8

The Structure of Hadrons

In Chapters 3 through 7, we have learned how to perform quantitative calculations for the electromagnetic interactions of leptons and quarks. The same techniques will allow us to compute the color interactions of quarks and gluons. There is, however, an immediate problem: experiments to study color (strong) interactions are performed with hadrons (e.g., proton beams or secondary π-beams interacting with nuclear targets), not with the quarks and gluons that are described by quantum field theory. This situation is similar to that encountered in atomic physics where experiments involving complex atoms have to be interpreted through the electromagnetic interactions of the constituent electrons. This analogy reveals the problem: we need to find the "wavefunctions" that describe, for example, a proton in terms of its constituent quarks and gluons. In this chapter, we present an experimental technique that allows us to determine the quark and gluon structure of hadrons; namely the deep inelastic scattering of leptons off hadronic targets. The structure functions so obtained will be presented in Chapter 9. Finally, in Chapters 10 and 11 we reach our goal and translate these quark–gluon QCD calculations into predictions for the results of experiments involving leptons and hadrons.

The discussion will soon reveal that these structure functions are not the static quark wavefunctions introduced in Chapter 2, although they are indirectly related to them.

8.1 Probing a Charge Distribution with Electrons. Form Factors

"Photographing" an object by scattering an electron beam off it is a well-proved technique in physics. Suppose we want to determine the charge distribution shown in Fig. 8.1, which could, for example, be the electron cloud of an atom. The procedure is to measure the angular distribution of the scattered electrons and compare it to the (known) cross section for scattering electrons from a point charge, in the form

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{point}} |F(q)|^2, \tag{8.1}$$

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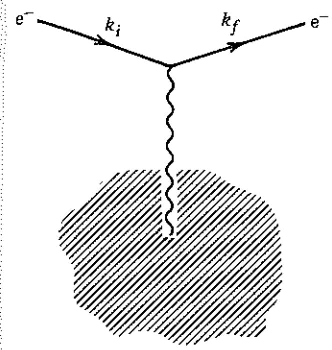

Fig. 8.1 Lowest-order electron scattering by a charge cloud.

where q is the momentum transfer between the incident electron and the target, $q = k_i - k_f$, see Fig. 8.1. We then attempt to deduce the structure of the target from the form factor $F(q)$ so determined.

We can gain insight into this technique by first looking at the scattering of unpolarized electrons of energy E from a static, spinless charge distribution $Ze\,\rho(\mathbf{x})$, normalized so that

$$\int \rho(\mathbf{x})\, d^3x = 1. \quad (8.2)$$

For a static target, it is found that the form factor in (8.1) is just the Fourier transform of the charge distribution

$$F(\mathbf{q}) = \int \rho(\mathbf{x}) e^{i\mathbf{q}\cdot\mathbf{x}}\, d^3x, \quad (8.3)$$

while the reference cross section for a structureless target is

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{point}} \equiv \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{(Z\alpha)^2 E^2}{4k^4 \sin^4 \dfrac{\theta}{2}}\left(1 - v^2 \sin^2 \frac{\theta}{2}\right), \quad (8.4)$$

where $k = |\mathbf{k}_i| = |\mathbf{k}_f|$, $v = k/E$, and θ is the angle through which the electron is scattered.

EXERCISE 8.1 It is useful practice of the techniques developed in the previous chapters to derive (8.3) and (8.4). We outline the various steps below. The electromagnetic field due to $Ze\,\rho(\mathbf{x})$ is $A^\mu = (\phi, \mathbf{0})$ where, using (6.59),

$$\nabla^2\phi = -Ze\,\rho(\mathbf{x}).$$

Use (6.4) and (6.6) to show that the scattering amplitude is (see also Section 7.1)

$$T_{fi} = -i2\pi\,\delta(E_f - E_i)(-e\bar{u}_f\gamma_0 u_i)\int e^{i\mathbf{q}\cdot\mathbf{x}}\phi(\mathbf{x})\,d^3x. \tag{8.5}$$

Justify

$$\int e^{i\mathbf{q}\cdot\mathbf{x}}\nabla^2\phi\,d^3x = -|\mathbf{q}|^2\int e^{i\mathbf{q}\cdot\mathbf{x}}\phi\,d^3x$$

and hence show that the integral in (8.5) is $Ze\,F(\mathbf{q})/|\mathbf{q}|^2$, see (7.9). Following the arguments of Section 4.3, verify that the differential cross section from a fixed target is

$$d\sigma = \frac{|T_{fi}|^2}{T}\,\frac{d^3k_f}{(2\pi)^3 2E_f}\left(\frac{1}{v2E_i}\right), \tag{8.6}$$

with

$$d^3k_f\,\delta(E_f - E_i) = kE\,d\Omega.$$

Summing final, and averaging initial, electron spins give

$$\frac{1}{2}\sum_{s_f,\,s_i}|\bar{u}_f\gamma_0 u_i|^2 = 4E^2\left(1 - v^2\sin^2\frac{\theta}{2}\right), \tag{8.7}$$

where θ is the angle introduced in Section 7.1. Check this answer with (6.25). Putting all this together yields the advertised result

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}}|F(\mathbf{q})|^2,$$

with the form factor given by (8.3).

EXERCISE 8.2 Show that if the electron beam is replaced by a beam of "point" spinless particles, the only change is that factor (8.7) is replaced by $4E^2$. This raises a question: why does the electron spin make no difference in the nonrelativistic limit, $v \to 0$? The remarks following (6.13) are the clue.

EXERCISE 8.3 By considering the electron helicity, explain why you would anticipate the $\cos^2(\theta/2)$ behavior of factor (8.7) in the extreme relativistic limit, see Section 6.6.

By virtue of the normalization condition, (8.2),

$$F(0) \equiv 1. \tag{8.8}$$

If $|\mathbf{q}|$ is not too large, we can expand the exponential in (8.3), giving

$$F(\mathbf{q}) = \int \left(1 + i\mathbf{q}\cdot\mathbf{x} - \frac{(\mathbf{q}\cdot\mathbf{x})^2}{2} + \cdots\right)\rho(\mathbf{x})\, d^3x$$
$$= 1 - \frac{1}{6}|\mathbf{q}|^2\langle r^2\rangle + \cdots, \tag{8.9}$$

where we have assumed that ρ is spherically symmetric, that is, a function of $r \equiv |\mathbf{x}|$ alone. The small-angle scattering therefore just measures the mean square radius $\langle r^2\rangle$ of the charge cloud. This is because in the small $|\mathbf{q}|$ limit the photon in Fig. 8.1 is soft and with its large wavelength can resolve only the size of the charge distribution $\rho(r)$ and is not sensitive to its detailed structure.

EXERCISE 8.4 If the charge distribution $\rho(r)$ has an exponential form, e^{-mr}, show, using (8.3), that the form factor

$$F(|\mathbf{q}|) \propto \left(1 - \frac{q^2}{m^2}\right)^{-2}$$

with $q^2 = -|\mathbf{q}|^2$.

8.2 Electron–Proton Scattering. Proton Form Factors

The above discussion cannot be applied directly to yield the structure of the proton. First, the proton's magnetic moment is involved in the scattering of the electron, not just its charge. Second, the proton is not static, but will recoil under the electron's bombardment. If, however, the proton were a point charge e with a Dirac magnetic moment $e/2M$, then we already know the answer. We can take over the result for electron–muon scattering, (6.50), and simply replace the mass of the muon by that of the proton:

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{lab}} = \left(\frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\right)\frac{E'}{E}\left\{\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right\}, \tag{8.10}$$

where the factor

$$\frac{E'}{E} = \frac{1}{1 + \frac{2E}{M}\sin^2\frac{\theta}{2}},$$

given by (6.48), arises from the recoil of the target.

Copying the calculation of the electron–muon cross section, the lowest-order amplitude for electron–proton elastic scattering, Fig. 8.2, is given by [see (6.8)]

$$T_{fi} = -i\int j_\mu\left(-\frac{1}{q^2}\right)J^\mu\, d^4x,$$

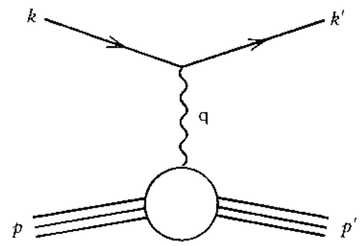

Fig. 8.2 Lowest-order electron–proton elastic scattering.

where $q = p' - p$ and the electron and proton transition currents are, respectively,

$$j^{\mu} = -e\,\bar{u}(k')\gamma^{\mu}u(k)\,e^{i(k'-k)\cdot x} \tag{8.11}$$

$$J^{\mu} = e\,\bar{u}(p')[\quad]u(p)\,e^{i(p'-p)\cdot x}, \tag{8.12}$$

see (6.6). Since the proton is an extended structure, we cannot replace the square brackets in (8.12) by γ^{μ}, as for point spin-$\frac{1}{2}$ particles in (8.11). But we know that J^{μ} must be a Lorentz four-vector, and so we must use the most general four-vector form that can be constructed from p, p', q and the Dirac γ-matrices sandwiched between $\bar{u}$ and u. There are only two independent terms, γ^{μ} and $i\sigma^{\mu\nu}q_{\nu}$, and their coefficients are functions of q^2 (q^2 is the only independent scalar variable at the proton vertex). Terms involving γ^5 are ruled out by the conservation of parity. Therefore, quite generally, we may write the square bracket of (8.12) in the form

$$[\quad] = \left[F_1(q^2)\gamma^{\mu} + \frac{\kappa}{2M}F_2(q^2)\,i\sigma^{\mu\nu}q_{\nu}\right] \tag{8.13}$$

where F_1 and F_2 are two independent form factors and κ is the anomalous magnetic moment (see Exercise 6.2).

EXERCISE 8.5 Show that current conservation, $\partial_{\mu}J^{\mu} = 0$, rules out $(p - p')^{\mu}$ as a possible four-vector. Why do we not show a term involving $(p + p')^{\mu}$ in (8.13)?

EXERCISE 8.6 Show that $p \cdot q$ is not an independent scalar variable by expressing it in terms of the variable q^2.

For $q^2 \to 0$, that is, when we probe with long-wavelength photons, it does not make any difference that the proton has structure at the order of 1 fermi. We effectively see a particle of charge e and magnetic moment $(1 + \kappa)e/2M$, where κ, the anomalous moment, is measured to be 1.79. The factors in (8.13) must therefore be chosen so that in this limit

$$F_1(0) = 1, \qquad F_2(0) = 1. \tag{8.14}$$

The corresponding values for the neutron are $F_1(0) = 0$, $F_2(0) = 1$, and experimentally $\kappa_n = -1.91$.

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If we use (8.13) to calculate the differential cross section for electron–proton elastic scattering, we find an expression similar to (8.10):

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{lab}} = \left(\frac{\alpha^2}{4E^2 \sin^4 \frac{\theta}{2}}\right) \frac{E'}{E} \left\{ \left(F_1^2 - \frac{\kappa^2 q^2}{4M^2} F_2^2 \right) \cos^2 \frac{\theta}{2} - \frac{q^2}{2M^2} (F_1 + \kappa F_2)^2 \sin^2 \frac{\theta}{2} \right\}, \tag{8.15}$$

see (6.50). This is known as the Rosenbluth formula. The two form factors, $F_{1,2}(q^2)$, parametrize our ignorance of the detailed structure of the proton represented by the blob in Fig. 8.2. These form factors can be determined experimentally by measuring $d\sigma/d\Omega$ as a function of θ and q^2. Note that if the proton were a point particle like the muon, then $\kappa = 0$ and $F_1(q^2) = 1$ for all q^2, and (8.15) would revert to (8.10).

In practice, it is better to use linear combinations of $F_{1,2}$,

$$\begin{aligned} G_E &\equiv F_1 + \frac{\kappa q^2}{4M^2} F_2 \\ G_M &\equiv F_1 + \kappa F_2, \end{aligned} \tag{8.16}$$

defined so that no interference terms, $G_E G_M$, occur in the cross section. Equation (8.15) becomes

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{lab}} = \frac{\alpha^2}{4E^2 \sin^4 \frac{\theta}{2}} \frac{E'}{E} \left(\frac{G_E^2 + \tau G_M^2}{1+\tau} \cos^2 \frac{\theta}{2} + 2\tau G_M^2 \sin^2 \frac{\theta}{2} \right), \tag{8.17}$$

with $\tau \equiv -q^2/4M^2$.

Now that interference terms have disappeared, these proton form factors may be regarded as generalizations of the nonrelativistic form factor introduced in Section 8.1, and so it would be nice if we could interpret their Fourier transforms as the charge and magnetic moment distributions of the proton. Unfortunately, the recoil of the proton makes this impossible. However, it is possible to show that the form factors $G_E(q^2)$ and $G_M(q^2)$ are closely related to the proton charge and magnetic moment distributions, respectively, in a particular Lorentz frame, called the Breit (or brick wall) frame, defined by $\mathbf{p}' = -\mathbf{p}$.

EXERCISE 8.7 Show that the proton transition current, $J^\mu(x)$ of (8.12), can be rewritten in the form

$$J^\mu(0) = e\,\bar{u}(p')\left[\gamma^\mu(F_1 + \kappa F_2) - \frac{(p^\mu + p'^\mu)}{2M}\kappa F_2\right] u(p). \tag{8.18}$$

Evaluate $J^\mu(0) \equiv (\rho, \mathbf{J})$ in the Breit frame ($\mathbf{p}' = -\mathbf{p}$). There is no energy

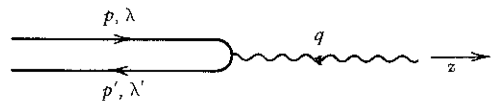


Fig. 8.3 The Breit or brick-wall frame, $\mathbf{p}' = -\mathbf{p}$.

transferred to the proton in this frame, and it behaves as if it had bounced off a brick wall, see Fig. 8.3. If the z axis is chosen along $\mathbf{p}$ and helicity spinors are used, show that

$$\rho = 2Me\, G_E(q^2) \qquad \text{for } \lambda = -\lambda',$$
$$J_1 \pm iJ_2 = \mp 2|\mathbf{q}|e\, G_M(q^2) \qquad \text{for } \lambda = \lambda' = \mp\tfrac{1}{2}, \tag{8.19}$$

and that all other matrix elements are zero; λ and λ' denote the initial and final proton helicities, respectively. Determine the corresponding values of the helicity of the virtual photon.

In generalizing the form factor of Section 8.1, we have replaced $F(|\mathbf{q}|)$ by $F(q^2)$. However, as long as $|\mathbf{q}|^2 \ll M^2$, we can take over the Fourier transform interpretation of Section 8.1.

EXERCISE 8.8 Show that for $|\mathbf{q}|^2 \ll M^2$, the form factors G_E and G_M are the Fourier transforms of the proton's charge and magnetic moment distributions, respectively.

G_E and G_M are referred to as the electric and magnetic form factors, respectively. The data on the angular dependence of ep → ep scattering can be used to separate G_E, G_M at different values of q^2, see (8.17). The result for $G_E(q^2)$ is

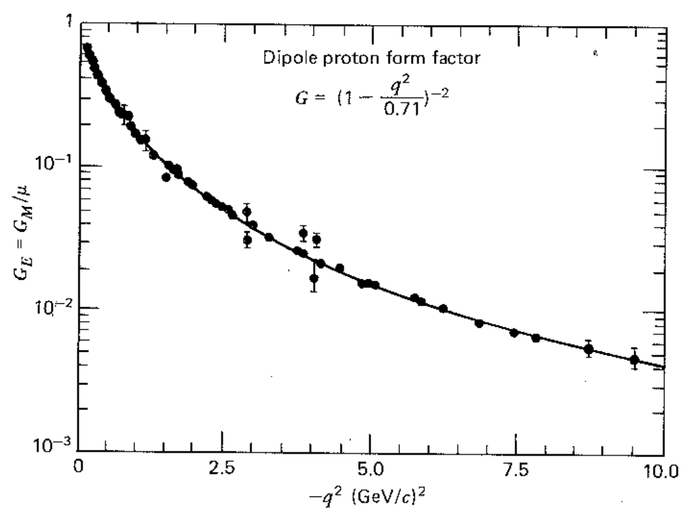


Fig. 8.4 The proton form factors as a function of q^2.

shown in Fig. 8.4. $G_M(q^2)$ has the same q^2 dependence. A closer look at Fig. 8.4 reveals that

$$G_E(q^2) \approx \left(1 - \frac{q^2}{0.71}\right)^{-2} \qquad \text{(in units of GeV}^2\text{)}. \tag{8.20}$$

The behavior for small $-q^2$ can be used to determine the residual terms in the expansion of (8.9). In particular, the mean square proton charge radius is

$$\langle r^2 \rangle = 6\left(\frac{dG_E(q^2)}{dq^2}\right)_{q^2=0} = (0.81 \times 10^{-13}\ \text{cm})^2. \tag{8.21}$$

The same radius of about 0.8 fm is obtained for the magnetic moment distribution. Using the result of Exercise 8.4, we conclude that the charge distribution of the nucleon has an exponential shape in configuration space.

8.3 Inelastic Electron–Proton Scattering ep → eX

Having measured the size of the proton, one might like to take a more detailed look at its structure by increasing the $-q^2$ of the photon to give better spatial resolution. This can be done simply by requiring a large energy loss of the bombarding electron. There is, however, a catch: because of the large transfer of energy, the proton will often break up, and the picture of Fig. 8.2 has to be generalized to Fig. 8.5. For modest $-q^2$, one might just excite the proton into a Δ-state and hence produce an extra π-meson, that is, ep → eΔ^+ → epπ^0. In these events, the invariant mass (see Fig. 8.5) is $W^2 \simeq M_\Delta^2$. When $-q^2$ is very large, however, the debris becomes so messy that the initial state proton loses its identity completely and a new formalism must be devised to extract information from the measurements. Figure 8.6 shows the invariant mass distribution. One notices the peak when the proton does not break up ($W = M$) and broader peaks when the target is excited to resonant baryon states. Beyond the resonances, the complicated multiparticle states with large invariant mass result in a smooth distribution in missing mass W.

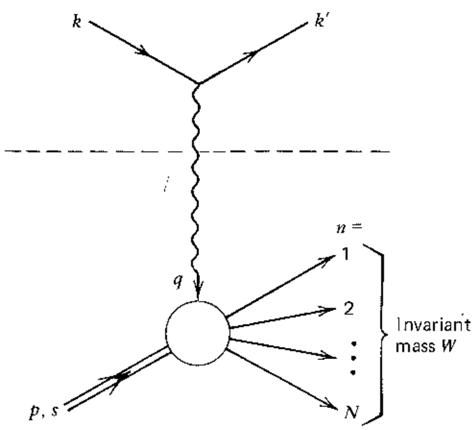


Fig. 8.5 Lowest-order diagram for ep → eX.

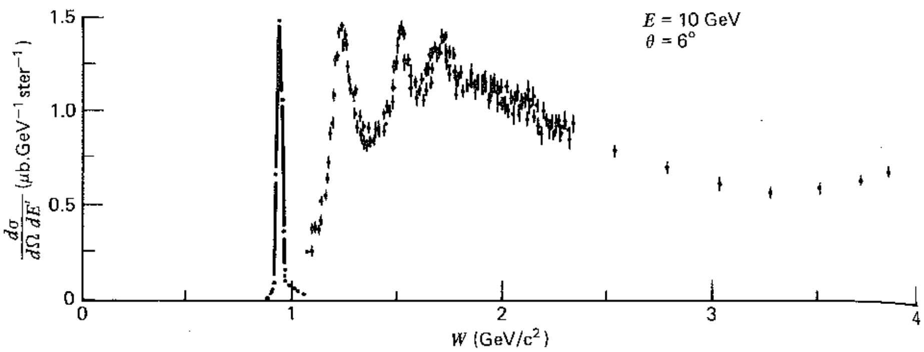


Fig. 8.6 The ep → eX cross section as a function of the missing mass W. Data are from the Stanford Linear Accelerator. The elastic peak at $W = M$ has been reduced by a factor of 8.5.

The problem now facing us is illustrated by recalling (8.11), (8.12), and Fig. 8.2. The switch from a muon to a proton target was made by replacing the lepton current j^μ ($\sim \bar{u}\gamma^\mu u$) by a proton current J^μ ($\sim \bar{u}\Gamma^\mu u$), and the most general form of Γ^μ was constructed. This is inadequate to describe the inelastic events of Fig. 8.5. Although everything above the dashed line in Fig. 8.5 remains unchanged (a fact which we shall exploit), the final state below it is not a single fermion described by a Dirac "$\bar{u}$" entry in the matrix element or current. Therefore, J^μ must have a more complex structure than (8.12). Instead, the expression for the cross section [see (6.18)]

$$d\sigma \sim L^e_{\mu\nu}(L^p)^{\mu\nu} \tag{8.22}$$

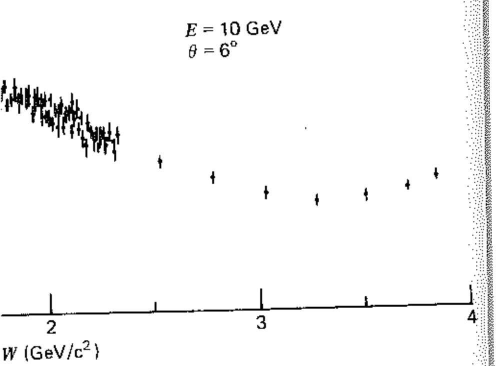
is directly generalized to

$$d\sigma \sim L^e_{\mu\nu}W^{\mu\nu}, \tag{8.23}$$

where $L^e_{\mu\nu}$ represents the lepton tensor of (6.20), since everything in the leptonic part of the diagram above the photon propagator in Fig. 8.5 is left unchanged. The hadronic tensor $W^{\mu\nu}$ serves to parametrize our total ignorance of the form of the current at the other end of the propagator. The most general form of the tensor $W^{\mu\nu}$ must now be constructed out of $g^{\mu\nu}$ and the independent momenta p and q ($p' = p + q$). γ^μ is not included, as we are parametrizing the cross section which is already summed and averaged over spins. We write

$$W^{\mu\nu} = -W_1 g^{\mu\nu} + \frac{W_2}{M^2}p^\mu p^\nu + \frac{W_4}{M^2}q^\mu q^\nu + \frac{W_5}{M^2}(p^\mu q^\nu + q^\mu p^\nu). \tag{8.24}$$

We have omitted antisymmetric contributions to $W^{\mu\nu}$, since their contribution to the cross section vanishes after insertion into (8.23) because the tensor $L^e_{\mu\nu}$ is symmetric. Note the omission of W_3 in our notation; this spot is reserved for a parity-violating structure function when a neutrino beam is substituted for the electron beam, so that the virtual photon probe is replaced by a weak boson; see Perl (1974), Close (1979), or Llewellyn Smith (1972).



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EXERCISE 8.9 Show that indeed $L^e_{\mu\nu} = L^e_{\nu\mu}$ and that

$$q^\mu L^e_{\mu\nu} = q^\nu L^e_{\mu\nu} = 0. \qquad (8.25)$$

EXERCISE 8.10 Show that current conservation at the hadronic vertex requires

$$q_\mu W^{\mu\nu} = q_\nu W^{\mu\nu} = 0. \qquad (8.26)$$

The proof may be left until after (8.39); it follows from $\partial_\mu \tilde{J}^\mu = 0$. As a result of (8.26), verify that

$$W_5 = -\frac{p \cdot q}{q^2} W_2,$$

$$W_4 = \left(\frac{p \cdot q}{q^2}\right)^2 W_2 + \frac{M^2}{q^2} W_1.$$

Thus, only two of the four inelastic structure functions of (8.24) are independent; so we may write

$$W^{\mu\nu} = W_1\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) + W_2 \frac{1}{M^2}\left(p^\mu - \frac{p \cdot q}{q^2} q^\mu\right)\left(p^\nu - \frac{p \cdot q}{q^2} q^\nu\right), \qquad (8.27)$$

where the W_i's are functions of the Lorentz scalar variables that can be constructed from the four-momenta at the hadronic vertex. Unlike elastic scattering, there are two independent variables, and we choose

$$q^2 \quad \text{and} \quad \nu \equiv \frac{p \cdot q}{M}. \qquad (8.28)$$

The invariant mass W of the final hadronic system is related to ν and q^2 by

$$W^2 = (p + q)^2 = M^2 + 2M\nu + q^2. \qquad (8.29)$$

EXERCISE 8.11 It is common to replace ν and q^2 by the dimensionless variables

$$x = \frac{-q^2}{2p \cdot q} = \frac{-q^2}{2M\nu}, \qquad y = \frac{p \cdot q}{p \cdot k}, \qquad (8.30)$$

where the four-momenta are shown on Fig. 8.5. Show that the allowed kinematic region for ep → eX is $0 \le x \le 1$ and $0 \le y \le 1$. Sketch this physical region in the ν, q^2 plane and check your answer with Fig. 9.3.

EXERCISE 8.12 Show that in the rest frame of the target proton,

$$\nu = E - E', \qquad y = \frac{E - E'}{E},$$

where E and E' are the initial and final electron energies, respectively.

Evaluation of the cross section for ep → eX is a straightforward repetition of the same calculation for $e^-\mu^- \to e^-\mu^-$ (or ep → ep) scattering with the substitution of $W_{\mu\nu}$, given by (8.27), for $L_{\mu\nu}^{\text{muon}}$ (or $L_{\mu\nu}^{p}$). Using the expression (6.25) for $(L^e)^{\mu\nu}$ and noting (8.25), we find

$$(L^e)^{\mu\nu} W_{\mu\nu} = 4W_1(k \cdot k') + \frac{2W_2}{M^2}\left[2(p \cdot k)(p \cdot k') - M^2 k \cdot k'\right]. \tag{8.31}$$

In the laboratory frame, this becomes

$$(L^e)^{\mu\nu} W_{\mu\nu} = 4EE'\left\{\cos^2\frac{\theta}{2} W_2(\nu, q^2) + \sin^2\frac{\theta}{2} 2W_1(\nu, q^2)\right\}, \tag{8.32}$$

see (6.44). By including the flux factor, (4.32), and the phase space factor for the outgoing electron, (4.24), we can obtain the inclusive differential cross section for inelastic electron–proton scattering, ep → eX,

$$d\sigma = \frac{1}{4\left((k \cdot p)^2 - m^2 M^2\right)^{1/2}} \left\{\frac{e^4}{q^4}(L^e)^{\mu\nu} W_{\mu\nu} 4\pi M\right\} \frac{d^3k'}{2E'(2\pi)^3}, \tag{8.33}$$

where $\overline{|\mathfrak{M}|^2}$ is given by the expression in the braces [recall (6.18)]. The extra factor of $4\pi M$ arises because we have adopted the standard convention for the normalization of $W^{\mu\nu}$. Inserting (8.32) in (8.33) yields

$$\boxed{\left.\frac{d\sigma}{dE'\,d\Omega}\right|_{\text{lab}} = \frac{\alpha^2}{4E^2 \sin^4\frac{\theta}{2}}\left\{W_2(\nu, q^2)\cos^2\frac{\theta}{2} + 2W_1(\nu, q^2)\sin^2\frac{\theta}{2}\right\},} \tag{8.34}$$

where, as usual, we neglect the mass of the electron.

8.4 Summary of the Formalism for Analyzing ep Scattering

Although we have achieved our objective, it is informative to take a second look at the formalism. The final result, (8.34), may be reexpressed in the form

$$\frac{d\sigma}{dE'\,d\Omega} = \frac{\alpha^2}{q^4}\frac{E'}{E}(L^e)^{\mu\nu}W_{\mu\nu}, \tag{8.35}$$

see (8.32) and (6.44). For comparison, recall the cross section for $e\mu^- \to e\mu^-$ scattering, (6.46),

$$d\sigma = \frac{1}{4ME}\frac{d^3k'}{(2\pi)^3 2E'}\frac{d^3p'}{(2\pi)^3 2p_0'}\left\{\frac{e^4}{q^4}(L^e)^{\mu\nu}L_{\mu\nu}^{\text{muon}}\right\}(2\pi)^4\delta^4(p+q-p'), \tag{8.36}$$

which can also be written in the form (8.35) with

$$W_{\mu\nu} = \frac{1}{4\pi M}\left(\frac{1}{2}\sum_s\sum_{s'}\right)\int\frac{d^3p'}{(2\pi)^3 2p_0'}\langle p,s|\tilde{J}_\mu^\dagger|p',s'\rangle$$
$$\times\langle p',s'|\tilde{J}_\nu|p,s\rangle(2\pi)^4\delta^4(p+q-p') \tag{8.37}$$

where

$$\langle p',s'|\tilde{J}_\nu|p,s\rangle \equiv \bar{u}^{(s')}(p')\gamma_\nu u^{(s)}(p). \tag{8.38}$$

Insertion of (8.37) into (8.35) reproduces (6.49). All we have done is to use a particular regrouping of the factors in our earlier derivation. If we replace (8.38) by $\bar{u}[\quad]u$ with $[\quad]$ given by (8.13), then we can recover the result for ep → ep scattering. The reason for all this is to note that $W_{\mu\nu}$ for ep → eX is nothing but a generalization of (8.37) to the case where a proton breaks up into many particles in the final hadronic state X. It can be formally written as

$$W_{\mu\nu} = \frac{1}{4\pi M}\sum_N\left(\frac{1}{2}\sum_s\right)\int\prod_{n=1}^{N}\left(\frac{d^3p_n'}{2E_n'(2\pi)^3}\right)\sum_{s_n}\langle p,s|\tilde{J}_\mu^\dagger|X\rangle$$
$$\times\langle X|\tilde{J}_\nu|p,s\rangle(2\pi)^4\delta^4\Big(p+q-\sum_n p_n'\Big), \tag{8.39}$$

where a sum over all possible many-particle (i.e., N particle) states X is included, see Fig. 8.5.

For future reference, it is useful to make a compendium of our results on form factors. We keep to the laboratory kinematics of Fig. 6.10 and neglect the mass of the electron. For all the reactions, the differential cross section in the energy (E') and angle (θ) of the scattered electron can be written in the form

$$\frac{d\sigma}{dE'\,d\Omega} = \frac{4\alpha^2 E'^2}{q^4}\{\quad\}. \tag{8.40}$$

First, for a muon target of mass m (or a quark target of mass m after substitution $\alpha^2 \to \alpha^2 e_q^2$ where e_q is the quark's fractional charge),

$$\{\quad\}_{e\mu\to e\mu} = \left(\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}\sin^2\frac{\theta}{2}\right)\delta\left(\nu + \frac{q^2}{2m}\right). \tag{8.41}$$

For elastic scattering from a proton target,

$$\{\quad\}_{ep\to ep} = \left(\frac{G_E^2 + \tau G_M^2}{1+\tau}\cos^2\frac{\theta}{2} + 2\tau G_M^2\sin^2\frac{\theta}{2}\right)\delta\left(\nu + \frac{q^2}{2M}\right), \tag{8.42}$$

where $\tau = -q^2/4M^2$ and M is the mass of the proton. Finally, for the case when the proton target is broken up by the bombarding electron,

$$\{\quad\}_{ep\to eX} = W_2(\nu, q^2)\cos^2\frac{\theta}{2} + 2W_1(\nu, q^2)\sin^2\frac{\theta}{2}. \tag{8.43}$$

Making use of the delta function, (8.41) and (8.42) can be integrated over E' with the result [see (6.50)]

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2\sin^4\frac{\theta}{2}}\frac{E'}{E}(\quad). \tag{8.44}$$

EXERCISE 8.13 The above results assume (lowest-order) single photon exchange is dominant. If two-photon exchange were significant, convince yourself that the e^-p and e^+p cross sections would not be equal.

8.5 Inelastic Electron Scattering as a (Virtual) Photon–Proton Total Cross Section

It is clear from the above discussion that the important issue is what happens below the dashed line in Fig. 8.5, where a (virtual) photon interacts with a proton. The role of the electron beam is simply that it is responsible for the presence of the virtual photon. It is useful to display these facts in our formalism. We start by writing the total cross section for scattering a *real* photon, with energy $q^0 = \nu \equiv K$ and (transverse) polarization ε, off the same unpolarized proton target producing two or more final-state particles. Using the Feynman rules and cross section kinematics which we have developed, we obtain

$$\sigma^{tot}(\gamma p \to X) = \frac{1}{(2K)(2M)}\sum_N\left(\frac{1}{2}\sum_s\right)\int\prod_{n=1}^{N}\left(\frac{d^3p_n'}{2E_n'(2\pi)^3}\right)$$
$$\times\sum_{s_n}(2\pi)^4\delta^4\left(p + q - \sum_n p_n'\right)\varepsilon^{\mu *}\varepsilon^{\nu}e^2\langle p, s|\tilde{J}_\mu^\dagger|X\rangle\langle X|\tilde{J}_\nu|p, s\rangle, \tag{8.45}$$

where, as before, a sum over all final states X is included. If W is the invariant mass of the final state, then

$$W^2 = (p+q)^2 = M^2 + 2MK. \tag{8.46}$$

We immediately note the striking similarity between (8.45) and the formal expression for $W_{\mu\nu}$ given by (8.39). Indeed, (8.45) appears to be simply

$$\sigma^{\text{tot}}(\gamma p \to X) = \frac{4\pi^2\alpha}{K}\varepsilon^{\mu *}\varepsilon^{\nu} W_{\mu\nu}. \tag{8.47}$$

However, there is a crucial proviso: for real photons, we must sum only over the two transverse polarizations of the incident photon. On the other hand, to interpret the ep $\to$ eX hadronic tensor $W_{\mu\nu}$ as a photon–proton total cross section, it is vital to remember that the photon is *virtual* and not limited to two polarization states. In fact, the cross section for virtual photons is not a well-defined concept. When $q^2 = 0$, the flux factor is the $4MK$ with $K = \nu$; but for virtual photons ($q^2 \neq 0$), the flux is arbitrary. The conventional choice is to require K to continue to satisfy (8.46), that is,

$$K = \frac{W^2 - M^2}{2M} = \nu + \frac{q^2}{2M}, \tag{8.48}$$

in the laboratory frame. This is known as the Hand convention. Another possible choice is $K = |\mathbf{q}|$.

To complete the interpretation of the structure functions, we must specify the polarization vectors $\varepsilon^{\mu}_{\lambda}$ of virtual photons (helicity λ). We take the z axis along $\mathbf{q}$ and use [see (6.92); a further discussion is given in Budnev et al. Phys. Rep *C15*, 181 (1975)]

$$\lambda = \pm 1: \varepsilon_{\pm} = \mp\sqrt{\tfrac{1}{2}}\,(0; 1, \pm i, 0), \tag{8.49}$$

$$\lambda = 0: \quad \varepsilon_0 = \frac{1}{\sqrt{-q^2}}\left(\sqrt{\nu^2 - q^2}\,; 0, 0, \nu\right). \tag{8.50}$$

EXERCISE 8.14 Verify that $q \cdot \varepsilon = 0$ for each λ, and show that, for a spacelike photon ($q^2 < 0$),

$$\sum (-1)^{\lambda+1} \varepsilon^{\mu *}\varepsilon^{\nu} = -g^{\mu\nu} + \frac{q^{\mu}q^{\nu}}{q^2}, \tag{8.51}$$

where the sum runs over the three polarization states of (8.49) and (8.50).
For $q^2 > 0$ the factor $(-1)^{\lambda+1}$ is omitted and the sum (8.51) with q^2 replaced by M^2 is identical to that over the spin states of a massive vector particle, see Section 6.12.

We can now evaluate the total cross sections for polarized photons (helicity λ) interacting with unpolarized protons:

$$\sigma^{\text{tot}}_{\lambda} = \frac{4\pi^2\alpha}{K}\varepsilon^{\mu *}_{\lambda}\varepsilon^{\nu}_{\lambda} W_{\mu\nu}. \tag{8.52}$$

Using (8.24) for $W_{\mu\nu}$, together with the above polarization vectors, we find that the transverse and longitudinal cross sections are, respectively,

$$\sigma_T \equiv \frac{1}{2}\left(\sigma_+^{\text{tot}} + \sigma_-^{\text{tot}}\right) = \frac{4\pi^2\alpha}{K} W_1(\nu, q^2) \tag{8.53}$$

$$\sigma_L \equiv \sigma_0^{\text{tot}} = \frac{4\pi^2\alpha}{K}\left[\left(1 - \frac{\nu^2}{q^2}\right) W_2(\nu, q^2) - W_1(\nu, q^2)\right] \tag{8.54}$$

EXERCISE 8.15 Verify eqs. (8.53) and (8.54). The calculation can be greatly simplified by writing the tensor decomposition for $W_{\mu\nu}$, (8.27), in the laboratory frame, where

$$p = (M; 0, 0, 0),$$

$$q = \left(\nu; 0, 0, \sqrt{\nu^2 - q^2}\right).$$

EXERCISE 8.16 Express the ep → eX differential cross section (8.34) in terms of $\sigma_{T,L}$. That is, show that

$$\left.\frac{d\sigma}{dE'\,d\Omega}\right|_{\text{lab}} = \Gamma(\sigma_T + \varepsilon\sigma_L), \tag{8.55}$$

where

$$\Gamma = \frac{\alpha K}{2\pi^2 |q^2|} \frac{E'}{E} \frac{1}{1-\varepsilon}, \tag{8.56}$$

$$\varepsilon = \left(1 - 2\frac{\nu^2 - q^2}{q^2} \tan^2 \frac{\theta}{2}\right)^{-1}. \tag{8.57}$$

EXERCISE 8.17 The formalism has been set up in such a way that, when $q^2 \to 0$,

$$\sigma_T \to \sigma^{\text{tot}}(\gamma \text{p})$$

$$\sigma_L \to 0, \tag{8.58}$$

where γ is a real photon and $\sigma^{\text{tot}}(\gamma\text{p})$ is given by (8.45).

Despite its appearance, convince yourself that $W_{\mu\nu}$ must not be singular at $q^2 = 0$. Hence, show that

$$W_2 \to 0 \qquad \text{and} \qquad \left(W_1 + \frac{\nu^2}{q^2} W_2 \right) \to 0 \tag{8.59}$$

as $q^2 \to 0$, and so establish that σ_L vanishes.

Can we extract additional information about the structure of the proton from these complex events where the proton breaks up? Both the structure of the events and their phenomenological interpretation look quite forbidding. The answer to this question is of great importance and is the subject of the next two chapters.