt{\tfrac{1}{2}}[(ud - du)u]\sqrt{\tfrac{1}{2}}[(\uparrow\downarrow - \downarrow\uparrow)\uparrow]\} \tag{10.50}$$

or

$$|p\uparrow\rangle = \sqrt{\tfrac{1}{18}}(2u\uparrow u\uparrow d\downarrow - u\downarrow u\uparrow d\uparrow - u\uparrow u\downarrow d\uparrow + 2d\downarrow u\uparrow u\uparrow - d\uparrow u\uparrow u\downarrow - d\uparrow u\downarrow u\uparrow + 2u\uparrow d\downarrow u\uparrow - u\downarrow d\uparrow u\uparrow - u\uparrow d\uparrow u\downarrow). \tag{10.51}$$

The proton magnetic moment is given by

$$\begin{aligned} \mu_p &= \langle p\uparrow|\mu_1 + \mu_2 + \mu_3|p\uparrow\rangle \\ &= 3 \times \tfrac{1}{18}[4\mu_u + 4\mu_u - 4\mu_d + (-\mu_u + \mu_u + \mu_d) + (\mu_u - \mu_u + \mu_d)] \\ &= \tfrac{1}{3}(4\mu_u - \mu_d). \end{aligned} \tag{10.52}$$

The magnetic moment of the neutron is obtained from (10.52) by interchanging u and d, so

$$\mu_u = \tfrac{1}{3}(4\mu_d - \mu_u). \tag{10.53}$$

Similar calculations can be performed for the other octet members. Using

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Table 10.7
Comparison between the predicted and observed magnetic dipole moments of the $\frac{1}{2}^+$ baryons

Baryon	Dipole moment	Predicted/μ_N	Observed/μ_N
p	$\frac{1}{3}(4\mu_u - \mu_d)$	2.79	2.793
n	$\frac{1}{3}(4\mu_d - \mu_u)$	−1.86	−1.913
Λ	μ_s	−0.58	-0.613 ± 0.004
Σ^+	$\frac{1}{3}(4\mu_u - \mu_s)$	2.68	2.42 ± 0.05
Σ^0	$\frac{1}{3}(2\mu_u + 2\mu_d - \mu_s)$	0.82	
Σ^-	$\frac{1}{3}(4\mu_d - \mu_s)$	−1.05	-1.157 ± 0.025
Ξ^0	$\frac{1}{3}(4\mu_s - \mu_u)$	−1.40	-1.250 ± 0.014
Ξ^-	$\frac{1}{3}(4\mu_s - \mu_d)$	−0.47	-0.679 ± 0.031

the constituent quark masses in table 10.6 we obtain the quark magnetic moments, in nuclear magnetons,

$$\mu_u = 1.863$$

$$\mu_d = -0.931$$

$$\mu_s = -0.582.$$

These values lead to the predicted moments for the baryons shown in table 10.7. The agreement between the predicted and measured magnetic moments is good for the p, n and Λ but less so for the other members of the octet. This may not be too surprising in view of the crudeness of the model. Here, we have dealt exclusively with the so-called valence quarks which endow the hadrons with their static properties. In chapter 12 we shall see that hadrons have much more complicated structures than implied in this chapter: small contributions to the magnetic moments should arise from the 'sea' of quark–antiquark pairs which exist in baryons in addition to the valence quarks.

10.7 *Heavy-meson spectroscopy*

In addition to the three light quarks u, d and s, which we have discussed so far in this chapter, two other much heavier quarks, charm c and bottom or beauty b, are known to exist. They were first discovered as 'hidden' flavours in the ψ (psi) and Υ (upsilon) mesons which are respectively $c\bar{c}$ and $b\bar{b}$ bound states. Although on vastly different energy scales, the observed spectroscopic levels in the ψ and Υ systems are very similar to the observed level scheme in the e^+e^- system (positronium). The close agreement between theoretical predictions based on simple potential models and experimental observations of the level spacings in the ψ and Υ systems strongly supports the validity of the quark model.