

# Physics 6B Homework 7 Solutions

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## Is Light Reflected or Refracted?

Part A: When light propagates from a material with a given index of refraction into a medium with a ~~smaller~~ <sup>smaller</sup> index of refraction, the speed of the light increases.

This is due to the relation  $n = \frac{c}{v}$ . As  $n$  increases,  $v$  decreases, and as  $n$  decreases,  $v$  increases.

Part B: What is the minimum value that the index of refraction can have?

The maximum value of the speed of light is  $c$ . Therefore the ~~maximum~~ <sup>minimum</sup> value of  $n$  is:

$$n = \frac{c}{c} = \boxed{1}$$

Part C: Now consider a ray of light that propagates from water ( $n = 1.33$ ) to air ( $n = 1$ ). If the incident ray strikes the water-air interface at an angle  $\theta_1 \neq 0$ , which of the following relations regarding the angle of refraction,  $\theta_2$ , is correct?

Using Snell's Law:

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

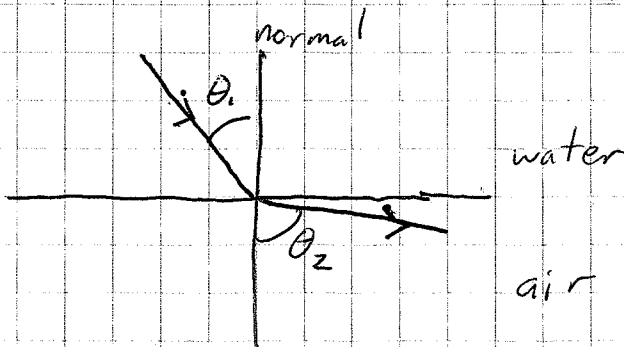
$$1.33 \sin \theta_{\text{water}} = \sin \theta_{\text{air}}$$

Since  $0 < \theta < 90^\circ$

$$\theta_{\text{water}} < \theta_{\text{air}}$$

$$\boxed{\theta_2 > \theta_1}$$

Light rays always bend away from the normal when going from high to low index of refraction.



Part D: Consider a ray of light that propagates (2) from water ( $n=1.33$ ) to glass. If the incident ray strikes the water-glass interface at an angle  $\theta_1 \neq 0$ , which of the following relations regarding the angle of refraction  $\theta_2$  is correct?

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

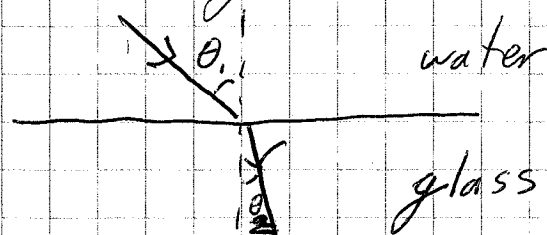
$$n_{\text{glass}} = n_2 = 1.52$$

$$1.33 \sin \theta_1 = 1.52 \sin \theta_2$$

$$.875 \sin \theta_1 = \sin \theta_2,$$

therefore  $\theta_1 > \theta_2$

Light rays always bend toward the normal when going from low to high index of refraction



Part E: Consider a ray of light that propagates from air ( $n=1$ ) to any one of the materials listed below. Assuming that the ray strikes the interface with any of the listed materials always at the same angle  $\theta_1$ , in which material will the direction of propagation of the ray change the most due to refraction?

For the largest change in direction you need the largest change in  $n$  at the interface. Therefore the answer is diamond, which has the largest  $n$  value.

Part F:

When  $n_1 > n_2$ ,  $\theta_2 > \theta_1$ , so as the incidence angle increases, the angle ~~angt~~ of refraction ( $\theta_2$ ) will increase to  $90^\circ$  maximum. Past this angle of incidence which causes this refraction, there will be no refracted light, only reflected. This angle of incidence is known as the critical angle ( $\theta_{\text{crit}}$ ).

Part G. What is the critical angle  $\theta_{crit}$  for light propagating from a material with index of refraction 1.50 to a material with index of refraction of 1.00. (3)

$$\text{At } \theta_1 = \theta_{crit}, \theta_2 = 90^\circ$$

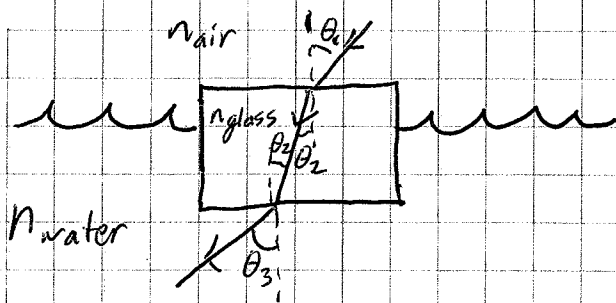
$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

$$\text{or } n_1 \sin \theta_{crit} = n_2 \sin 90^\circ = n_2$$

$$\theta_{crit} = \sin^{-1}\left(\frac{n_2}{n_1}\right) = \sin^{-1}\left(\frac{1.00}{1.50}\right) = \boxed{.730 \text{ radians}}$$

### Re fraction Through Glass and Water

A plate of glass with parallel faces having a refractive index of 1.45 is resting on the surface of water in a tank. A ray of light coming from above in air makes an angle of incidence  $38.5^\circ$  with the normal to the top surface of the glass.



$$\theta_1 = 38.5^\circ \quad n_{air} = 1 \quad n_{glass} = 1.45$$

$$n_{water} = 1.33$$

What angle  $\theta_3$  does the ray refracted into the water make with the normal to the surface? Use 1.33 for the index of water.

Here you have to use Snell's Law twice, first to find  $\theta_2$  and then to find  $\theta_3$ .

$$n_{air} \sin \theta_1 = n_{glass} \sin \theta_2$$

$$\theta_2 = \sin^{-1}\left(\frac{n_{air} \sin \theta_1}{n_{glass}}\right) = \sin^{-1}\left(\frac{1 \sin(38.5^\circ)}{1.45}\right) = 25.4^\circ$$

$$n_{glass} \sin \theta_2 = n_{water} \sin \theta_3$$

$$\theta_3 = \sin^{-1}\left(\frac{n_{glass} \sin \theta_2}{n_{water}}\right) = \sin^{-1}\left(\frac{1.45 \sin(25.4^\circ)}{1.33}\right) = \boxed{27.9^\circ}$$

(4)

You could also leave everything as symbols, until the end (i.e. not calculate  $\theta_2$  explicitly).

$$n_{\text{glass}} \sin \theta_2 = n_{\text{water}} \sin \theta_3$$

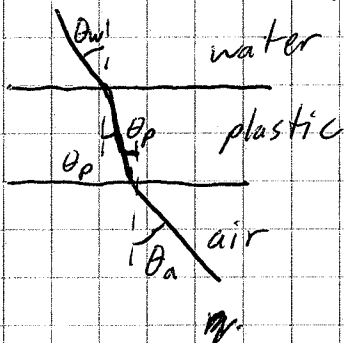
$$n_{\text{glass}} \left( \frac{n_{\text{air}} \sin \theta_1}{n_{\text{glass}}} \right) = n_{\text{water}} \sin \theta_3$$

$$\theta_3 = \sin^{-1} \left( \frac{n_{\text{air}} \sin \theta_1}{n_{\text{water}}} \right) = \sin^{-1} \left( \frac{1}{1.33} \sin(38.5^\circ) \right) = \boxed{27.9^\circ}$$

note that this expression does not depend on  $n_{\text{glass}}$ , it is as if the light ray went straight from the air to water

## Underwater Optics

Consider a flat piece of plastic ( $n_p$ ) with water ( $n_w$ ) on one side and air on the other.



Part A: If the light strikes the plastic (from the water) at an angle  $\theta_w$ , at what angle  $\theta_a$  does it emerge from the plastic (into the air)?

This is the same calculation as the previous problem. Finding  $\theta_p$ , and then  $\theta_a$ ...

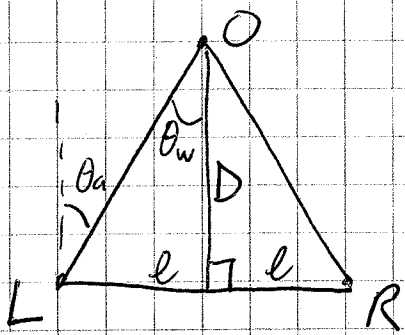
$$n_w \sin \theta_w = n_p \sin \theta_p \Rightarrow \theta_p = \sin^{-1} \left( \frac{n_w \sin \theta_w}{n_p} \right)$$

$$n_p \sin \theta_p = n_a \sin \theta_a$$

$$n_p \left( \frac{n_w \sin \theta_w}{n_p} \right) = n_a \sin \theta_a \Rightarrow \sin \theta_a = \frac{n_w \sin \theta_w}{n_a}$$

$$\theta_a = \sin^{-1} \left( \frac{n_w \sin \theta_w}{n_a} \right)$$

One of the main ways humans perceive distance is through triangulation, in which the angle at which light strikes each eye is used to determine distance. This process is sketched in the figure below: (5)



L and R represent your left and right eyes,  $2l$  is the distance between your eyes, and the distance to the object is  $D$ .

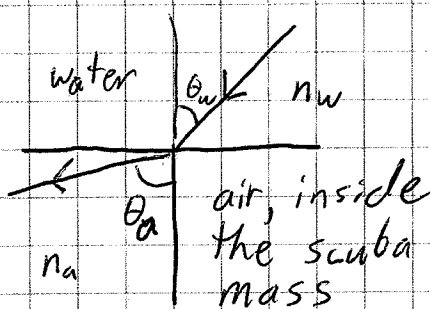
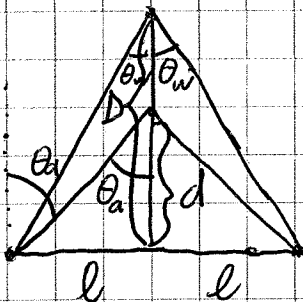
Part B: What is the distance to the object in terms of  $\theta_w$  and  $l$ ?

Using trigonometry:  $\tan \theta_w = \frac{l}{D} \Rightarrow \boxed{D = \frac{l}{\tan(\theta_w)}}$

Part C: If the distance to the object is more than about 4m, then you can use the small angle approximation  $\tan(\theta) \approx \theta$ . What is  $D$  if you use this approximation?

Making the substitution  $\tan \theta = \theta$ ,  $\boxed{D = \frac{l}{\theta_w}}$

If you are wearing a scuba mask underwater, the refraction of light when it goes from water to air will cause  $\theta_a$  to increase, causing your brain to perceive the object closer than it actually is.



$d$  is the ~~per~~ distance at which your brain perceives the object

Part D: Use Find the ratio  $d/D$ . Use small angle approximations  $\sin(x) \approx x$ ,  $a \sin(x) \approx x$ , and express answer in terms of  $n_w$  and  $n_a$ .

First find  $\theta_a$ .  $n_a \sin \theta_a = n_w \sin \theta_w \approx n_a \theta_a = n_w \theta_w \Rightarrow \theta_a = \frac{n_w}{n_a} \theta_w$

Now find  $d$ .

(6)

$$d + n_a \theta_a = \frac{l}{d} \approx \theta_a = \frac{l}{d} \Rightarrow d = \frac{l}{\theta_a} = \frac{n_a l}{n_w \theta_w}$$

Using the previous result for  $D$ :

$$\frac{d}{D} = \frac{n_a l}{n_w \theta_w} \cdot \frac{\theta_w}{l} = \boxed{\frac{n_a}{n_w}}$$

Part E: Given that  $n_w = 1.33$  and  $n_a = 1.00$ , by what percent do objects underwater appear closer than they actually are?

We need to find  $\frac{D-d}{D} \times 100\%$

$$= \left(1 - \frac{d}{D}\right) \times 100\% = \left(1 - \frac{n_a}{n_w}\right) \times 100\% = \left(1 - \frac{1.00}{1.33}\right) \times 100\% = \boxed{25\%}$$

### Total Internal Reflection Conceptual Question

Consider scenarios A  $\rightarrow$  F in which a ray of light traveling in material 1 is incident onto the interface with material 2.

|   | Material 1 ( $n_1$ ) | Material 2 ( $n_2$ ) |
|---|----------------------|----------------------|
| A | air (1.00)           | water (1.33)         |
| B | water (1.33)         | air (1.00)           |
| C | diamond (2.42)       | air (1.00)           |
| D | air (1.00)           | quartz (1.46)        |
| E | benzene (1.50)       | water (1.33)         |
| F | diamond (2.42)       | water (1.33)         |

Part A: For which of these scenarios is total internal reflection possible?

For <sup>total</sup> internal reflection to occur the refracted angle with respect to the normal needs to be larger than the incident angle. This happens when  $n_2 < n_1$ . Therefore you will have total internal reflection for BCE.

Part B: Rank the scenarios by their critical angle, from largest critical angle to smallest. (7)

$\theta_c$  occurs when  $\theta_2 = 90^\circ$ . Using Snell's law:

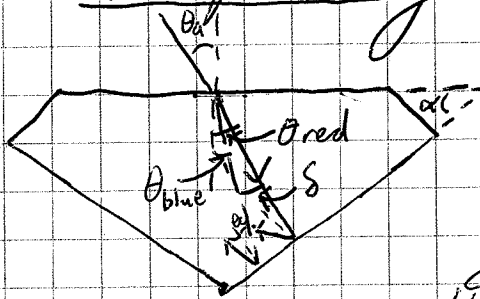
$$n_1 \sin \theta_c = n_2 \sin 90^\circ = n_2$$

$\sin \theta_c = \frac{n_2}{n_1}$ , so the scenario with the largest value of  $\frac{n_2}{n_1}$  will have the largest critical angle.

The ranking is (from largest  $\theta_c$  to smallest):

|   |                          |
|---|--------------------------|
| E | $\frac{n_2}{n_1} = .886$ |
| B | $\frac{n_2}{n_1} = .752$ |
| F | $\frac{n_2}{n_1} = .550$ |
| C | $\frac{n_2}{n_1} = .413$ |

### A Sparkling Diamond I



red  
 blue

A beam of white light is incident upon a diamond at  $\theta_0$

Since the index of refraction depends on the light's wavelength, the different colors that compose white light will spread out as they pass through the diamond.

$$n_{\text{red}} = 2.410 \quad n_{\text{blue}} = 2.450 \quad n_{\text{air}} = 1.000$$

$$c = 2.998 \times 10^8 \text{ m/s}$$

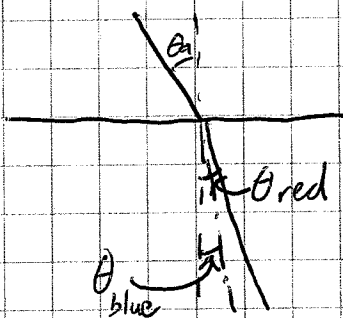
Part A: Calculate  $v_{\text{red}}$ , the speed of red light in the diamond.

$$v_{\text{red}} = \frac{c}{n_{\text{red}}} = \frac{2.998 \times 10^8 \text{ m/s}}{2.410} = 1.244 \times 10^8 \text{ m/s}$$

Part B: Calculate  $v_{\text{blue}}$

$$v_{\text{blue}} = \frac{c}{n_{\text{blue}}} = \frac{2.998 \times 10^8 \text{ m/s}}{2.450} = 1.224 \times 10^8 \text{ m/s}$$

Part C: Derive a formula for  $\delta$ , the angle between the red and blue refracted rays.



Find  $\theta_{red}$  and  $\theta_{blue}$  via Snell's Law.

$$n_{red} \sin \theta_{red} = n_{air} \sin \theta_a$$

$$\theta_{red} = a \sin \left( \frac{1}{n_{red}} \sin \theta_a \right)$$

$$n_{blue} \sin \theta_{blue} = n_{air} \sin \theta_a$$

$$\theta_{blue} = a \sin \left( \frac{1}{n_{blue}} \sin \theta_a \right)$$

Because  $n_{blue} > n_{red}$ ,  $\theta_{blue} < \theta_{red}$ . Therefore

$$\delta = \theta_{red} - \theta_{blue} = a \sin \left( \frac{1}{n_{red}} \sin \theta_a \right) - a \sin \left( \frac{1}{n_{blue}} \sin \theta_a \right)$$

$$\boxed{a \sin \left( \frac{1}{n_{red}} \sin \theta_a \right) - a \sin \left( \frac{1}{n_{blue}} \sin \theta_a \right)}$$

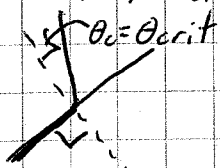
Part D: Calculate  $\delta$  numerically for  $\theta_a = 45^\circ$ .

$$\delta = a \sin \left( \frac{1}{2.410} \sin(45^\circ) \right) - a \sin \left( \frac{1}{2.450} \sin(45^\circ) \right)$$

$$= \boxed{.287^\circ}$$

## A Sparkling Diamond II

Part A: Consider  $\theta_c$ , the angle at which the blue reflected ray hits the bottom surface of the diamond. Find the  $\theta_c$  at which the light ray goes from being refracted out into the air to being totally internally reflected into the diamond ( $\theta_{crit}$ )

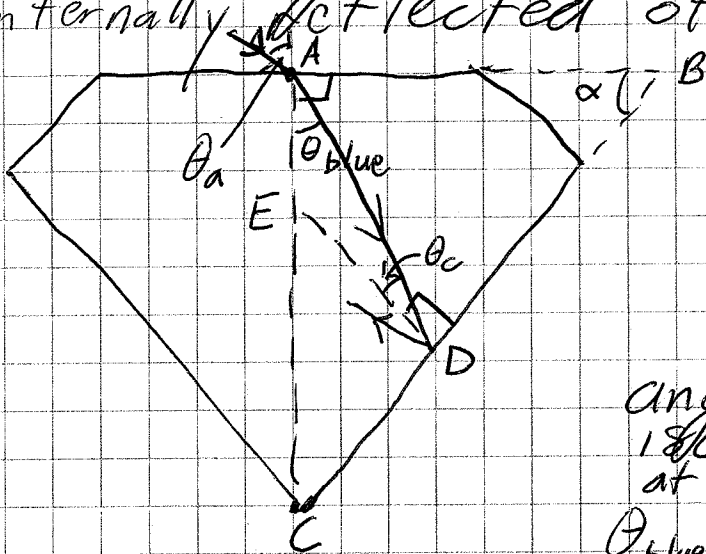


$$n_{blue} \sin \theta_{crit} = n_{air} \sin(90^\circ)$$

$$\theta_{crit} = a \sin \left( \frac{1}{n_{blue}} \right) = a \sin \left( \frac{1}{2.450} \right) = 24.09^\circ$$



Part B: A diamond is cut such that the angle between its top surface and its bottom surface is  $\alpha$ . For  $\alpha = 45^\circ$ , find the largest possible value of the incident angle such that the blue light is totally internally reflected off the bottom surface. (9)



First we have to do some geometry. Knowing the angles of a triangle add to  $180^\circ$ , we see that  $\angle C = 90^\circ - \alpha$  and therefore  $\angle DEC = \alpha$ . Knowing the angles in a line add to  $180^\circ$ ,  $\angle DEA = 180^\circ - \alpha$ . Looking at triangle AED now, we see

$$\theta_{\text{blue}} + \theta_c + 180^\circ - \alpha = 180^\circ$$

$$\theta_{\text{blue}} + \theta_c = \alpha$$

Now using Snell's Law to relate  $\theta_a$  and  $\alpha$ .

$$n_{\text{air}} \sin \theta_a = n_{\text{blue}} \sin \theta_{\text{blue}} = n_{\text{blue}} \sin (\alpha - \theta_c)$$

$$\theta_a = \text{asin}^{**} (n_{\text{blue}} \sin (\alpha - \theta_c))$$

We want the largest value of  $\theta_a$ , which occurs when  $\theta_c$  is the minimal value that produces total internal reflection,  $\theta_{\text{crit}}$ . Therefore

$$\theta_a \underset{\uparrow}{\text{max}} = \text{asin}^{**} (n_{\text{blue}} \sin (\alpha - \theta_{\text{crit}})) = \text{asin}^{**} (2.450 \sin (45 - 24.09^\circ)) = \boxed{60.97^\circ}$$