Nuclear and Particle Physics

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8 *Symmetry transformations and conservation laws*

8.1 Introduction

The intimate connection between symmetry and conservation laws is perhaps nowhere so evident as in elementary particle physics; indeed the ideas of unification, briefly sketched in chapter 7, are based upon symmetry principles. The symmetries we shall deal with fall into two broad categories. The first are space–time symmetries which arise due to the existence of *equivalent* space–time frames of reference in which the laws of physics can be formulated and in which they have equal validity. Examples are translations and rotations in space. Symmetries belonging to the second category are known as internal symmetries of which isospin is an example; we shall meet others such as charge conjugation and SU(3) symmetry in due course. We shall call a symmetry transformation one which connects equivalent frames of reference.

Perhaps the most economical way to appreciate the connection between a symmetry transformation and an associated conservation law is to compare the Schrödinger and Heisenberg approaches to quantum mechanics. In the former the dynamical state of a system is represented by a wavefunction or state vector $|\psi(t)\rangle$ which evolves in time and dynamical variables are represented by operators. In the Heisenberg approach the state vectors are considered fixed while the dynamical variables evolve

in time. If a system is prepared at time t_0 in the state $|\psi(t_0)\rangle$ then at some later time t the state will evolve to $|\psi(t)\rangle$. Formally, this is written

$$|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle. \tag{8.1}$$

The evolution operator $U(t, t_0)$ is subject to the initial condition $U(t_0, t_0) = 1$. If we postulate that the principle of linear superposition is maintained in the evolution of the system then $U(t, t_0)$ must be a linear operator. On comparing equation (8.1) with the time-dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H|\psi(t)\rangle \tag{8.2}$$

where H is the Hamiltonian of the system, we deduce that

$$U(t, t_0) = \exp\left[\frac{-iH(t - t_0)}{\hbar}\right]. \tag{8.3}$$

Evidently $U(t, t_0)$ is a unitary operator since its adjoint $U^\dagger$ is equal to its inverse U^{-1}, or $U^\dagger U = UU^\dagger = 1$. We note that the transformation (8.1) preserves the norms of the state vectors, i.e. if $|\psi(t_0)\rangle$ is normalized then so is $|\psi(t)\rangle$; probability is conserved. 8.2

Let us now consider how a dynamical variable D evolves when the state vectors evolve according to (8.1). We are specifically interested in conserved quantities. If D is a constant of the motion its expectation value at any time t will be the same as that at time t_0. To admit the possibility of a time dependence we add a subscript 0 when the expectation value is determined in the initial state. In order to avoid cumbersome equations we omit the time dependence of the evolution operator. Then,

$$\langle\psi(t)|D|\psi(t)\rangle = \langle\psi(t_0)|U^\dagger DU|\psi(t_0)\rangle = \langle\psi(t_0)|D_0|\psi(t_0)\rangle.$$

The last equality is tantamount to the Heisenberg description in which the time dependence has been transferred from the state vectors to the dynamical variables. It follows that

$$U^\dagger DU = D_0$$

or, since U is unitary,

$$D = UD_0U^\dagger. \tag{8.4}$$

If D is indeed a conserved quantity we must have $dD/dt = 0$ for all t.

On differentiating (8.4) and using the explicit form (8.3) for $U(t, t_0)$ we obtain

$$\frac{dD}{dt} = \frac{1}{i\hbar}[H, D] + \frac{\partial D}{\partial t} \tag{8.5}$$

where $[H, D] = HD - DH$ is the commutator of D with the Hamiltonian. If D does not depend explicitly on the time, $\partial D/\partial t = 0$ and then *D is a conserved quantity if it commutes with the Hamiltonian*, i.e. if $[H, D] = 0$. In order to make the connection with a symmetry transformation we observe that if $HD = DH$ then

$$D^\dagger HD = H \tag{8.6}$$

if D is unitary. If the Hamiltonian is invariant under the transformation generated by D then D is a conserved quantity. We now consider some specific examples.

8.2 *Translations in space*

The operator corresponding to an infinitesimal translation ϵ along the x axis is $D(\epsilon) = (1 + \epsilon\,\partial/\partial x)$. Its effect on a state vector $|\psi(x)\rangle$ is

$$D(\epsilon)|\psi(x)\rangle = \left(1 + \epsilon\frac{\partial}{\partial x}\right)|\psi(x)\rangle = |\psi(x)\rangle + \epsilon\frac{\partial}{\partial x}|\psi(x)\rangle = |\psi(x + \epsilon)\rangle.$$

The operator corresponding to the x component of momentum is $p_x = -i\hbar\,\partial/\partial x$, so that

$$D(\epsilon) = 1 + i\frac{\epsilon p_x}{\hbar}$$

or, generalizing to three dimensions,

$$D(\epsilon) = 1 + i\frac{\boldsymbol{\epsilon}\cdot\boldsymbol{p}}{\hbar}. \tag{8.7}$$

A finite translation $\boldsymbol{r}$ can be generated by successive application of the operator in (8.7), giving

$$D(\boldsymbol{r}) = \operatorname*{Lt}_{\substack{\epsilon\to 0\\ n\to\infty}}\left(1 + i\frac{\boldsymbol{\epsilon}\cdot\boldsymbol{p}}{\hbar}\right)^n = \exp\left(i\frac{\boldsymbol{r}\cdot\boldsymbol{p}}{\hbar}\right) \tag{8.8}$$

where $n\epsilon \to \boldsymbol{r}$ as $n \to \infty$ and $\epsilon \to 0$.

The transformation generated by D is a symmetry transformation since it connects equivalent frames: the laws of physics must be independent of the origin of a reference frame. The linear momentum $\boldsymbol{p}$ is the *generator* of translations and its conservation results from a symmetry of space: space is homogeneous. Alternatively, momentum conservation may be viewed as a symmetry of the system, for if the Hamiltonian of the system is invariant under the transformation D, the dynamical variable corresponding to D, the linear momentum in the present case, is conserved. The unitary operators (8.8) form a *representation* of the translation symmetry group and since the operators commute the group is Abelian.

8.3 *Rotations in space*

For rotations of the frame of reference, or equivalently the rotation of a physical system within a fixed frame, the commutative law is generally not obeyed and the rotation group is non-Abelian. If we assume Cartesian axes for the frame the coordinate transformation resulting from an infinitesimal rotation ϵ around the z axis is

$$x \to x' = x + \epsilon y \qquad y \to y' = y - \epsilon x \qquad z \to z' = z \tag{8.9}$$

and the unitary operator $D(\epsilon)$ transforming the state function is

$$\begin{aligned} D(\epsilon)|\psi(x, y, z)\rangle &= |\psi(x + \epsilon y, y - \epsilon x, z)\rangle \\ &\approx |\psi(x, y, z)\rangle + \epsilon\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)|\psi(x, y, z)\rangle \\ &= \left(1 - \mathrm{i}\frac{\epsilon L_z}{\hbar}\right)|\psi(x, y, z)\rangle \end{aligned} \tag{8.10}$$

where L_z is the z component of the orbital angular momentum $\boldsymbol{L}$ of the system. As in the case of a translation a finite rotation ϕ can be generated by successive operations and

$$D(\phi) = \operatorname*{Lt}_{\substack{\epsilon \to 0 \\ n \to \infty}} \left(1 - \mathrm{i}\frac{\epsilon L_z}{\hbar}\right)^n = \exp\left(-\mathrm{i}\frac{\phi L_z}{\hbar}\right). \tag{8.11}$$

Generally, a rotation θ about an axis specified by a unit vector $\boldsymbol{n}$ is generated by the operator

$$D(\theta) = \exp\left(-\mathrm{i}\theta\frac{\boldsymbol{n}\cdot\boldsymbol{L}}{\hbar}\right) \tag{8.12}$$

where now the angular momentum $\boldsymbol{L}$ is the generator of the transformation.

Since the laws of physics must be independent of the orientation of the coordinate axes in space the frames of reference linked by the rotation operators are equivalent and the rotations form a symmetry group. The conservation of angular momentum can thus be regarded as being due to the isotropy of space. Alternatively, it can be regarded as a symmetry of the system; if the Hamiltonian of the system is invariant under rotations, angular momentum is conserved.

The state functions so far used have been scalar functions corresponding to scalar fields. If we extend the formalism to general fields for which angular momenta $\boldsymbol{J}$ are defined as the generators of rotations by equation (8.12) with $\boldsymbol{L} \to \boldsymbol{J}$, then for a vector field, such as that of electromagnetic radiation,

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{s} \tag{8.13}$$

where s is interpreted as an *intrinsic spin* associated with the field. For a vector field the field quanta have spin quantum number $s = 1$; the spin of the photon for instance is 1. For a scalar field $s = 0$.

The rotation (translation) group has the property that finite rotations (translations) can be generated as the product of successive infinitesimal rotations (translations). The group is then completely defined by consideration of infinitesimal transformations and such groups are known as Lie groups. The Lie algebra of the rotation group is based on the commutation relations for the angular momentum operators

$$[J_j, J_k] = \mathrm{i}\epsilon_{jkl} J_l \tag{8.14}$$

where $\epsilon_{jkl} = +1\ (-1)$ if j, k, l are cyclic (anticyclic) and $\epsilon_{jkl} = 0$ otherwise. The coefficients ϵ_{jkl} are known as the *structure constants* of the rotation group. The commutation relations (8.14) result from the fact that rotations in space do not commute.

An operator that commutes with all the generators of a group is called a *Casimir operator*, or invariant. For the rotation group the total angular momentum operator

$$J^2 = J_x^2 + J_y^2 + J_z^2 \tag{8.15}$$

is the only Casimir operator:

$$[J^2, J_i] = 0 \qquad \text{for } i = x, y, z. \tag{8.16}$$

We can therefore construct simultaneous eigenstates $|j, m\rangle$ of J^2 and *one* of the generators, conventionally J_z. The labels j and m are the quantum numbers of the total angular momentum and the component along the

z axis. The eigenvalue equations are

$$\begin{aligned} J^2|j, m\rangle &= j(j+1)|j, m\rangle \\ J_z|j, m\rangle &= m|j, m\rangle \end{aligned} \tag{8.17}$$

with $m = -j, -j+1, \ldots, j-1, j$. The quantum number j can have one of the values $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

It is convenient to form linear combinations of the other generators, J_x and J_y, which have useful commutation relations with J_z. For the rotation group these are

$$J_+ = J_x + \mathrm{i}J_y$$

the 'raising operator', and

$$J_- = J_x - \mathrm{i}J_y$$

the 'lowering operator', which satisfy the commutation relations

$$[J_z, J_\pm] = \pm J_\pm. \tag{8.18}$$

Consider a state $|j, m\rangle$, i.e. a state with total angular momentum j and component m along the quantization axis. From (8.18) we have

$$\begin{aligned} J_z J_-|j, m\rangle &= (J_- J_z - J_-)|j, m\rangle \\ &= J_-(J_z - 1)|j, m\rangle \\ &= (m-1)J_-|j, m\rangle. \end{aligned}$$

Thus

$$J_z[J_-|j, m\rangle] = (m-1)[J_-|j, m\rangle]. \tag{8.19}$$

Equation (8.19) is an eigenvalue equation showing that the new state $J_-|j, m\rangle$ is an eigenstate of J_z with eigenvalue $m-1$. It is for this reason that J_- is called a lowering operator. Similarly, the state $J_+|j, m\rangle$ is an eigenstate of J_z with eigenvalue $m+1$. Thus we can write

$$\begin{aligned} J_-|j, m\rangle &= C_-|j, m-1\rangle \\ J_+|j, m\rangle &= C_+|j, m+1\rangle \end{aligned}$$

with the conditions that $C_- = 0$ for $m = -j$ and $C_+ = 0$ for $m = j$. The values of C_+ and C_-, up to an arbitrary phase, can be determined as follows. We have

$$J_- J_+|j, m\rangle = J_- C_+(m)|j, m+1\rangle = C_-(m+1)C_+(m)|j, m\rangle.$$

In order to preserve the normalization of the states $|j, m\rangle$ with different m values we must have $C_-(m+1) = C_+(m)$. Then

$$J_-J_+|j, m\rangle = C_+^2(m)|j, m\rangle.$$

From the definitions of J_+ and J_- and the commutation relations (8.14) we obtain

$$J_-J_+ = J^2 - J_z(J_z + 1)$$

so

$$C_+ = [j(j+1) - m(m+1)]^{1/2}.$$

Similarly

$$C_- = [j(j+1) - m(m-1)]^{1/2}.$$

In summary,

$$\begin{aligned} J^2|j, m\rangle &= j(j+1)|j, m\rangle \\ J_z|j, m\rangle &= m|j, m\rangle \\ J_\pm|j, m\rangle &= [j(j+1) - m(m \pm 1)]^{1/2}|j, m \pm 1\rangle. \end{aligned} \tag{8.20}$$

The relations (8.20) define the matrix elements of the angular momentum operators in a basis in which J^2 and J_z are diagonal. Since all the operators in (8.20) are diagonal in j, matrices for each j are generally written with the rows labelled with m and the columns with m'. The matrix elements are

$$\begin{aligned} \langle j, m|J^2|j, m'\rangle &= j(j+1)\delta_{mm'} \\ \langle j, m|J_z|j, m'\rangle &= m\delta_{mm'} \\ \langle j, m|J_\pm|j, m'\rangle &= [j(j+1) - m(m \pm 1)]^{1/2}\delta_{m,m' \pm 1}. \end{aligned} \tag{8.21}$$

The set of states $|j, m\rangle$ with $|m| \leqslant j$ are said to form a *multiplet*. For a given j there are $2j + 1$ states corresponding to the $2j + 1$ different m values or different orientations of j. For $j = 0, \frac{1}{2}$ and 1 the multiplets are a singlet, doublet and triplet respectively.

The sum of two angular momenta, $\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2$, is also an angular momentum satisfying the commutation relations (8.14). It may be described in terms of the basis $|j_1m_1\rangle|j_2m_2\rangle$ or alternatively in terms of $|jm\rangle$. These two bases are connected by a unitary transformation

$$|jm\rangle = \sum_{m_1m_2} C^{jm}_{m_1m_2}|j_1m_1\rangle|j_2m_2\rangle \tag{8.22}$$

where the vector addition or Clebsch–Gordan coefficients can be calculated

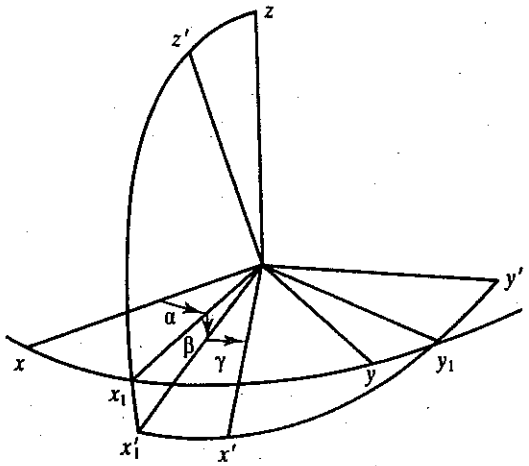


Figure 8.1
Definition of the Euler angles.

by using the raising or lowering operators but for convenience are tabulated in appendix G.

In all the foregoing angular momentum equations we have measured the angular momentum in units of $\hbar$; the absolute value of an angular momentum j is $\sqrt{[j(j+1)]}\hbar$.

Finally in this section we consider representations of the rotation group. A general rotation is normally specified in terms of the Euler angles (see figure 8.1.), which are defined by three successive rotations:

(a) a rotation through an angle α about the z axis generated by the unitary operator $D(\alpha) = \exp(-i\alpha J_z)$;
(b) a rotation through β about the new y axis y_1 generated by $D(\beta) = \exp(-i\beta J_{y_1})$;
(c) a rotation through γ about the z' axis and generated by $D(\gamma) = \exp(-i\gamma J_{z'})$.

8.4

These rotations are anticlockwise when looking along the rotation axis towards the origin and the three rotations together have taken us from the original axes labelled xyz to the final axes $x'y'z'$. The unitary operator for this arbitrary rotation is thus

$$D(\alpha, \beta, \gamma) = \exp(-i\gamma J_{z'}) \exp(-i\beta J_{y_1}) \exp(-i\alpha J_z). \tag{8.23}$$

This expression is inconvenient because it refers to different coordinate frames but a little thought shows that the same transformation of axes results from first a rotation γ about the *original* z axis followed by a rotation β about the *original* y axis and finally a rotation α about the *original* z axis. We therefore arrive at the important result

$$D(\alpha, \beta, \gamma) = \exp(-i\alpha J_z) \exp(-i\beta J_y) \exp(-i\gamma J_z) \tag{8.24}$$

which now refers only to the original coordinate axes.

As usual we choose a set of basis vectors $|jm\rangle$ which are simultaneously eigenstates of J^2 and J_z. Since J^2 is invariant under a rotation the matrix of D in this representation is diagonal in j. We therefore confine ourselves to the matrix elements

$$D^{(j)}_{mm'}(\alpha, \beta, \gamma) = \langle jm|D(\alpha, \beta, \gamma)|jm'\rangle$$
$$= \exp(-i\alpha m) d^{(j)}_{mm'}(\beta) \exp(-i\gamma m') \tag{8.25}$$

where

$$d^{(j)}_{mm'}(\beta) = \langle jm| \exp(-i\beta J_y)|jm'\rangle. \tag{8.26}$$

Explicit calculations of some of these matrix elements are given in appendix H. The matrix $D^{(j)}$ has dimension $(2j + 1)$ and is said to be the $(2j + 1)$-dimensional irreducible representation of the rotation group. The lowest dimension non-trivial representation of the rotation group $D^{(1/2)}$ is the *fundamental* representation in the sense that all higher dimension representations can be constructed from it. An example of the use of these rotation matrices is given in section 14.3.8.

8.4 *The group SU(2)*

Because of its importance in the description of both spin and isospin we review here the main properties of the special unitary group in two dimensions, SU(2).

For a spin $\frac{1}{2}$ particle the discussion of section 8.3 makes it clear that there are two eigenstates $|s, m_s\rangle$ of the commutating operators s^2 and s_z and that these have $m_s = \pm\frac{1}{2}$, namely spin 'up' and spin 'down'. If there is no preferred direction in space, so that all frames of reference that differ only by a rotation of axes are equivalent, these states are indistinguishable. They may be connected by the raising and lowering operators $s_\pm = s_x \pm is_y$ and if the eigenstates are written in the form of column vectors $\binom{1}{0}, \binom{0}{1}$ these operators have the simple matrix form

$$s_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad s_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{8.27}$$

from which we may conclude that the spin component matrices are

$$s_x = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \qquad s_y = \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix} \qquad s_z = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}. \tag{8.28}$$

These are just the *Pauli matrices* σ divided by 2.

The set of all unitary 2×2 matrices is known as the group U(2). A general spin state

$$|\psi\rangle \equiv \begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

is transformed into $|\psi'\rangle$ under the action of the unitary operator U: $|\psi\rangle \to |\psi'\rangle = U|\psi\rangle$. Since U is unitary, $UU^\dagger = U^\dagger U = 1$, probability is preserved under the transformation. Then,

$$\det(UU^\dagger) = (\det U)(\det U^\dagger) = (\det U)(\det U)^* = |\det U|^2 = 1$$

so that

$$\det U = \exp(i\theta)$$

where θ is real. This overall phase factor is relatively uninteresting: it corresponds to a global rotation of the state $|\psi\rangle$. The smaller group of transformations, with $\det U = 1$, is known as the special,* or unimodular, unitary group in two dimensions, SU(2). Now, for any hermitian matrix X, say,

$$\det[\exp(iX)] = \exp[i\ \mathrm{Tr}(X)]$$

where the trace, $\mathrm{Tr}(X)$, is the sum of the diagonal elements. If the unitary matrix $\exp(iX)$ is unimodular, $\mathrm{Tr}(X) = 0$. The Pauli spin matrices are hermitian and traceless and the set of matrices

$$U_i(\theta) = \exp(-\tfrac{1}{2}i\sigma_i\theta) \qquad \text{with } i = x, y, z \tag{8.29}$$

form a representation of the group SU(2). The group of phase transformations mentioned above, U(1), and the group SU(2) are subgroups of U(2) and the relation between these groups is written $\mathrm{U(2)} = \mathrm{SU(2)} \otimes \mathrm{U(1)}$. The Pauli matrices (cf. (8.28)) form the fundamental, two-dimensional representation of SU(2). The algebra of the group is just the algebra of the generators, $J_i = \frac{1}{2}\sigma_i$, summarized in the commutation relations

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\epsilon_{ijk}\frac{\sigma_k}{2}. \tag{8.30}$$

As before (equation (8.14)) the constants ϵ_{ijk} are the structure constants of the group.

Higher dimensional representations can be formed from the fundamental representation. For example, two spin $\frac{1}{2}$ particles can couple to

* The special requirement is equivalent to choosing the phase $\theta = 0$.

give a total of four states which we write symbolically as $|\uparrow\rangle|\uparrow\rangle$, $|\downarrow\rangle|\downarrow\rangle$, $|\uparrow\rangle|\downarrow\rangle$ and $|\downarrow\rangle|\uparrow\rangle$. The total spin can have the values 1 and 0 only, and these four states reduce to a triplet ($s = 1$) and a singlet ($s = 0$). Specifically

$$\begin{aligned}
|1, 1\rangle &= |\tfrac{1}{2}, \tfrac{1}{2}\rangle|\tfrac{1}{2}, \tfrac{1}{2}\rangle && |\uparrow\rangle|\uparrow\rangle \\
|1, 0\rangle &= \frac{1}{\sqrt{2}}(|\tfrac{1}{2}, \tfrac{1}{2}\rangle|\tfrac{1}{2}, -\tfrac{1}{2}\rangle + |\tfrac{1}{2}, -\tfrac{1}{2}\rangle|\tfrac{1}{2}, \tfrac{1}{2}\rangle) && \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle) \\
|1, -1\rangle &= |\tfrac{1}{2}, -\tfrac{1}{2}\rangle|\tfrac{1}{2}, -\tfrac{1}{2}\rangle && |\downarrow\rangle|\downarrow\rangle \\
|0, 0\rangle &= \frac{1}{\sqrt{2}}(|\tfrac{1}{2}, \tfrac{1}{2}\rangle|\tfrac{1}{2}, -\tfrac{1}{2}\rangle - |\tfrac{1}{2}, -\tfrac{1}{2}\rangle|\tfrac{1}{2}, \tfrac{1}{2}\rangle) && \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle).
\end{aligned} \tag{8.31}$$

This decomposition has led to an *irreducible* representation and in the notation of group theory is written

$$2 \otimes 2 = 4 = 3 \oplus 1.$$

This combination of representations is a simple example of equation (8.22) – the factors $1/\sqrt{2}$ are Clebsch–Gordan coefficients. This procedure can be generalized to higher dimensions and the results[1] adhere to the familiar rule that the coupling of two irreducible representations with angular momenta j_1 and j_2 yields representations with j given by

$$|j_1 - j_2| \leqslant j \leqslant j_1 + j_2. \tag{8.32}$$

8.5 *Systems of identical particles*

Before discussing an important example of the SU(2) group, isospin, we would like to introduce a fundamental and far-reaching symmetry connected with the behaviour of systems of identical particles under particle interchange.

If a system consists of identical particles the principle of indistinguishability asserts that there is no measurement which can be made which will distinguish a system of these particles from one in which two (or more) of them have been interchanged. For simplicity, consider a system of two identical particles, labelled with the collective quantum numbers ξ_1 and ξ_2, say. We write the state vector as $|\xi_1\xi_2\rangle$. On interchange of the two particles, the state vector can at most be multiplied by a phase factor which is unobservable in any measurement. Then,

$$|\xi_2\xi_1\rangle = e^{i\alpha}|\xi_1\xi_2\rangle$$

and, on a second interchange,

$$|\xi_1\xi_2\rangle = e^{i\alpha}|\xi_2\xi_1\rangle = e^{2i\alpha}|\xi_1\xi_2\rangle.$$

Consequently,

$$e^{2i\alpha} = 1 \qquad e^{i\alpha} = \pm 1.$$

Then, on interchange we have

$$|\xi_2\xi_1\rangle = \pm|\xi_1\xi_2\rangle. \tag{8.33}$$

Systems for which the state vector does not change sign are said to be symmetric with respect to particle interchange; those which change sign are antisymmetric. The principle is readily extended to systems containing any number of identical particles. The conditions (8.33) can be satisfied in a manner reminiscent of the equation for the coupling of two angular momentum states. The composite state vector $|\xi_1\xi_2\rangle$ can be written in terms of the individual particle state vectors $|\xi_1\rangle$ and $|\xi_2\rangle$; then

$$|\xi_1\xi_2\rangle = \frac{1}{\sqrt{2}}(|\xi_1\rangle|\xi_2\rangle + |\xi_2\rangle|\xi_1\rangle) \tag{8.34}$$

is symmetric and

$$|\xi_1\xi_2\rangle = \frac{1}{\sqrt{2}}(|\xi_1\rangle|\xi_2\rangle - |\xi_2\rangle|\xi_1\rangle) \tag{8.35}$$

is antisymmetric.

We recall that identical fermions obey the Pauli exclusion principle; no two identical fermions can exist in the same quantum state. If $|\xi_1\rangle = |\xi_2\rangle$ then, in equation (8.35), $|\xi_1\xi_2\rangle$ vanishes, whereas in equation (8.34) it does not. Consequently, a state consisting of identical fermions must be described by an antisymmetric state function. Identical bosons on the other hand are described by symmetric state functions.

8.6 *Parity*

The parity transformation introduced in section 3.7, unlike translation and rotation in space, is a discrete transformation. No sequence of continuous changes can convert a left-handed frame of reference to a right-handed one. If, however, this is a symmetry transformation in the

sense that the Hamiltonian of a system is invariant under the transformation then we recognize parity P as a conserved quantity and the basis of a good quantum number. As already pointed out in section 3.7 it is *multiplicative* rather than additive as for the continuous transformations. Some of the uses of parity in complex systems such as nuclei have already been explored in Part I of this book, which accepts that nucleons are conventionally assigned even parity. Here we examine the question of the parity of such single particles, which we take to be an intrinsic property.

Hadrons were seen in chapter 7 to be composite objects and if the internal motion in such an object has a definite reflection symmetry it seems reasonable to assign an *intrinsic* parity which is quite separate from that appropriate to spatial motion of the hadron as a whole. For instance, the lowest-lying mesons, the pions, are considered to consist of a quark–antiquark pair (figure 8.2) in a state of relative orbital angular momentum $L = 0$. Since parity is multiplicative the overall parity is given by $(-1)^L$ times the intrinsic parities of quark and antiquark. We will see quite generally in chapter 11 that the intrinsic parity of a fermion and antifermion are opposite and it follows that the intrinsic parity of the pion is odd ($P = -1$). Such particles, with $J^P = 0^-$, are described as *pseudoscalar* because they behave as scalars under coordinate rotation but have a wavefunction which changes sign on reflection. A *scalar* particle has zero spin and even parity $J^P = 0^+$. We note that this argument is assigning an *absolute* intrinsic parity to the pion.

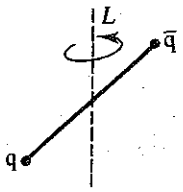


Figure 8.2
A quark–antiquark system with relative orbital angular momentum L.

The intrinsic parity of the pion was measured long before the invention of quarks. In the experiments of Panofsky *et al.*[2] and later Chinowski and Steinberger,[3] in which negative pions were stopped in deuterium, the reaction

$$\pi^- + \mathrm{d} \rightarrow \mathrm{n} + \mathrm{n}$$

was observed, whereas

$$\pi^- + \mathrm{d} \rightarrow \mathrm{n} + \mathrm{n} + \pi^0$$

was not. Later studies of the mesonic X-rays emitted showed that the capture takes place from an atomic S state of deuterium and it is assumed that the deuterium nucleus has $J^P = 1^+$ (section 3.11.2). Since the pion has spin 0 the total angular momentum in the initial state is $J = 1$. The only properly antisymmetrized state for two neutrons with a total angular momentum equal to one is the $^3\mathrm{P}_1$ state with odd parity (regardless of the intrinsic parity of the neutron). Since the reaction is a strong interaction the conservation of parity requires an odd intrinsic parity for the π^-. It is a general result of quantum field theory that if the negative pion has odd parity then so does the positive pion. This is true of all bosons; the parity of the particle and its antiparticle are the same, in marked contrast to the situation concerning fermions and antifermions.

The intrinsic parity of the π^0 can be determined in principle from measurements on the two-photon decay mode, $\pi^0 \to \gamma\gamma$. A classical electromagnetic wave is transverse, the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields are perpendicular to the direction of propagation of the wave. So, we suppose that a (real) photon has associated with it a polarization vector $\boldsymbol{\epsilon}$ which is perpendicular to the photon momentum $\boldsymbol{k}$; $\boldsymbol{\epsilon}\cdot\boldsymbol{k} = 0$. The following argument, due to Yang,[4] shows how the scalar or pseudoscalar nature of the π^0 can be determined. In the decay of a spin 0 pion, at rest, into two photons (figure 8.3) the photons must be collinear and the two-photon wave function must be a scalar under rotations ($J = 0$). The only vectors in the problem are the polarization vectors $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ of the two photons and their relative momentum vector $\boldsymbol{k}$. From these we can form the scalar quantities

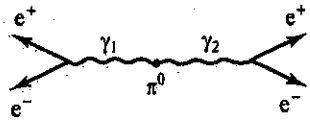


Figure 8.3
The decay $\pi^0 \to 2\gamma \to 2(e^+e^-)$ viewed in the π^0 rest frame.

$$\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2 \qquad \text{and} \qquad (\boldsymbol{\epsilon}_1 \wedge \boldsymbol{\epsilon}_2)\cdot\boldsymbol{k}. \tag{8.36}$$

The former is even under the parity transformation while the latter is odd. Conservation of parity in the electromagnetic decay requires that only one of the possibilities (8.36) is allowed. Thus by measuring the relative orientation of the polarization vectors of the two photons the intrinsic parity of the π^0 can be determined. The cross-section for electron–positron pair production is sensitive to the photon polarization vector and by measuring the relative orientation of the two production planes ($\boldsymbol{k}_{e^-} \wedge \boldsymbol{k}_{e^+}$) that of the polarization vectors may be ascertained. This is an extremely difficult experiment but it was shown by Kroll and Wada[5] that the correlation between the production planes persists in the so-called double Dalitz decay mode

$$\pi^0 \to (e^+ + e^-) + (e^+ + e^-)$$

in which the virtual photons are internally converted into pairs. If ϕ is the relative orientation of the planes the frequency distribution has the theoretical expectation

$$F(\phi) = 1 + \alpha\cos(2\phi) \qquad \text{for a scalar } \pi^0$$

and

$$F(\phi) = 1 - \alpha\cos(2\phi) \qquad \text{for a pseudoscalar } \pi^0.$$

The results of Plano *et al.*[6] are given in figure 8.4 and they show that the intrinsic parity of the π^0 is odd, like that of the π^+ and π^-.

The situation regarding the parity of the nucleons is not as clear cut as for the pions: the neutron and proton are arbitrarily assigned positive intrinsic parity. Since to a very high degree of accuracy baryon number is conserved, the intrinsic parities of the baryons in the initial state of a reaction will always be the same as in the final state, regardless of the assigned intrinsic parity. This is not so for pions for which no such conservation law holds.

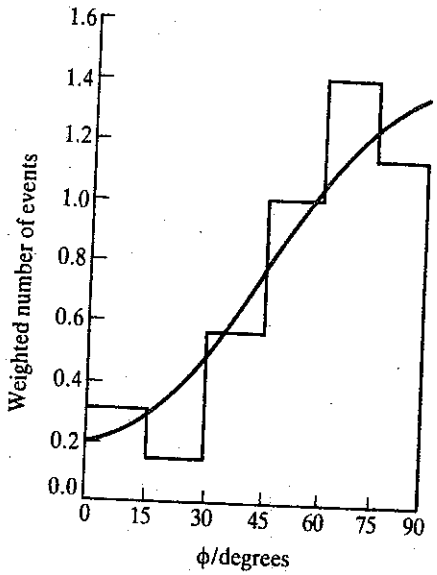


Figure 8.4 Plot of the weighted number of events as a function of the angle between the planes of polarization. Each event is weighted by the square of the correlation coefficient α for that event. The curve is a fit to the data and is given by $F(\phi) = 1 - (0.75 \pm 0.42)\cos(2\phi)$, consistent with expectations for a pseudoscalar π^0. (After Plano R *et al.* 1959 *Phys Rev Lett* **3** (525).)

The intrinsic parity of other stable baryons must be determined by some process (a) which is parity conserving and (b) in which the particle in question is either created or destroyed. Since parity is violated in the weak interactions the decay modes

$$K^0 \rightarrow \pi^+\pi^- \qquad \text{and} \qquad \Lambda^0 \rightarrow p\pi^-$$

cannot be used to determine the intrinsic parities of the K^0 meson and the Λ^0 hyperon. However, in the strong associated production process

$$\pi^- + p \rightarrow K^0 + \Lambda^0$$

the *relative* parity of the K^0 and Λ^0 can be determined. The intrinsic parity of the Λ^0 is conventionally assumed to be the same as that of the proton. Then, a detailed analysis of the angular momentum states involved in the interaction yields the result that the K^0 has odd intrinsic parity. From the rule that for bosons the parity of the particle and its antiparticle are the same, the $\bar{K}^0$ also has odd intrinsic parity. It turns out that all the *stable* mesons have odd intrinsic parity and all the stable baryons have even intrinsic parity. This does not mean, however, that *all* mesons and *all* baryons adhere to this pattern. We will see in chapter 9 that some meson *resonances* have even intrinsic parity and some baryon *resonances* have odd intrinsic parity.

8.7 Isospin: an example of the SU(2) group

8.7.1 Introduction

The concept of isospin was introduced in section 3.8 and used in sections 3.9, 3.11.4 and 7.3 without explicit development of the formalism. In terms of the arguments presented earlier in this chapter, however, it is a quantity governed by the transformations of the group SU(2).

As in the case of ordinary spin we use the two commuting variables, the isospin I (as I^2) and its third component I_3, to label states. The two possible orientations of the isospin $I = \frac{1}{2}$, corresponding to the proton and neutron, are represented by column vectors $\binom{1}{0}$ and $\binom{0}{1}$. Isospin raising and lowering operators can be constructed by forming linear combinations of the other components I_1 and I_2 of I. Then

$$I_+ = I_1 + \mathrm{i}I_2 \qquad \text{and} \qquad I_- = I_1 - \mathrm{i}I_2$$

and the matrix elements of the four isospin-operators $\hat{I}^2$, $\hat{I}_3$, $\hat{I}_+$ and $\hat{I}_-$ with respect to the states $|I, I_3\rangle$ as a basis are given by

$$\begin{aligned}
\langle I, I_3'|\hat{I}^2|I, I_3\rangle &= I(I+1)\delta_{I_3,I_3'} \\
\langle I, I_3'|\hat{I}_3|I, I_3\rangle &= I_3\delta_{I_3,I_3'} \\
\langle I, I_3'|\hat{I}_\pm|I, I_3\rangle &= [I(I+1) - I_3(I_3 \pm 1)]^{1/2}\delta_{I_3',I_3\pm 1}
\end{aligned} \tag{8.37}$$

in complete analogy with equations (8.21).

The fundamental, two-dimensional representation of the isospin rotation group is based on the three matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8.38}$$

which are identical to the Pauli spin matrices but in order to stress that we are dealing with the isospin SU(2) group we have labelled them τ_i, $i = 1, 2, 3$. These operate on the isospin doublet $\binom{1}{0}$ and $\binom{0}{1}$, corresponding to the states $|\frac{1}{2}, \frac{1}{2}\rangle$ and $|\frac{1}{2}, -\frac{1}{2}\rangle$ respectively. As before, the algebra of the group is just the algebra of the generators, $I_i = \frac{1}{2}\tau_i$, summarized in the commutation relations

$$\left[\frac{\tau_i}{2}, \frac{\tau_j}{2}\right] = \mathrm{i}\epsilon_{ijk}\frac{\tau_k}{2}. \tag{8.39}$$

The matrices

$$U_i(\theta) = \exp(-\tfrac{1}{2}\mathrm{i}\tau_i\theta) \qquad \text{with } i = 1, 2, 3 \tag{8.40}$$

form the fundamental representation of the isospin rotation group. Other representations of the SU(2) group, the one-, three-, four- etc. dimensional representations, corresponding to $I = 0, 1, \frac{3}{2}, \ldots$, can be constructed from the fundamental representation.

Again, the rules for the coupling of two isospins, I' and I'' say, are identical to those for the coupling of two angular momenta. If $I = I' + I''$, the possible values of I are restricted to

$$|I' - I''| \leqslant I \leqslant I' + I''$$

and for a given I

$$I_3 = I'_3 + I''_3.$$

The coupling involves the Clebsch–Gordan coefficients in exactly the same way as in the coupling of angular momentum states, thus

$$|I, I_3\rangle = \sum_{I'_3 I''_3} C^{II_3}_{I'_3 I''_3} |I', I'_3\rangle |I'', I''_3\rangle. \tag{8.41}$$

The basis states are now vectors in isospin space but otherwise equation (8.41) is identical to equation (8.22).

8.7.2 The extended Pauli principle

Let us re-examine the concept of the charge independence of the nucleon–nucleon force in the light of the isospin formalism introduced above. If we regard the neutron and proton as identical particles, as far as the strong interactions are concerned, the wavefunction for two nucleons may be written symbolically (see section 3.9.1) as

$$\Psi = \psi(\text{space})\chi(\text{spin})I(\text{isospin})$$

and for two 'identical' fermions this must be antisymmetric with respect to interchange of the two nucleons. The isospin states may be written down with the aid of equation (8.41). For two nucleons, with $I = \frac{1}{2}$, the total isospin can be 1 or 0. The $I = 1$ triplet is

$$|1, 1\rangle = |\tfrac{1}{2}, \tfrac{1}{2}\rangle |\tfrac{1}{2}, \tfrac{1}{2}\rangle \equiv |p\rangle|p\rangle$$

$$|1, 0\rangle = \frac{1}{\sqrt{2}} (|\tfrac{1}{2}, \tfrac{1}{2}\rangle |\tfrac{1}{2}, -\tfrac{1}{2}\rangle + |\tfrac{1}{2}, -\tfrac{1}{2}\rangle |\tfrac{1}{2}, \tfrac{1}{2}\rangle) \equiv \frac{1}{\sqrt{2}} (|p\rangle|n\rangle + |n\rangle|p\rangle)$$

$$|1, -1\rangle = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \equiv |n\rangle|n\rangle$$

and the $I = 0$ singlet is

$$|0, 0\rangle = \frac{1}{\sqrt{2}}(|\tfrac{1}{2}, \tfrac{1}{2}\rangle|\tfrac{1}{2}, -\tfrac{1}{2}\rangle - |\tfrac{1}{2}, -\tfrac{1}{2}\rangle|\tfrac{1}{2}, \tfrac{1}{2}\rangle) \equiv \frac{1}{\sqrt{2}}(|p\rangle|n\rangle - |n\rangle|p\rangle).$$

The factors $1/\sqrt{2}$ are the appropriate Clebsch–Gordan coefficients. It is evident that the triplet states are symmetric with respect to interchange of the two nucleons while the singlet state is antisymmetric. There is no *a priori* reason why the force between two nucleons should be the same in the $I = 1$ state and the $I = 0$ state. Indeed the antisymmetry of the wavefunction Ψ, required by the extended Pauli principle, implies that ψ(space)χ(spin) is symmetric for $I = 0$ (antisymmetric) and antisymmetric for $I = 1$ (symmetric). Because of the different space–spin wavefunctions one would expect that the force between two nucleons in the $I = 1$ state would differ from that in the $I = 0$ state. Charge independence of the nucleon–nucleon force (section 3.11.4) then applies only to the isospin triplet state and states that the pp, np and nn force depends on the total isospin but not on the third component, provided that the nucleon pairs are in the same antisymmetric spin and orbital angular momentum states. There is no corresponding argument for two nucleons in an $I = 0$ state because there is only one charge state.

8.7.3 Some consequences of isospin conservation

Perhaps the most obvious consequence of isospin conservation in the strong interactions is that, as already mentioned in sections 3.8 and 7.3, all hadrons belong to isospin multiplets. We note from table 7.2 that the masses of particles in a given multiplet are very nearly the same (approximately 4 per cent difference for the pion triplet). Now the mass of a particle may be regarded as an eigenvalue of its Hamiltonian evaluated in the particle rest frame and the Hamiltonian may be written

$$H = H_S + H_E + H_W$$

where S, E and W represent the strong, electromagnetic and weak interactions of the particle. Since the strong interactions conserve isospin the part of the mass arising from H_S will be constant within a multiplet. Unfortunately, the electromagnetic contribution cannot yet be calculated but it seems certain to be greater than that of the weak interaction and is probably responsible for the multiplet mass splitting.

It was stated in chapter 7 that the difference between the so-called stable hadrons and the resonances is that the latter decay via the strong interactions whereas the former are either genuinely stable or decay with much slower rates via the electromagnetic or weak interactions. Again, since isospin is conserved in the strong interactions, the relative decay

rates of a resonance, into channels involving particles in the same isospin multiplets, will depend only on the isospin coupling rules (equation (8.41)), i.e. they will depend on the ratios of the appropriate Clebsch–Gordan coefficients. To make this statement more concrete consider the decay of the $\Delta^+(1232)$, i.e. the singly charged pion–nucleon resonance with mass 1232 MeV. Generally the rate of decay is given by 'Fermi's golden rule' as

$$\Gamma \propto |\langle f|H|i\rangle|^2 \rho(E) \tag{8.42}$$

where $\langle f|H|i\rangle$ is the matrix element for the decay from the initial state $|i\rangle$ to the final state $|f\rangle$ and the statistical factor $\rho(E)$ is the density of states or 'phase space' available for the reaction at the energy E. There are four different charge states of the $\Delta(1232)$ and therefore the isospin is $I = \frac{3}{2}$. The resonance decays into a pion, with $I = 1$, and a nucleon, with $I = \frac{1}{2}$. The possible decay channels of the $\Delta^+(1232)$ are

$$\Delta^+(1232) \to \mathrm{p}\pi^0 \qquad \text{and} \qquad \Delta^+(1232) \to \mathrm{n}\pi^+$$

and since isospin is conserved the $\mathrm{N}\pi$ final states must have $I = \frac{3}{2}$; the $I = \frac{1}{2}$ combination is forbidden. The matrix elements $\langle f|H|i\rangle$ will be the product of an amplitude, $A_{3/2}$ say, which is independent of the particular final state, and the appropriate Clebsch–Gordan coefficients for the coupling of the $I = \frac{1}{2}$ nucleon states and the $I = 1$ pion states to give the $I = \frac{3}{2}$ Δ state. Considering first the decay $\Delta^+(1232) \to \mathrm{p}\pi^0$ and labelling states as $|I, I_3\rangle$ we have

$$\Delta^+(1232) = |\tfrac{3}{2}, \tfrac{1}{2}\rangle$$
$$\mathrm{p} = |\tfrac{1}{2}, \tfrac{1}{2}\rangle$$
$$\pi^0 = |1, 0\rangle.$$

With reference to the tables of Clebsch–Gordan coefficients in appendix G for the coupling $\frac{1}{2} \otimes 1$ we see that

$$|\tfrac{3}{2}, \tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}|\tfrac{1}{2}, \tfrac{1}{2}\rangle|1, 0\rangle.$$

Therefore,

$$\Gamma[\Delta^+(1232) \to \mathrm{p}\pi^0] \propto \tfrac{2}{3}|A_{3/2}|^2 \rho(E).$$

Turning to the second channel, $\Delta^+(1232) \to \mathrm{n}\pi^+$, we have

$$\mathrm{n} = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$
$$\pi^+ = |1, 1\rangle$$

and the appropriate Clebsch–Gordan coefficient in this case is $\sqrt{(1/3)}$.

Therefore,

$$\Gamma[\Delta^+(1232) \to n\pi^+] \propto \tfrac{1}{3}|A_{3/2}|^2\rho(E).$$

Since the masses of the particles involved in the two decay channels are essentially the same, the phase space factor $\rho(E)$ is the same in each case and so the ratio of the decay rates is

$$\frac{\Gamma[\Delta^+(1232) \to p\pi^0]}{\Gamma[\Delta^+(1232) \to n\pi^+]} = 2.$$

Similar relationships hold for cross-sections of *reactions* involving members of the same isospin multiplets. As an illustration consider the π, N, K and Σ multiplets, with $I = 1, \frac{1}{2}, \frac{1}{2}$ and 1 respectively, and specifically the reactions

$$\pi^- p \to K^0\Sigma^0 \tag{i}$$

$$\pi^- p \to K^+\Sigma^- \tag{ii}$$

$$\pi^+ p \to K^+\Sigma^+. \tag{iii}$$

The cross-sections σ are proportional to the square of the matrix element, or amplitude, connecting the initial and final states,

$$\sigma \propto |\langle f|H|i\rangle|^2\rho(E)$$

where, as before, the factor $\rho(E)$ is the phase space available for the reaction. Both the initial and final states involve particles with $I = \frac{1}{2}$ and $I = 1$ and therefore the total isospin can in general be $I = \frac{1}{2}$ or $I = \frac{3}{2}$. Since isospin is conserved in these strong interactions there are two channels with two independent amplitudes, one corresponding to $I = \frac{1}{2}$, $A_{1/2}$ say, and the other corresponding to $I = \frac{3}{2}$, $A_{3/2}$. We can think of the operator H as consisting of two independent parts,

$$H = H_{1/2} + H_{3/2}$$

where $H_{1/2}$ causes transitions in the $I = \frac{1}{2}$ channel and $H_{3/2}$ in the $I = \frac{3}{2}$ channel. Then,

$$\sigma = K|\langle f|H_{1/2} + H_{3/2}|i\rangle|^2 \tag{8.43}$$

where K is a constant at a given energy E. (Again, since the masses of the particles in a given isospin multiplet are very nearly the same, the phase space factors for a particular incident energy are equal to a good approximation.) In order to calculate relations between cross-sections for the different processes we must write each initial and each final state in terms of the total isospin with the aid of the tables of Clebsch–Gordan

coefficients. We have

$$|\pi^-\rangle|\mathrm{p}\rangle = |1,-1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle = \sqrt{\tfrac{1}{3}}|\tfrac{3}{2},-\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$
$$|\pi^+\rangle|\mathrm{p}\rangle = |1,1\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle = |\tfrac{3}{2},\tfrac{3}{2}\rangle$$
$$|\mathrm{K}^0\rangle|\Sigma^0\rangle = |\tfrac{1}{2},-\tfrac{1}{2}\rangle|1,0\rangle = \sqrt{\tfrac{2}{3}}|\tfrac{3}{2},-\tfrac{1}{2}\rangle + \sqrt{\tfrac{1}{3}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$
$$|\mathrm{K}^+\rangle|\Sigma^-\rangle = |\tfrac{1}{2},\tfrac{1}{2}\rangle|1,-1\rangle = \sqrt{\tfrac{1}{3}}|\tfrac{3}{2},-\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$
$$|\mathrm{K}^+\rangle|\Sigma^+\rangle = |\tfrac{1}{2},\tfrac{1}{2}\rangle|1,1\rangle = |\tfrac{3}{2},\tfrac{3}{2}\rangle.$$

Using these isospin states* we may now calculate the cross-sections for the processes (i)–(iii).

(i) $\pi^-\mathrm{p} \to \mathrm{K}^0\Sigma^0$ On inserting the initial and final states into equation (8.43) we obtain

$$\sigma = K|(\sqrt{\tfrac{2}{3}}\langle\tfrac{3}{2},-\tfrac{1}{2}| + \sqrt{\tfrac{1}{3}}\langle\tfrac{1}{2},-\tfrac{1}{2}|)H_{1/2} + H_{3/2}(\sqrt{\tfrac{1}{3}}|\tfrac{3}{2},-\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle)|^2.$$

Remembering that $H_{1/2}$ and $H_{3/2}$ act only on $I = \frac{1}{2}$ and $I = \frac{3}{2}$ states respectively, we have

$$\sigma = K\tfrac{2}{9}|A_{3/2} - A_{1/2}|^2$$

where $A_{3/2}$ and $A_{1/2}$ are the two (complex) amplitudes

$$A_{3/2} = \langle f|H_{3/2}|i\rangle \quad \text{and} \quad A_{1/2} = \langle f|H_{1/2}|i\rangle.$$

(ii) $\pi^-\mathrm{p} \to \mathrm{K}^+\Sigma^-$ In this case,

$$\sigma = K|(\sqrt{\tfrac{1}{3}}\langle\tfrac{3}{2},-\tfrac{1}{2}| - \sqrt{\tfrac{2}{3}}\langle\tfrac{1}{2},\tfrac{1}{2}|)H_{1/2} + H_{3/2}(\sqrt{\tfrac{1}{3}}|\tfrac{3}{2},-\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle)|^2$$
$$= K\tfrac{1}{9}|A_{3/2} + 2A_{1/2}|^2.$$

(iii) $\pi^+\mathrm{p} \to \mathrm{K}^+\Sigma^+$ Since both initial and final states are pure $I = \frac{3}{2}$ in this case, we have

$$\sigma = K|\langle\tfrac{3}{2},\tfrac{3}{2}|H_{1/2} + H_{3/2}|\tfrac{3}{2},\tfrac{3}{2}\rangle|^2$$
$$= K|A_{3/2}|^2.$$

Thus, the conservation of isospin implies that the cross-sections for these processes are in the ratios

$$\sigma_{(\mathrm{i})} : \sigma_{(\mathrm{ii})} : \sigma_{(\mathrm{iii})} = \tfrac{2}{9}|A_{3/2} - A_{1/2}|^2 : \tfrac{1}{9}|A_{3/2} + 2A_{1/2}|^2 : |A_{3/2}|^2.$$

* A little thought and/or experience allows the individual particle isospin states to be written down immediately. Alternatively, application of the Gell-Mann–Nishijima relation, equation (7.5), gives the I_3 values.

8.8 Charge conjugation

The charge conjugation operation, like the parity transformation, is a discrete symmetry operation. It consists of changing particle to antiparticle in any process while maintaining the dynamical variables, momentum, spin etc., unchanged. If a process and its charge conjugate behave in the same way, or satisfy the same physical laws, the process is said to be charge conjugation invariant. Alternatively, C or C-parity is said to be conserved in the process. If we specify a state by $|A, p, \sigma \ldots\rangle$ where A stands for the additive quantum numbers such as charge Q, baryon number B, lepton number L, strangeness S, etc. and p and σ represent the dynamical quantities such as momentum and spin, then the charge conjugation operation C changes the sign of the additive quantum numbers, if non-zero, and leaves p, σ, etc. unchanged. A second application of the charge conjugation operator must return the system to its original state so that, as in the case of parity, if a system is an eigenstate of C its eigenvalue is either $+1$ or -1. That is,

$$C|A, p, \sigma\rangle \rightarrow |\bar{A}, p, \sigma\rangle = e^{i\alpha}|A, p, \sigma\rangle$$

$$C^2|A, p, \sigma\rangle = C|\bar{A}, p, \sigma\rangle = e^{2i\alpha}|A, p, \sigma\rangle = 1|A, p, \sigma\rangle.$$

Therefore, $e^{i\alpha} = \pm 1$ and $C|A, p, \sigma\rangle = \pm 1|A, p, \sigma\rangle$. We use the notation $\bar{A}$ to signify the change particle $\rightarrow$ antiparticle under charge conjugation.

The observation that, under charge conjugation, $Q \rightarrow -Q$, $B \rightarrow -B$, $L \rightarrow -L$, $S \rightarrow -S$, etc. imposes severe restrictions on the possible eigenstates of C. We recall that in order that C be a constant of the motion it must commute with the Hamiltonian of the system, $[H, C] = 0$. Now, Q, B and L are all conserved quantities and, at least as far as the strong interactions are concerned, so is the strangeness S. They all commute with the Hamiltonian but none of them commute with C. For example, consider a state with $B = 1$ and $Q = L = S = 0$. If we introduce a 'baryon number operator' $\hat{B}$, which counts the number of baryons, then

$$\hat{B}|B = 1\rangle = +1|B = 1\rangle.$$

Now,

$$C\hat{B}|B = 1\rangle = +1C|B = 1\rangle = +1|B = -1\rangle.$$

On the other hand,

$$\hat{B}C|B = 1\rangle = \hat{B}|B = -1\rangle = -1|B = -1\rangle$$

hence

$$C\hat{B} \neq \hat{B}C,$$

i.e. B and C do not commute. A similar argument applies for operators

corresponding to Q, L and S and we are thus led to the important conclusion* that a state can be an eigenstate of C only if it has $Q = B = L = S = 0$. The neutral pion satisfies this condition so that the π^0 and its antiparticle are indistinguishable. On the other hand, the K^0 and its antiparticle the $\bar{K}^0$ are distinct, they are not eigenstates of C because the strangeness quantum number S is non-zero.

Of course while not many isolated particles are eigenstates of C, *systems* of particles can be if they have $Q = B = L = S = 0$. For instance, a proton–antiproton system in a state of definite orbital angular momentum l and spin s is an eigenstate of C, which we now show. With reference to figure 8.5 let us assume that initially the antiproton has spatial coordinate x_1 and is in a spin state σ_1 with the proton at x_2 in a spin state σ_2. We have

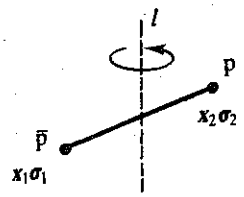


Figure 8.5
A proton–antiproton system with relative orbital angular momentum l.

$$C|\bar{p}x_1\sigma_1 px_2\sigma_2\rangle = |px_1\sigma_1\bar{p}x_2\sigma_2\rangle. \tag{8.44}$$

Equation (8.44) is not an eigenvalue equation but we can make it into one by interchanging the space and spin coordinates. The proton and antiproton both have spin $\frac{1}{2}$ and we have already seen that when two spin $\frac{1}{2}$ particles couple to give a total spin $s = 1$ or $s = 0$ the triplet $s = 1$ state is symmetric with respect to interchange of the spin coordinates and the singlet $s = 0$ state is antisymmetric. Thus on interchange of spin coordinates a factor $(-1)^{s+1}$ is introduced. Interchange of space coordinates is just the parity transformation which, for the system under consideration, gives a factor $(-1)^l$ from the orbital motion and another factor -1 from the opposite intrinsic parities of the proton and antiproton. Therefore, on interchange we have

$$C|\bar{p}x_1\sigma_1 px_2\sigma_2\rangle = (-1)^{s+1}(-1)^{l+1}|\bar{p}x_1\sigma_1 px_2\sigma_2\rangle \tag{8.45}$$

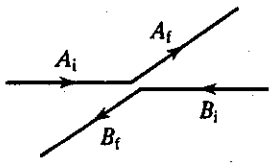
which is an eigenvalue equation, with eigenvalue equal to $(-1)^{l+s}$.

Both the strong and the electromagnetic interactions are invariant under charge conjugation but, as was the case with parity, the weak interactions are not. We shall see in chapter 11 that although the weak interactions violate P and C separately, to a very good approximation they are invariant under the combined operation CP.

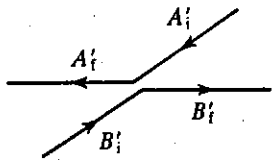
8.9 *Time reversal*[7]

The time reversal transformation should not be regarded as a running of time backwards into the past. Classically we may think of a collision

* To the list of additive quantum numbers should be added the further quark flavours or quantum numbers charm, bottom and top.



(a)



(b)

Figure 8.6
(a) A collision between two molecules and (b) the time-reversed process.

between two molecules A and B (figure 8.6(a)) with initial and final velocities A_i, B_i, A_f, B_f. The time-reversed process would be that shown in figure 8.6(b) with A_f and B_f reversed to A_i' and B_i' leading to final velocities A_f' and B_f' which are the reverse of A_i and B_i. In a gas the probabilities of the two processes shown in figures 8.6(a) and 8.6(b) would be equal and the process is described as time-reversal invariant. This is known as the principle of *microscopic reversibility*.

To describe the time reversal transformation in quantum mechanical terms we must recall that there are two types of observables. First, there is the probability that, having prepared a system in the state $|\psi\rangle$, it will be found on measurement to be in the state $|\phi\rangle$, given by $|\langle\phi|\psi\rangle|^2$. Secondly, there are the expectation values of dynamical quantities D in the state $|\psi\rangle$, i.e. $\langle D\rangle = \langle\psi|D|\psi\rangle$. In the symmetry transformations we have discussed so far the transformation operators have all been unitary, i.e. if under the transformation U

$$|\psi\rangle \rightarrow |\psi'\rangle = U|\psi\rangle$$

then the 'overlap' $\langle\phi|\psi\rangle \rightarrow \langle\phi'|\psi'\rangle = \langle\phi U^\dagger|U\psi\rangle = \langle\phi|\psi\rangle$ provided $U^\dagger U = 1$. The expectation values transform as

$$\begin{aligned}\langle\psi'|D'|\psi'\rangle &= \langle\psi U^\dagger|D'|U\psi\rangle \\ &= \langle\psi|U^\dagger D' U|\psi\rangle \\ &= \langle\psi|D|\psi\rangle\end{aligned}$$

again, provided that U is unitary.

This is not the case with time reversal. Let U be a unitary operator which changes t to $-t$. A system is time reversal invariant if U commutes with the Hamiltonian of the system, i.e. if

$$U^\dagger H U = H. \tag{8.46}$$

If the system is in a state $|\psi\rangle$ the time development is given by the Schrödinger equation

$$H|\psi\rangle = \mathrm{i}\hbar\frac{\partial}{\partial t}|\psi\rangle. \tag{8.47}$$

Now,

$$UH|\psi\rangle = U\mathrm{i}\hbar\frac{\partial}{\partial t}|\psi\rangle = -\mathrm{i}\hbar\frac{\partial}{\partial t}(U|\psi\rangle)$$

since U changes t to $-t$. Since $UH = HU$ we have

$$H(U|\psi\rangle) = -\mathrm{i}\hbar\frac{\partial}{\partial t}(U|\psi\rangle). \tag{8.48}$$

Thus, if $|\psi'\rangle = U|\psi\rangle$, we see that $|\psi'\rangle$ does not satisfy the Schrödinger equation. Instead it satisfies

$$H|\psi'\rangle = -i\hbar \frac{\partial}{\partial t}|\psi'\rangle.$$

We note, however, that $|\psi\rangle$, the solution to the Schrödinger equation, is not an observable and the symmetry can be restored by defining an operator $T = UK$, where the operator K carries out complex conjugation on anything that stands to the right of it. If we operate on (8.47) with T we have, since the Hamiltonian is real,

$$TH|\psi\rangle = HT|\psi\rangle = -i\hbar T \frac{\partial}{\partial t}|\psi\rangle = i\hbar \frac{\partial}{\partial t}(T|\psi\rangle).$$

Thus the state $T|\psi\rangle$ is a solution to the Schrödinger equation, i.e. if $|\psi(t)\rangle$ is a solution to the Schrödinger equation then so is $|\psi^*(-t)\rangle$. The operator T has been called 'antilinear' by Wigner, i.e.

$$T(C_1|\psi\rangle + C_2|\phi\rangle) = C_1^* T|\psi\rangle + C_2^* T|\phi\rangle$$

and 'antiunitary'

$$\langle\psi'|\phi'\rangle = \langle\psi T^\dagger | T\phi\rangle = \langle\psi|\phi\rangle^* = \langle\phi|\psi\rangle.$$

We note, however, that since

$$|\langle\psi'|\phi'\rangle| = |\langle\phi|\psi\rangle| = |\langle\psi|\phi\rangle|$$

the operator T leaves the physical content of quantum mechanics unchanged. Thus the quantum mechanical equivalent of the classical time reversal transformation is $t \to -t$ and complex conjugation, $i \to -i$. Under this transformation the quantum mechanical principle of microscopic reversibility holds.

Since, under time reversal, $\boldsymbol{x} \to \boldsymbol{x}$, $\boldsymbol{p} \to -\boldsymbol{p}$ and $i \to -i$, the basic commutation relation

$$[x_i, p_j] = i\hbar\delta_{ij}$$

still holds. The orbital angular momentum commutation relations are also preserved since $\boldsymbol{L} \to -\boldsymbol{L}$ and $i \to -i$, then

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k.$$

In table 8.1 we list the transformation properties of some common variables under time reversal.

Table 8.1
The transformation properties of some common variables under time reversal

Variable	Behaviour
Position	$r \to r$
Momentum	$p \to -p$
Angular momentum	$L \to -L$
Spin	$S \to -S$
Electric field E	$E \to E$
Magnetic field B	$B \to -B$

We end this section on time reversal with a brief discussion of an important class of experiments which test the validity of time reversal invariance in electromagnetic interactions and, as it turns out, invariance under the parity transformation. These are experiments designed to measure the electric-dipole moments of particles. The orientation of a particle can be specified by the orientation of its spin with respect to some axis. If we choose a z axis so that the spin is aligned along it, the electric-dipole moment μ_e is given by

$$\mu_e = \int \rho z \, d\tau. \tag{8.49}$$

In this expression z is measured from the centre of mass of the particle, ρ is the electric charge density and $d\tau$ is a volume element. If the particle possesses a net electric charge, the existence of an electric-dipole moment implies that the centre of charge and the centre of mass do not coincide. If, on the other hand, the particle is electrically neutral, a non-zero electric-dipole moment can arise if there is an asymmetry in the charge distribution with a net positive charge in one 'hemisphere' and a corresponding net negative charge in the other. If the particle is assumed to possess a magnetic-dipole moment μ_m, then in the presence of an electromagnetic field the interaction Hamiltonian is

$$H_{int} = \mu_m \boldsymbol{\sigma} \cdot \boldsymbol{B} + \mu_e \boldsymbol{\sigma} \cdot \boldsymbol{E} \tag{8.50}$$

where $\boldsymbol{\sigma}$ is the particle spin. Under time reversal T we have

$$\boldsymbol{\sigma} \cdot \boldsymbol{B} \xrightarrow{T} \boldsymbol{\sigma} \cdot \boldsymbol{B} \quad \text{and} \quad \boldsymbol{\sigma} \cdot \boldsymbol{E} \xrightarrow{T} -\boldsymbol{\sigma} \cdot \boldsymbol{E} \tag{8.51a}$$

and under the parity transformation

$$\boldsymbol{\sigma} \cdot \boldsymbol{B} \xrightarrow{P} \boldsymbol{\sigma} \cdot \boldsymbol{B} \quad \text{and} \quad \boldsymbol{\sigma} \cdot \boldsymbol{E} \xrightarrow{P} -\boldsymbol{\sigma} \cdot \boldsymbol{E}. \tag{8.51b}$$

In (8.51a) the negative sign arises because of the behaviour of spin under time reversal ($\boldsymbol{E}$ remains unchanged), whereas in (8.51b) the negative sign arises because under the parity transformation the spin remains unaltered

but $\boldsymbol{E}$ changes sign. The magnetic interaction is unchanged by either P or T. The existence of a particle with an electric-dipole moment would then be evidence for the violation of time reversal invariance and parity conservation in the electromagnetic interactions.

A description of the experiments to measure the electric-dipole moments of particles is given in the review article by Ramsey.[8] The most sensitive tests of P and T violation are measurements of the electric-dipole moment of the neutron and the current experimental limit is[9]

$$\mu_e < 1.2 \times 10^{-25} e \text{ cm}.$$

If the neutron possesses an electric-dipole moment at all, it is exceedingly small. With one very important exception, the weak decays of the K mesons, to which we shall return in chapter 11, all interactions are time reversal invariant.

8.10 *The CPT theorem*

There are no fundamental reasons why the forces in nature should be invariant under the transformations C, P and T separately, but taken together the combined operation of time reversal, space inversion and charge conjugation appears to be a fundamental symmetry transformation which has important and very general consequences. The CPT theorem[10–12] is based on very general assumptions of quantum field theory and relativity and states that any Hamiltonian which is invariant under proper Lorentz transformations is also invariant under the combined operation CPT, whether or not it is invariant under C, P or T separately.

The consequences of the CPT theorem are that the masses and lifetimes of particle and antiparticle should be exactly the same. No exception to this prediction has ever been found. Another consequence of the theorem is that if any individual or pair of symmetries is violated there must be a compensating asymmetry in the other operations, or operation, so that exact symmetry under CPT is upheld. As mentioned above, we shall see in chapter 11 that the weak interactions violate C and P separately but in general they are invariant under the combined operation of C and P. In the weak decays of the K^0 mesons a small but definite CP violation is evident but this is accompanied by a corresponding violation of T invariance so that CPT invariance is respected even in these decays.

8.11 *G-parity*

We recall that eigenstates of the charge conjugation operator must have $Q = B = L = S = 0$ so that, although particles such as the π^0, η^0, γ and

neutral, particle–antiparticle combinations such as e^+e^- and $\bar{p}p$ systems can be eigenstates of C, the number of eigenstates of C is somewhat restricted by the above condition. It was pointed out by Lee and Yang[13] that a useful conservation law for the strong interactions can be set up by combining the operation of charge conjugation with a rotation in isospin space. We will see that this removes the restriction that Q must be zero.

A rotation θ about an axis defined by a unit vector $\hat{\boldsymbol{n}}$ in isospin space is given by

$$R(\theta) = \exp(-i\boldsymbol{\tau}\cdot\hat{\boldsymbol{n}}\theta) \tag{8.52}$$

where τ is the isospin operator. For an isospin doublet, for example, τ_1, τ_2 and τ_3 are just the Pauli matrices. The operation G is defined as a rotation π about the y or 2 axis in isospin space followed by charge conjugation

$$G = C\exp(-i\tau_2\pi). \tag{8.53}$$

Consider the triplet of π mesons π^+, π^0, π^-. Charge conjugation has the effect of transforming $\pi^+ \to \pi^-$, $\pi^- \to \pi^+$ and $\pi^0 \to \pi^0$. The π^0 is an eigenstate of C with eigenvalue $+1$. For the charged pions we can have

$$C|\pi^+\rangle = \pm 1|\pi^-\rangle \qquad \text{and} \qquad C|\pi^-\rangle = \pm 1|\pi^+\rangle; \tag{8.54}$$

there is an arbitrary phase.

It follows from the commutation relations that the isospin operators have the same algebraic properties as the ordinary angular momentum operators. Under a rotation π about the y axis an angular momentum state $|j, m\rangle$ transforms as

$$\exp(-iJ_y\pi)|j, m\rangle = (-1)^{j-m}|j, -m\rangle.$$

Similarly, for rotations in isospin space we have

$$\exp(-i\tau_2\pi)|I, I_3\rangle = (-1)^{I-I_3}|I, -I_3\rangle.$$

Thus, for a rotation π about the 2 axis in isospin space we have

$$R_2(\pi)|\pi^+\rangle = |\pi^-\rangle \qquad R_2(\pi)|\pi^-\rangle = |\pi^+\rangle$$

and

$$R_2(\pi)|\pi^0\rangle = (-1)|\pi^0\rangle.$$

For the neutral pion the G-parity is therefore unambiguously equal to -1. Since the strong interactions conserve isospin and are invariant under

8.12

the charge conjugation operation, one might expect that the G-parity of the charged pions is the same as that of the neutral pion. In order to ensure this the phases in the charge conjugation operation (8.54) on the π^+ and π^- are chosen so that

$$C|\pi^{\pm}\rangle = (-1)|\pi^{\pm}\rangle.$$

Then, under the G-transformation

$$G|\pi^{+,-,0}\rangle = (-1)|\pi^{+,-,0}\rangle \tag{8.55}$$

so that the G-parity of the pion is -1. G-parity is a multiplicative quantum number and the G-parity of a system of n pions is $(-1)^n$. G-parity is a good quantum number for non-strange mesons and is conserved in the strong interactions. Its conservation leads to selection rules, for instance, in nucleon–antinucleon annihilation into pions. The $N\bar{N}$ system can have isospin $I = 0, 1$ and the G-parity of an $N\bar{N}$ system in a state with relative orbital angular momentum l and total spin s is

$$G(N\bar{N}) = (-1)^{I+l+s}.$$

Thus, if $I + l + s$ is even (odd) annihilation is possible only into an even (odd) number of pions.

8.12 *The electromagnetic field*[14,15]

We summarize in this section some of the properties of the electromagnetic field that are important in both nuclear and particle physics. The basic equations can be found in the references quoted.

8.12.1 Gauge invariance and Maxwell's equations

The classical electromagnetic field is described by Maxwell's equations in terms of the vectors $\boldsymbol{E}$, the electric component of the field, and $\boldsymbol{B}$, the magnetic component, together with ρ the charge density and $\boldsymbol{j}$ the current density. The equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0 \tag{8.56}$$

expresses the fact that electric charge is conserved and since the equation