

HOMEWORK 3 SOLUTIONS - PH221

Oops! Beware: this problem has
Problem 1 been changed a bit.

The idea here is to find the energy-independent nuclear interaction length, and then look up the corresponding energy in the PDG range-energy plot.

For Fe, the nuclear interaction length is $\lambda_s = 132 \text{ g/cm}^2$. Now, refer to the "Mean Range and Energy Loss in Lead..." plot in the PDG. In the periodic table, Fe is very close to Cu, so we can use the Cu result. Thus, we see that a pion with incident momentum of about 400 MeV/c has a range of about 130 g/cm². Since the π mass is substantially less than this, this is also the approximate energy.

400 MeV

The astute reader will notice that, in actuality, this is in a region where baryonic resonances (Δ 's) are produced, so the cross section is not in its asymptotic near-constant form, and so corrections to the above number may be substantial.

Problem 2

From the Particle Data Booklet, the energy loss rate for minimum-ionizing muons passing through aluminum is $1.6 \text{ MeV-g}^{-1}\text{-cm}^2$,

and through carbon it is $1.75 \text{ MeV-g}^{-1}\text{-cm}^2$.

Si and

O lie on either side of

aluminum in the periodic

table, with a weighted

mean somewhat less than

aluminum. Thus, approximately,

Element Z

C 6

O 8

Al 13

Si 14

$$-\frac{dE}{dx} \approx 1.65 \frac{\text{MeV}}{\text{g}} \cdot \text{cm}^2$$

For a 1,000 GeV (1 TeV) muon, then, this would correspond to

$$\Delta x = \frac{10^6 \text{ MeV}}{1.65 \frac{\text{MeV}}{\text{g}} \cdot \text{cm}^2} = 6.1 \times 10^5 \text{ g/cm}^2$$

In terms of actual distance travelled, this amounts to

$$\Delta d = \frac{\Delta x}{\rho} = \frac{6.1 \times 10^5 \text{ g/cm}^2}{2.64 \text{ g/cm}^3} = 2.3 \times 10^5 \text{ cm} \approx 2.3 \text{ km!}$$

where the density ρ is also available from the PDG booklet.

Now, for the electron component, the front will radiate all but $1/e$ of its energy ~~after~~ for each radiation length traversed, i.e.,

$$E(x) = E(0) \cdot e^{-x/x_0}$$

From the PDG, for Si, $x_0 \approx 22 \text{ g/cm}^2$, and for O₂, $x_0 \approx 34 \text{ g/cm}^2$. Thus, for SiO₂, $x_0 \approx 30 \text{ g/cm}^2$, and since $\rho = 2.6 \text{ g/cm}^3$, $x_0 \approx 12 \text{ cm}$. Thus, at this depth (2.3 km)

$$E(x) = E(0) \cdot e^{-2.3 \times 10^5 / 12} = E(0) \cdot e^{-1.9 \times 10^4} = 0$$

The electron's (electromagnetic) component has been completely eliminated.

Problem 3

From the Particle Data Booklet, the attenuation length of a 3 MeV γ is $\sim 30 \text{ g/cm}^2$ in Al. Thus, the attenuation length in cm is

$$\lambda = \frac{30 \text{ g/cm}^2}{2.7 \text{ g/cm}^3} = 11.1 \text{ cm.}$$

Now, 90% attenuation \Rightarrow 10% retention

$$\Rightarrow e^{-x/\lambda} = 0.1$$

$$\Rightarrow x = -\lambda \ln(0.1) = \boxed{25.6 \text{ cm}} \quad 3 \text{ MeV } \gamma's.$$

Problem 4

For two variables x_1 and x_2 , we have the joint p.d.f.

$$f(x_1, x_2) = \frac{1}{2\pi} |S|^{-1/2} \exp \left[-\frac{1}{2} \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix}^T S^{-1} \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix} \right]$$

with the "error matrix" S given by (S^{-1} is known as the "weight matrix")

$$S = \begin{pmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix}$$

Taking the determinant,

$$|S| = (1-\rho^2) \sigma_1^2 \sigma_2^2$$

yielding the weight matrix

$$(S)^{-1} = \frac{1}{(1-\rho^2)} \begin{pmatrix} 1/\sigma_1^2 & -\rho/\sigma_1 \sigma_2 \\ -\rho/\sigma_1 \sigma_2 & 1/\sigma_2^2 \end{pmatrix}$$

Putting this all together, we find

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-g^2}} \exp \left\{ -\frac{1}{2(1-g^2)} \left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} - \frac{2g(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} \right] \right\}$$

Note that for $g=0$, this becomes

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2} \exp \left\{ -\frac{1}{2} \left[\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} \right] \right\}$$

$$\begin{aligned} &= \frac{1}{\sqrt{2\pi}\sigma_1} \exp \left\{ -\frac{1}{2} \frac{(x_1-\mu_1)^2}{\sigma_1^2} \right\} \frac{1}{\sqrt{2\pi}\sigma_2} \exp \left\{ -\frac{1}{2} \frac{(x_2-\mu_2)^2}{\sigma_2^2} \right\} \\ &= f(x_1) f(x_2) \end{aligned}$$

the product of two independent gaussians.

Finally, in the easier-to-calculate case $\sigma_1 = \sigma_2 = 1$ and $\mu_1 = \mu_2 = 0$, we have

$$f(x_1, x_2) = \frac{1}{2\pi\sqrt{\beta}} \exp\left\{-\frac{1}{2\beta}[x_1^2 - 2gx_1x_2 + x_2^2]\right\}$$

where $\beta = 1 - g^2$. Thus, the covariance $\langle (x_1 - \mu_1)(x_2 - \mu_2) \rangle$ is

$$\langle x_1 x_2 \rangle = \frac{1}{2\pi\sqrt{\beta}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\beta}[x_1^2 - 2gx_1x_2 + x_2^2]\right\} x_1 x_2 dx_1 dx_2$$

Completing the square,

$$\langle x_1 x_2 \rangle = \frac{1}{2\pi\sqrt{\beta}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\beta}[(x_1 - gx_2)^2 + \beta x_2^2]\right\} x_1 x_2 dx_1 dx_2$$

Let $x_1 \rightarrow x_1 - gx_2$, so $dx_1 \rightarrow dx_1$, and

$$\langle x_1 x_2 \rangle = \frac{1}{2\pi\sqrt{\beta}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\beta}[x_1^2 + \beta x_2^2]\right\} (x_1 + gx_2) dx_1 x_2 dx_2$$

$$= \frac{1}{2\pi\sqrt{\beta}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cancel{\exp\left\{-\frac{x_1^2}{2\beta}\right\}} x_1 dx_1 x_2 dx_2 +$$

$$+ \frac{1}{2\pi\sqrt{\beta}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{-\frac{x_1^2}{2\beta}\right\} \exp\left\{-\frac{x_2^2}{2}\right\} gx_2^2 dx_1 dx_2$$

Now, let $X \rightarrow X/\sqrt{\beta}$, so $dx \rightarrow \sqrt{\beta} x$ and

$$\begin{aligned}\langle x_1 x_2 \rangle &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\exp\left\{-\frac{x_1^2}{2}\right\}}_{\sqrt{\frac{1}{2\pi}}} \exp\left\{-\frac{x_2^2}{2}\right\} g x_2^2 dx_1 dx_2 \\ &= \frac{g}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{x_2^2}{2}\right\} g x_2^2 dx_2\end{aligned}$$

But, we know that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{x_2^2}{2}\right\} x_2^2 dx_2 = \sigma_2^2 = 1, \text{ so}$$

$$\boxed{\langle x_1 x_2 \rangle = g}$$

as desired.

Problem 5

Consider an event that is equally likely to happen anywhere on the unit interval. The rms of this distribution is given by

$$\langle x^2 \rangle = \frac{\int_{-1/2}^{1/2} x^2 dx}{\int_{-1/2}^{1/2} dx} = \frac{x^3/3 \Big|_{-1/2}^{1/2}}{x \Big|_{-1/2}^{1/2}} = \frac{\frac{1}{24} + \frac{1}{24}}{1} = \frac{1}{12}$$

so

$$\sigma = \sqrt{\langle x^2 \rangle - \frac{1}{12}}$$

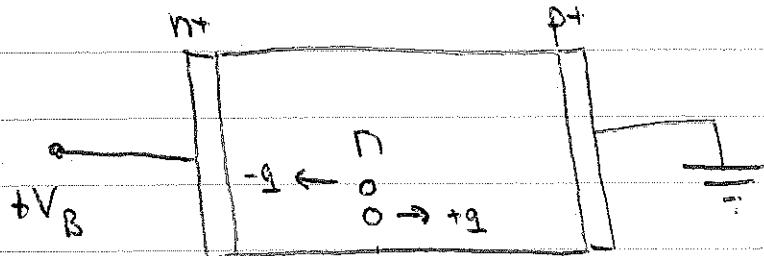
Thus, for 2mm spacing

$$\sigma_x = \frac{2\text{mm}}{\sqrt{12}} = 580\mu\text{m}$$
 is
the resolution.

Sharing only helps the resolution - if two adjacent wires are hit, the passage of the particle is localized to whatever the overlap region is. If pulse-height information is available, taking a weighted mean can give even better resolution.

Problem 6

The detector described in the problem looks like



the diagram at left.

Assuming that the detector is fully depleted, we know from class that $E=0$

at the n/p junction, rising linearly to its maximum value

$$|E| = eN_d d / \epsilon \text{ at}$$

the n/p junction, with the sign of E given by the direction

of motion of the positive charge. Here, N_d is the number density of donors, and ϵ the dielectric constant of Si.

Recalling our result for the conductivity σ (or resistivity ρ) for N-type materials

$$\sigma = 1/\rho = e N_d \mu_e$$

We can write

$$E = \frac{e N_d}{\epsilon} x = \frac{\sigma}{\mu_e \epsilon} x = \frac{x}{\mu_e \tau} \quad \tau = \epsilon / \sigma = \epsilon \rho$$

analogous to the expression $E = -\frac{x}{\mu_e \tau}$ for p-type.

(3.10)

The other difference between n-type and p-type detectors is that the electron (holes) are being swept into the low (high) field region for n-type, opposite to the p-type case. Let's see explicitly how this plays out:

Electrons

$$V = \frac{dx}{dt} = -\mu_e E = -\frac{\mu_e}{\mu_B} \frac{X}{I} = -\frac{X}{\tau}$$

and so

$$x_e(t) = x_0 \exp\left(-\frac{t}{\tau}\right)$$

Paralleling the development from class,

$$\frac{dQ}{dt} = \frac{q}{d} \frac{dx}{dt} = \left(-\frac{e}{d}\right) \left(-\frac{x_0}{\tau}\right) \exp\left(-\frac{t}{\tau}\right)$$

$$Q_0(t) = \frac{ex_0}{d} \int_0^t \exp\left(-\frac{t'}{\tau}\right) dt' = \frac{ex_0}{d} \int_0^{t/\tau} \exp(-y) dy$$

$$= -\frac{ex_0}{d} \left[\exp(-y) \right]_0^{t/\tau} = \boxed{\frac{ex_0}{d} \left[1 - \exp\left(-\frac{t}{\tau}\right) \right]}$$

Holes

$$V = \frac{dx}{dt} = \mu_n E = \frac{\mu_n}{\mu_e} \frac{x}{T}$$

$x_h(t) = x_0 \exp\left(\frac{\mu_n}{\mu_e} \frac{E}{T}\right)$. To make things simpler,
define

$$A = \frac{\mu_n}{\mu_e T}$$

- so

$$x_h(t) = x_0 \exp(At)$$

Then,

$$\frac{dQ}{dt} = \frac{e}{d} \frac{dx}{dt} = \frac{ex_0}{d} A \exp(At) \quad \text{and so}$$

$$Q_h(t) = \int_0^t \frac{ex_0}{d} A \exp(At') dt' = \int_0^{At} \frac{ex_0}{d} \exp(y) dy$$

$$= \frac{ex_0}{d} \left[\exp(y) \right]_0^{At} = \frac{ex_0}{d} \left[\exp(At) - 1 \right]$$

until the time at which the hole reaches the implant,
given by

$$d = x_0 e^{At} \Rightarrow At = \ln d/x_0, \text{ so until}$$

$$t = \frac{1}{A} \ln \frac{d}{x_0} = \frac{\mu_e}{\mu_h} \tau \ln \left(\frac{d}{x_0} \right)$$

the pulse develops according to the hole-related Eq.

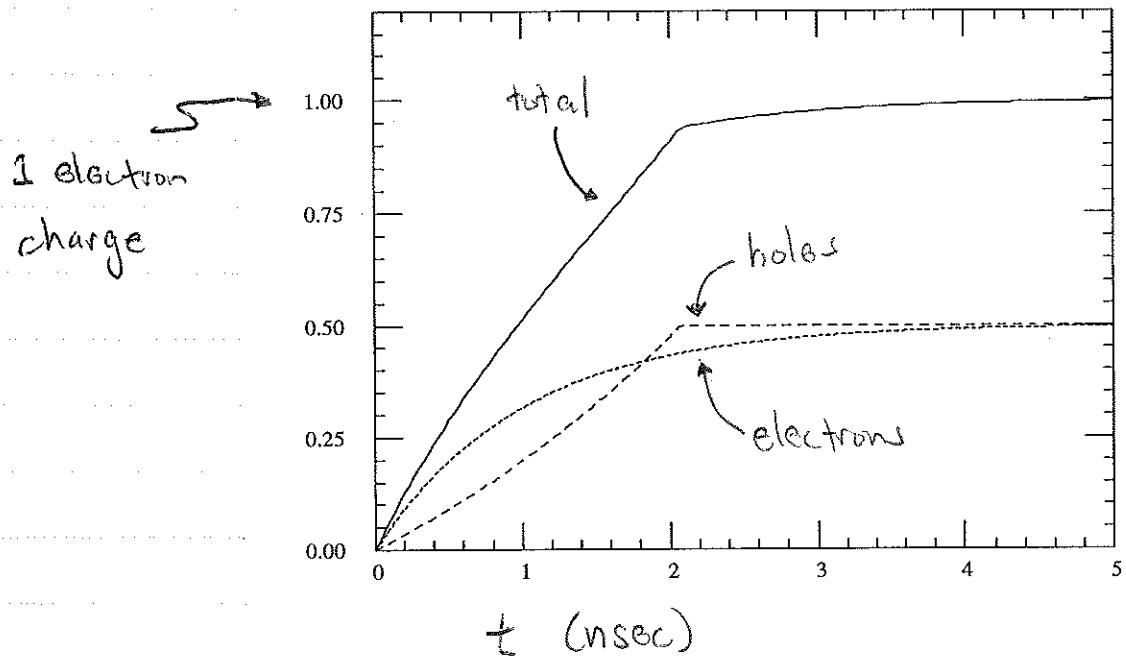
$$Q_h(t) = \frac{e x_0}{d} \left[\exp \left(\frac{\mu_h t}{\mu_e \tau} \right) - 1 \right]$$

Plugging in $\mu_e = 1500 \text{ cm}^2/\text{V-s}$, $\mu_h = 500 \text{ cm}^2/\text{V-s}$, and $\tau = \tau_p = 10^{-9} \text{ s}$, we have

$$Q_h(t) = \frac{e}{2} \left[1 - \exp(-t/\tau) \right]$$

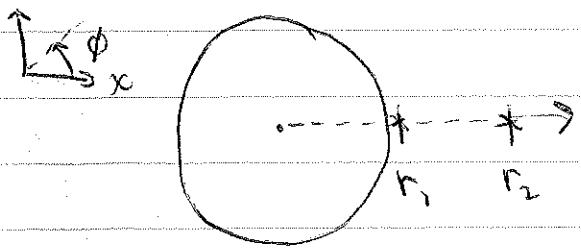
$$Q_h(t) = \frac{e}{2} \left[\exp \left(\frac{1}{3} \frac{t}{\tau} \right) - 1 \right] \text{ until } t = 3\tau \ln(2)$$

These two functions, plus the total, is shown on the next page. The signal development in n-type is similar for both electrons + holes, w/out the extremely fast electron component observed for p-type.



Problem 7

Without loss of generality, assume the track is exiting the beam pipe at $\phi = 0$.



Thus, the track is constrained by a measurement of the y coordinate y_1 at r_1 , and the y coordinate y_2 at r_2 .

The reconstructed slope is then just

$$m = \frac{y_2 - y_1}{r_2 - r_1}$$

and the impact parameter b is then

$$b = y_1 - mr_1 = y_1 - \frac{y_2 - y_1}{r_2 - r_1} r_1 = \frac{y_1 r_2 - y_1 r_1 - y_2 r_1 + y_1 r_1}{r_2 - r_1}$$

$$= \frac{y_1 r_2 - y_2 r_1}{r_2 - r_1}$$

So, then

$$db = \frac{r_2}{r_2 - r_1} dy_1 + \frac{-r_1}{r_2 - r_1} dy_2$$

and so

$$\sigma_b^2 = \left(\frac{r_2}{r_2 - r_1} \right)^2 \sigma_{y_1}^2 + \left(\frac{-r_1}{r_2 - r_1} \right)^2 \sigma_{y_2}^2$$

$$= \frac{r_2^2 + r_1^2}{(r_2 - r_1)^2} \sigma_y^2$$

in the limit that $\sigma_{y_1} = \sigma_{y_2}$.

In our example, $r_1 = 2.5\text{cm}$ $r_2 = 5\text{cm}$, and $\sigma_y = 5\mu\text{m}$, so

$$\sigma_b^2 = \frac{(2.5)^2 + (5)^2}{(2.5)^2} (5\mu\text{m})^2 = \sqrt{125}\mu\text{m}^2$$

$$\boxed{\sigma_b = 11\mu\text{m}}$$

Note that we have assumed here that $p_T = \infty$, so that the contribution from multiple Coulomb scattering is 0.

In the case that the particle has $p = 0.5\text{GeV}/c$, we must consider multiple scattering. We use the approximate formula

$$\sigma_\phi^{\text{plane}} = \frac{14.5\text{MeV}}{\beta p c} \sqrt{\frac{L}{L_{\text{RAD}}}}$$

$\boxed{3.15}$

Since the particle traverses the cylindrical-shell beam pipe perpendicular to its axis, its total path length in aluminum is just 3mm. A moderation length of aluminum is 89mm, so

$$\frac{L}{L_{RAD}} = \frac{1}{89} = 0.011$$

and

$$\sigma_\phi^{\text{plane}} = \frac{14.5 \text{ MeV}}{\beta(500 \text{ MeV})} \sqrt{0.011}$$

Now, $\beta = p/E$, and $E = p^2 + m^2 = (500)^2 + (135)^2 = (518)^2$,
 so $\beta = .97$, and

$$\sigma_\phi^{\text{plane}} = 3.1 \text{ mmrad}$$

This angular deflection will smear the extrapolation back to the beamline by

$$\sigma_b^{\text{MS}} = l \sigma_\phi^{\text{plane}} \quad \text{where } l = 2.5 \text{ cm is the}$$

extrapolation level - or, so

$$\sigma_b^{\text{MS}} = (25 \text{ mm})(0.0031) = 78 \mu\text{m}$$

$$\sigma_b^{\text{TOT}} = \sqrt{(78 \mu\text{m})^2 + (11 \mu\text{m})^2} = 79 \mu\text{m}$$

mult scat

msmt error
from last stop

3.16

This is no longer valid!

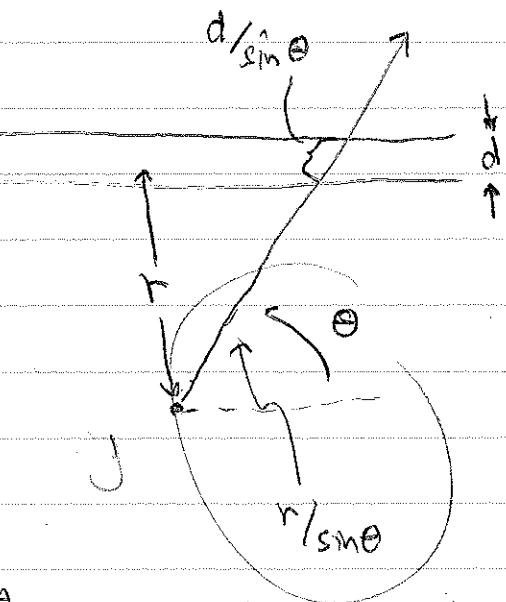
Note that real silicon trackers have some thickness in the detectors themselves (~ 0.1 mrad per layer), and so this is optimistic. The SLD achieves roughly 50 nm w/ a 2.5 cm bumppipe.

In the case that the track still exits \perp to the bumppipe, but at a different momentum, the MS term will vary as $1/p$. Since this term dominates for tracks below $16\text{GeV}/c$, $\sigma_b \propto 1/p$. However, we must also account for tracks which exit at different angles.

If d is the pipo thickness, and r its radius, we see that the material path length varies as $d/\sin\theta$, yielding a $(\sin\theta)^{-1/2}$ term in σ_b .

In addition, the extrapolation length also is lengthened to $r/\sin\theta$, giving an additional $(\sin\theta)^{-1}$ term in σ_b . Thus, does this mean

$$\sigma_b \propto \frac{1}{p \sqrt{\sin\theta} \sin\theta} = \frac{1}{p (\sin\theta)^{3/2}} = \frac{1}{p_2 r \sin\theta} \quad \text{for 2d impact parameter?}$$



Problem 8

For the Poisson distribution, the probability of observing 0 events when the mean no. of events expected in the interval is μ is given by

$$P(0) = e^{-\mu} \frac{\mu^0}{0!} = e^{-\mu}$$

Thus, if in 60.7% of the 10s intervals you observe no energy, and thus no events,

$$.607 = e^{-\mu} \Rightarrow \mu = 0.50$$

Now, let E_α be the α energy. The mean energy observed in the detector for intervals containing 1 or more α -particle depositions will then be

$$\begin{aligned} m &= \frac{\sum_{n=1}^{\infty} (n E_\alpha) [e^{-\mu} \frac{\mu^n}{n!}]}{\sum_{n=1}^{\infty} e^{-\mu} \frac{\mu^n}{n!}} = \frac{E_\alpha \sum_{n=1}^{\infty} e^{-\mu} \frac{\mu^n}{(n-1)!}}{\left[\sum_{n=0}^{\infty} e^{-\mu} \frac{\mu^n}{n!} \right] - e^{-\mu}} \\ &= \frac{\mu E_\alpha e^{-\mu} \sum_{n=1}^{\infty} \frac{\mu^{n-1}}{(n-1)!}}{1 - e^{-\mu}} = \frac{\mu E_\alpha \left[e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \right]}{1 - e^{-\mu}} = \frac{\mu E_\alpha}{1 - e^{-\mu}} \end{aligned}$$

$$\Rightarrow E_\alpha = \frac{[1 - e^{-\mu}]m}{\mu} = (\text{for } \mu = 0.50) \frac{(3.93)}{.50} (6.97) = \boxed{5.5 \text{ MeV}}$$