Physics 6B – Winter 2011

Homework 5 Solutions

Standard Expression for a Traveling Wave

$y(x,t)=A\sin(kx-\omega t)$ gives the y position of a point on a sine wave traveling in the x direction at a speed v for all positions x and times t. Hence x and t are the independent variables. The phase of the wave is defined at the argument of the sine function $kx-\omega t$.

The parameters:

- Since the maximum absolute value of the sine function is 1; A is the maximum displacement or amplitude of the wave.
- The argument of the sine function must be unit-less so the parameter k must have units of inverse distance. Also, at a fixed time, the sine function repeats after a distance equal to the wavelength from all points. Considering the fixed time t=0 and x going from 0 to lambda (the wavelength) it follows that $k=\frac{2\pi}{\lambda}$; this is referred to as the wavenumber.
- Similarly, omega must ensure that, at a fixed x, when a time equal to a period goes by, a point on the y axis returns to where it started, therefore $\omega=\frac{2\pi}{T}$, which is the angular frequency.
- A point on the x axis moves a distance equal to one wavelength in one period, therefore

$$v=\frac{\lambda}{T}=\frac{\lambda}{2\pi}\frac{2\pi}{T}=\frac{(2\pi/T)}{(2\pi/\lambda)}=\frac{\omega}{k}$$

Properties of Ocean Waves

A fisherman notices that his boat is moving up and down periodically without any horizontal motion, owing to waves on the surface of the water. It takes a time of 2.10s for the boat to travel from its highest point to its lowest, a total distance of 0.610m. The fisherman sees that the wave crests are spaced a horizontal distance of 5.50m apart.

A) How fast are the waves traveling?

We're give the time it takes to move through a half of the period ("from its highest point to its lowest") so we know that $T=2.10\cdot 2\text{s}=4.20\text{s}$. We're also given the wavelength $\lambda=5.50\text{m}$. From the first problem we know that the velocity can be written in terms of the period and wavelength:

$$v=\frac{\lambda}{T}=\frac{5.50\text{m}}{4.20\text{s}}=1.31\text{m}/s \quad .$$

B) What is the amplitude A of each wave?

We're also given twice the amplitude in the same statement quoted in part A, so,

$$A=\frac{0.610\text{m}}{2}=0.305\text{m}\quad.$$

Surface Waves

The waves on the ocean are surface waves: They occur at the interface of water and air, extending down into the water and up into the air at the expense of becoming exponentially reduced in amplitude. They are neither transverse nor longitudinal. The water both at and below the surface travels in vertical circles, with exponentially smaller radius as a function of depth.

Both empirical measurements and calculations beyond the scope of introductory physics give the propagation speed of water waves as

$$v=\sqrt{\frac{g}{k}}$$

where $g=9.8\text{m}/s^2$ is the magnitude of the acceleration due to gravity and k is the wavenumber.

This relationship applies only when the following three conditions hold:

1. The water is several times deeper than the wavelength.
2. The wavelength is large enough that the surface tension of the waves can be neglected.
3. The ratio of wave height to wavelength is small.

The restoring force (analogous to the tension in a string) that restores the water surface to flatness is due to gravity, which explains why these waves are often called "gravity waves."

A) Find the speed v of water waves in terms of the wavelength lambda.

Using the relationship between k and lambda derived in the first problem we find

$$v=\sqrt{\frac{g}{k}}=\sqrt{\frac{g}{(2\pi/\lambda)}}=\sqrt{\frac{g\lambda}{2\pi}}\quad.$$

B) Find the speed v of a wave of wavelength $\lambda=8.0\text{m}$.

Plugging in the value for lambda into part A gives $v=\sqrt{\frac{9.8\text{m}/s^2\cdot 8.0\text{m}}{2\pi}}=3.5\text{m}/s$ **.**

C) Find the period T for a wave of wavelength λ.

Express the period in terms of π, λ, and g.

From the first problem we can write the velocity in terms of the angular frequency and the wavenumber, so equating this to the given formula for ocean waves, we have

$$v=\sqrt{\frac{g}{k}}=\frac{\omega}{k}=\frac{2\pi}{Tk}\rightarrow\sqrt{\frac{g}{k}}=\frac{2\pi}{Tk}\quad,$$

solving this for T **shows us that**

$$T=\sqrt{\frac{k}{g}\frac{2\pi}{k}}=\sqrt{\frac{k(2\pi)^2}{gk^2}}=\sqrt{\frac{(2\pi)^2}{gk}}=\sqrt{\frac{(2\pi)^2}{g(2\pi/\lambda)}}=\sqrt{\frac{2\pi\lambda}{g}} \quad .$$

D) On the East Coast of the United States, the National Weather Service frequently reports waves with a period of 4.0 s. Find the wavelength λ and speed v of these waves.

Express your answers numerically as an ordered pair separated by a comma. Give an accuracy of two significant figures.

To find the wavelength solve the expression for the period given in part B for lambda, then plug in the given value for the period.

$$T=\sqrt{\frac{2\pi\lambda}{g}}\rightarrow T^2=\frac{2\pi\lambda}{g}\rightarrow\lambda=\frac{T^2 g}{2\pi}$$

Plugging in our values

$$\lambda=\frac{(4.0\mathrm{s})^2 9.8\mathrm{m}/s^2}{2\pi}=25\mathrm{m}$$

The velocity can be found using the given velocity formula and the relationship between the wavenumber k and the wavelength lambda.

$$v=\sqrt{\frac{g}{k}}=\sqrt{\frac{g\lambda}{2\pi}}$$

Substituting our expression for the wavelength gives

$$v=\sqrt{\frac{g}{2\pi}\frac{T^2 g}{2\pi}}=\frac{Tg}{2\pi} \quad .$$

Finally, plugging in our given value for the period gives

$$v=\frac{4.0\mathrm{s}\,9.8\mathrm{m}/s^2}{2\pi}=6.2\mathrm{m}/s \quad .$$

E) On the West Coast of the United States, the National Weather Service frequently reports waves (really swells) with a period of 15 s. Find the wavelength λ and speed v of these waves.

Express your answers numerically as an ordered pair separated by a comma. Give an accuracy of two significant figures.

Just like in part D

$$\lambda=\frac{T^2 g}{2\pi}=\frac{(15\mathrm{s})^2 9.8\mathrm{m}/s^2}{2\pi}=350\mathrm{m}$$

and

$$v=\frac{Tg}{2\pi}=\frac{15\mathrm{s}\,9.8\mathrm{m}/s^2}{2\pi}=23\mathrm{m}/s \quad .$$

An Intense Car Stereo

A popular car stereo has four speakers, each rated at 60 W. In answering the following questions, assume that the speakers produce sound at their maximum power.

A) Find the intensity I of the sound waves produced by one 60-W speaker at a distance of 1.0 m.

We're told in the problem that the intensity I of a wave is defined as the ratio of the power P of a wave to an area A

$$I=\frac{P}{A}=\frac{P}{4\pi r^2}$$

The second equal sign is if we're interested in the intensity on the surface of a sphere a distance r away from the center. If the source emits waves uniformly in all possible directions (produces spherical waves) this is the total intensity a distance r away. We'll use this approximation for the car speakers, therefore,

$$I=\frac{60\mathrm{W}}{4\pi(1\mathrm{m})^2}=4.8\mathrm{W}/m^2 \quad .$$

B) Find the intensity I of the sound waves produced by one 60-W speaker at a distance of 1.5 m.

Express your answer numerically in watts per square meter. Use two significant figures.

Again $I=\frac{P}{4\pi r^2}=\frac{60\mathrm{W}}{4\pi(1.5\mathrm{m})^2}=2.1\mathrm{W}/m^2 \quad .$

C) Find the intensity I of the sound waves produced by four 60-W speakers as heard by the driver. Assume that the driver is located 1.0 m from each of the two front speakers and 1.5 m from each of the two rear speakers.

Express your answer numerically in watts per meter squared.

The total intensity is just the sum of the 4 individual intensities:

$$I_{total}=I_1+I_2+I_3+I_4$$

Since two pairs of the distances are the same and all the speakers emit the same power we can write this as

$$I=\frac{P}{4\pi r_1^2}+\frac{P}{4\pi r_1^2}+\frac{P}{4\pi r_2^2}+\frac{P}{4\pi r_2^2}=\frac{P}{4\pi}\left(\frac{2}{r_1^2}+\frac{2}{r_2^2}\right)=\frac{P}{2\pi}\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}\right) \quad .$$

Plugging in our values for the distances and power we have

$$I=\frac{60\mathrm{W}}{2\pi}\left(\frac{1}{(1\mathrm{m})^2}+\frac{1}{(1.5\mathrm{m})^2}\right)=13.8\mathrm{W}/m^2$$

D) The *threshold of hearing* is defined as the minimum discernible intensity of the sound. It is approximately 10^{-12} W/m^2. Find the distance d from the car at which the sound from the stereo can still be discerned. Assume that the windows are rolled down and that each speaker actually produces 0.06 W of sound, as suggested in the last follow-up comment.

Express your answer numerically in meters.

To find this distance solve the expression for intensity for r:

$$I=\frac{P}{4\pi r^2}\rightarrow r^2=\frac{P}{4\pi I}\rightarrow r=\sqrt{\frac{P}{4\pi I}}$$

Here four times the power of one speaker is the total power. Our values give

$$r=\sqrt{\frac{4\cdot 0.06\mathrm{W}}{4\pi 10^{-12}W/m^2}}=1.38\times 10^5 m$$

(This nonphysical answer is due to the fact that we're not considering long distance effects such as energy dissipation, interfering sound sources, etc.)

<u>The Decibel Scale</u>

We're given the following info: The general formula for the sound intensity level, in decibels, corresponding to intensity I is

$$\beta = 10\log\left(\frac{I}{I_0}\right)\mathrm{dB},$$

where I_0 is a reference intensity. For sound waves, I_0 is taken to be $10^{-12}\,\mathrm{W/m^2}$. Note that $\log$ refers to the logarithm to the base 10.

A) What is the sound intensity level β, in decibels, of a sound wave whose intensity is 10 times the reference intensity (i.e., $I = 10I_0$)?

Express the sound intensity numerically to the nearest integer.

Using the given expression for intensity we find that

$$I=10\log\left(\frac{10\mathrm{I}_0}{I_0}\right)dB=10\log(10)\,dB=10\mathrm{dB}$$ **.**

B) What is the sound intensity level β, in decibels, of a sound wave whose intensity is 100 times the reference intensity (i.e. $I = 100I_0$)?

Express the sound intensity numerically to the nearest integer.

$$I=10\log\left(\frac{100\mathrm{I}_0}{I_0}\right)dB=10\log(100)\,dB=10\cdot 10\mathrm{dB}=100\mathrm{dB}$$

C) Calculate the change in decibels ($\Delta\beta_2$, $\Delta\beta_4$, and $\Delta\beta_8$) corresponding to $m = 2$, $m = 4$, and $m = 8$.

Give your answers, separated by commas, to the nearest integer--this will give an accuracy of 20%, which is good enough for sound.

We're asked to find three different changes in intensities in decibels where m is the factor of the intensity I to the reference intensity I_0, so,

$$I_m = 10\log(\frac{mI_0}{I_0})\,dB = 10\log(m)\,dB$$ **and,**

$$I_2 = 10\log(2)\,dB = 3\text{dB}$$
$$I_4 = 10\log(4)\,dB = 6\text{dB}$$
$$I_8 = 10\log(8)\,dB = 9\text{dB}$$

Ear Damage from a Small Firecracker

The expression for intensity in dB is given by $\beta = 10\log(\frac{I}{I_0})\,dB$ **; we can invert this expression to get an expression that converts a given intensity in dB to W/m^2.**

$$\beta = 10\log(\frac{I}{I_0})\,dB \rightarrow 10^{\beta} = 10^{10\log(\frac{I}{I_0})dB} = 10^{\log(\frac{I}{I_0})^{10\text{dB}}} = (\frac{I}{I_0})^{10\text{dB}} \rightarrow 10^{\frac{\beta}{10\text{dB}}} = \frac{I}{I_0} \rightarrow I = I_0\, 10^{\frac{\beta}{10\text{dB}}}$$

In this formula, I_0 is a reference intensity, which, for sound waves, is taken to be $10^{-12}\ \mathrm{W/m^2}$. This constant must be used to convert a particular physical intensity into a sound intensity level measured in decibels.

Once we know the sound intensity level (in decibels) at a certain reference distance from a sound source, the $1/r^2$ decrease of intensity with distance can be accounted for by *subtracting* the decibel value appropriate to the ratio of the new distance to the reference distance.

In this problem you will use the decibel scale to analyze a small firecracker that emits 1200 W of peak power. To avoid confusion, intensities denoted by I are in units of watts per meter squared; intensities denoted by β are in units of decibels.

A) What is the peak intensity β in decibels at a distance of 1 m from the firecracker?

Express β in decibels to the nearest integer.

To find the peak intensity in dB we must first determine the intensity in W/m^2 that the firecracker is emitting:

$$I = \frac{P}{4\pi r^2} = \frac{1200\text{W}}{4\pi(1\text{m})^2} = 96\text{W}/m^2$$

Next we can calculate the intensity in decibels using the dB conversion equation given above:

$$\beta = 10\log(\frac{I}{I_0})\,dB = 10\log\left(\frac{96\text{W}/m^2}{10^{-12}\,W/m^2}\right)dB = 10\log(9.6\times 10^{13})\,dB = 140\text{dB}$$

B) It takes a sound intensity of about 160 dB to rupture the human eardrum. How close must the firecracker described in the introduction be to the ear to rupture the eardrum?

Express the distance D_{rupture}, in meters, to one decimal place.

We can use our inverted decibel equation to find this distance:

$$I = I_0 10^{\frac{\beta}{10\text{dB}}} = \frac{P}{4\pi r^2} \rightarrow r^2 = \frac{P}{4\pi I_0} 10^{-\frac{\beta}{10\text{dB}}} \rightarrow r = \sqrt{\frac{P}{4\pi I_0} 10^{-\frac{\beta}{10\text{dB}}}}$$

We're given the peak power and the sound intensity to rupture your human eardrum. Plugging in the given peak power, reference intensity and rupture intensity, we get

$$D_{rupture} = \sqrt{\frac{P_{peak}}{4\pi I_0} 10^{-\frac{\beta_{rupture}}{10\text{dB}}}} = \sqrt{\frac{1200\text{W}}{4\pi 10^{-12} W/m^2} 10^{-\frac{160\text{dB}}{10\text{dB}}}} = 0.1\text{m} \quad .$$

C) Will this firecracker produce temporary loss of hearing in someone who sets if off and stands 3 m away from the explosion? Momentary sounds above 120 dB produce such loss.

The sound intensity 3m from the firecracker is

$$I = \frac{P}{4\pi r^2} = \frac{1200\text{W}}{4\pi (3\text{m})^2} = 10\text{W}/m^2$$

In decibels this intensity is

$$\beta = 10\log(\frac{I}{I_0})dB = 10\log\left(\frac{10\text{W}/m^2}{10^{-12} W/m^2}\right)dB = 10\log(10^{13})dB = 10\cdot 13\log(10)dB = 130\text{dB} > 120\text{dB}$$

Since this intensity is greater than 120dB, the intensity to cause temporary loss of hearing, this firecracker can indeed produce temporary hearing loss if you're 3m away when it goes off.

Can the Firecracker Be Heard?

We apply the dB scale to a small firecracker that has an intensity of 140 dB at a distance of 1 m.

Some useful info given: The change in dB corresponding to the change in distance may be found by replacing I_0 in the formula given here (the familiar decibel formula) with the intensity at the reference distance and replacing I with the intensity at the new distance.

A) A child sets off the firecracker at a distance of 100 m from the family house. What is the sound intensity β_{100} at the house?

Express the sound intensity in decibels.

We can use the useful info given in the problem to find this intensity.

Note on the useful info above: A property of logarithms is that the difference between two logs (with *same* base) equals the log of the ratio

$$\log(x) - \log(y) = \log(\frac{x}{y})$$

Using this relationship we see that

$$\beta_{100} - \beta_1 = 10\log(\frac{I_{100}}{I_0})dB - 10\log(\frac{I_1}{I_0})dB = 10\left[\log(\frac{I_{100}}{I_0}) - \log(\frac{I_1}{I_0})\right]dB = 10\log(\frac{I_{100}}{I_1})dB$$

The ratio of intensities can be found using the expression for I in W/m^2:

$$\frac{I_{100}}{I_1}=\frac{\left(\frac{P}{4\pi(100\text{m})^2}\right)}{\left(\frac{P}{4\pi(1\text{m})^2}\right)}=\frac{1}{100^2}=10^{-4}$$

So the *change* in intensity from 1m to 100m in decibels is:

$$\beta_{100}-\beta_1=10\log(10^{-4})\,dB=-40\text{dB}$$

Therefore the intensity at 100m is given by:

$$\beta_{100}=\beta_1-40\text{dB}=140\text{dB}-40\text{dB}=100\text{dB}$$

The child's parents are inside the house talking. A typical house attenuates (reduces the intensity of) outside sounds by a factor of around 40 dB--more at high frequencies (when double-pane windows are closed) and less at low frequencies. Also, a typical conversation has an average sound level of 65 dB.

B) Will the firecracker sound louder than the parents' conversation?

The house will attenuate the intensity of the firecracker by a factor of 40dB so the intensity that makes it into the house will be $\beta_{house}=100\text{dB}-40\text{dB}=60\text{dB}$ **. If the conversation is 65dB the firecracker won't sound louder.**

<u>Problem 14.28</u>

A rope is stretched between supports 12 m apart; its tension is 35 N. If one end of the rope is tweaked, the resulting disturbance reaches the other end 0.45 s later.

A) What is the total mass of the rope?

Express your answer using two significant figures.

The following is an expression for the velocity of a wave traveling on a stretched rope:

$$v=\sqrt{\frac{T}{\mu}}\quad .$$

where v is the velocity, T is the tension on the rope and mu is the mass per unit length. From the information given we know that $v=12\text{m}/0.45\text{s}$ **. We can solve the velocity equation for the linear density mu and then plug in the given values for v and T.**

$$v^2=\frac{T}{\mu}\rightarrow\mu=\frac{T}{v^2}$$

To find the total mass of the rope multiply the linear density by the total length L.

$$m=\mu L=\frac{T}{v^2}L$$

Plugging in our values gives

$$m=\frac{35\text{N}}{(12\text{m}/0.45\text{s})^2}12\text{m}=590\text{g}\quad .$$

Ant on a Tightrope

A large ant is standing on the middle of a circus tightrope that is stretched with tension T_s. The rope has mass per unit length μ. Wanting to shake the ant off the rope, a tightrope walker moves her foot up and down near the end of the tightrope, generating a sinusoidal transverse wave of wavelength λ and amplitude A. Assume that the magnitude of the acceleration due to gravity is g.

A) What is the minimum wave amplitude $A_{\rm min}$ such that the ant will become momentarily weightless at some point as the wave passes underneath it? Assume that the mass of the ant is too small to have any effect on the wave propagation.

Express the minimum wave amplitude in terms of T_s, μ, λ, and g.

The ant will feel "weightless" when the only force acting on it is the force of gravity. This will be true if the rope's acceleration in the downward direction is equal to or greater than the acceleration of gravity. This would be similar to being in a falling elevator or in orbit around a body; everything around you accelerates at the same rate and thus appears "weightless". However we know that the influence of gravity stretches over infinity at the speed of light.

Anyway, we're interested in the amplitude of a wave traveling on the rope that corresponds to a y acceleration of exactly g, since this would be the minimum amplitude. The acceleration in the y direction can be found by differentiating the y position function with respect to time twice.

$$y(x,t)=A\sin(kx-\omega t)$$

$$v_y=\frac{dy}{dt}=-\omega A\cos(kx-\omega t)$$

$$a_y=\frac{dv_y}{dt}=\frac{d^2y}{dt^2}=-\omega^2 A\sin(kx-\omega t)$$

The maximum downward acceleration occurs when the sine function equals 1. We want this maximum acceleration to equal that of gravity.

$$a_{y(max)}=-\omega^2 A=-g\rightarrow A=\frac{g}{\omega^2}$$

We can write this in terms of the tension, linear mass density, wavelength and acceleration of gravity by revisiting the very first problem of this homework. Recall that

$$v=\frac{\omega}{k}\rightarrow\omega=vk \quad , \quad k=\frac{2\pi}{\lambda} \quad ,$$

and that for a rope under tension T_s and with mass per unit length mu

$$v=\sqrt{\frac{T_s}{\mu}} \quad .$$

Putting all of this together we see that

$$A=\frac{g}{\omega^2}=\frac{g}{(vk)^2}=\frac{g}{\left(\sqrt{\frac{T_s}{\mu}}\frac{2\pi}{\lambda}\right)^2}=g\left(\sqrt{\frac{\mu}{T_s}}\frac{\lambda}{2\pi}\right)^2=\frac{g\mu}{T_s}\left(\frac{\lambda}{2\pi}\right)^2 \quad .$$