**Problem 1.** The black holes identified with “Active Galactic Nuclei” have masses about \(10^9 M_\odot\). Estimate their Schwarzschild radius, in kilometers.

**Solution:**

\[ M_\odot \approx 1.5\, \text{km}, \]

so

\[ r_s = 2 M \approx 3 \times 10^9 \text{km} \]

**Problem 3.** About how many minutes are there in a year? Seconds? About how large is a light year, expressed in kilometers?

**Solution:** There are 525,600 minutes in a 365 day year,

[Check out http://www.youtube.com/watch?v=-r2xXtSsPV0](http://www.youtube.com/watch?v=-r2xXtSsPV0)

or about \(3 \times 10^7\) seconds. So in a year, light travels

\[ 3 \times 10^7 \text{ sec} \times 3 \times 10^5 \text{ km/sec} \approx 10^{13}\text{km}. \]

**Problem 4.** Which of the following are *not true* in Einstein’s theory of general relativity:

(a.) Photons travel at the speed of light

(b.) The gravitational force between antiparticles is repulsive

(c.) Over small regions of space-time, acceleration and gravitation are equivalent

(d.) The laws of nature look the same in coordinate systems related by general coordinate transformations.

**Solution:** The answer is (b); the gravitational force is always attractive.

**Problem 5.** What is the approximate age of the universe in years? In seconds?

**Solution:** The universe is about 14 billion years old. This is

\[ t = 14 \times 10^9 \text{ yrs} \times 3 \times 10^7 \text{ sec/yr} \approx 4 \times 10^{17} \text{ sec}. \]

**Problem 6.** The metric, in Kruskal coordinates, is

\[ ds^2 = \frac{32 M^3}{r} e^{-r/2M}(-dV^2 + dU^2) + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \]

For fixed angles, \(\theta\) and \(\phi\), describe the paths of photons in terms of \(U\) and \(V\).

**Solution:** \(ds^2 = 0\) gives \(dU = \pm dV\), so \(U = \pm V + \text{constant}\), i.e. they are \(45^\circ\) lines.
The Friedman-Robertson-Walker metric
\[ ds^2 = -dt^2 + a(t)^2 d\vec{x}^2 \] (1)
is invariant under translations, \( \vec{x} \rightarrow \vec{x} + \vec{\delta} \). Can you construct three Killing vectors associated with this symmetry? Is the metric invariant under translations in time? If so, construct the Killing vector. If not, explain why not.

**Solution:** Under translations, \( d\vec{x} \rightarrow d\vec{x} \), since \( \vec{\delta} \) is constant. So the metric is unchanged. Under corresponding shifts of \( t \), \( a(t) \) is not invariant, so the metric is not invariant. The Killing vectors for translations are \( \xi^T = (0, 1, 0, 0) \), etc.

**Problem 7.**
For the Eddington-Finkelstein coordinates,
\[ ds^2 = -(1 - \frac{2M}{r})dv^2 + 2dvdr + r^2(d\theta^2 + d\phi^2) \] (2)
construct two Killing vectors. Construct the conserved quantities associated with these.

**Solution:** The lagrangian has symmetries \( v \rightarrow v + c \) and \( \phi \rightarrow \phi + c \). Correspondingly, the Killing vectors are:
\[ \xi^v = (1, 0, 0, 0) \quad \xi^\phi = (0, 0, 0, 1) \]
Correspondingly, the conserved quantities are:
\[ e = -u \cdot \xi^v = (1 - \frac{2M}{r})\frac{\partial v}{\partial \tau} + \frac{\partial r}{\partial \tau} \quad \ell = u \cdot \xi^\phi = r^2 \frac{\partial \phi}{\partial \tau} \]
Note the second term in \( e \), which arises because \( g_{vr} = 1 \).

**Problem 8.**
During the matter dominated era, we saw that the scale factor of the universe (the quantity \( a(t) \) in eqn. 1) behaves as \( a(t) = a_0 t^{2/3} \). (3)
If typical galaxies, are separated by a distance \( d \) at \( t = 1 \) billion years, how much are they separated at 8 billion years? (it may seem obvious, but just in case, here’s a hint: figure out by what factor \( a \) grows, and think about what happens to distances in eqn. 1).

**Solution:** Over this time, the scale factor has grown by a factor of 4. So distances have all increase by a factor of four.

**Problem 9.**
We argued that as time passes, the temperature behaves as \( T \propto 1/a(t) \). Consider, as in the previous problem, the matter dominated era. How old was the universe when it was just hot enough to ionize atoms (remember from your quantum mechanics that the ionization energy of hydrogen is about 13 eV).

**Solution:** A typical ionization energy is about 10 eV, corresponding to about \( 10^5 \) °K. Remembering that the temperature behaves inversely with \( a \), we see that the scale factor was about \( 10^4 \) times smaller than today; in the matter dominated area, \( a \sim t^{2/3} \), so the age of the universe was about \( 10^6 \) times younger than today. But a better number turns out to be 1 eV. In this case, the universe was about \( 10^{4.5} \) times as young, or roughly 100,000 years old.

**Problem 10.** For the Schwarschild metric
\[ ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \] (4)
(a.) Write the components of the metric tensor, $g_{\mu\nu}$ and its inverse, $g^{\mu\nu}$.

Solution:

$g_{tt} = -(1 - \frac{2M}{r})$  
$g_{rr} = (1 - \frac{2M}{r})^{-1}$  
$g_{\theta\theta} = r^2$  
$g_{\phi\phi} = r^2 \sin^2 \theta$

(b.) Calculate the Christoffel symbols $\Gamma^r_{rr}$, $\Gamma^r_{tt}$.

Solution: This can be done using either the formula for the Christoffel symbols, below, or the action method. I'll do the former:

$$\Gamma^r_{rr} = \frac{1}{2} g^{rr} \left( \frac{\partial}{\partial r} g_{rr} \right) = \frac{-2M}{r^2} (1 - \frac{2M}{r})^{-1}$$

$$\Gamma^r_{tt} = \frac{1}{2} g^{tt} - \frac{1}{2} \frac{\partial}{\partial r} g_{tt} = -\frac{2M}{r^2} (1 - \frac{2M}{r})^{-1}$$

Problem 9.

(a.) What are the conserved quantities for the motion of a massive particle in the Schwarschild metric?

Solution: “Energy” and angular momentum:

$$e = -\xi^t \cdot u = \frac{\partial t}{\partial r} (1 - \frac{2M}{r})$$

$$\ell = \xi^\phi \cdot u = r^2 \frac{\partial \phi}{\partial r}$$

(b.) For a motion purely in the radial ($r$) direction, do any of the conserved quantities vanish? Write the equation $u^\mu u^\nu g_{\mu\nu} = -1$, for a motion purely in the radial direction, in terms of the remaining conserved quantities.

Solution: For radial motion, the angular momentum vanishes. Then

$$u^2 = -(1 - \frac{2M}{r}) \left( \frac{\partial t}{\partial r} \right)^2 + (1 - \frac{2M}{r})^{-1} \left( \frac{\partial r}{\partial r} \right)^2$$

$$= (1 - \frac{2M}{r})^{-1} (-e^2 + \left( \frac{\partial r}{\partial r} \right)^2 = -1$$

(c.) Using the result of (b), write an expression for the proper time for a particle to pass from $r = 6M$ to $r = 3M$, assuming that it starts at rest. You don’t have to do the integral.

Solution:

$$\left( \frac{\partial r}{\partial \tau} \right)^2 = e^2 - (1 - \frac{2M}{r}).$$

This can be integrated to give $\tau$ as a function of $r$. $e$ is evaluated by writing this expression for $r = 6M$ with $\frac{dr}{d\tau} = 0$.

Useful formulae

1. $\frac{1}{1+\epsilon} \approx 1 - \epsilon \quad (1 + \epsilon)^a \approx 1 + a\epsilon$.
2. Lorentz transformation: $x' = \gamma(x + vt)$  
$t' = \gamma(t + vx)$
3. Distance element in spherical coordinates: $ds^2 = dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$
4. Distance element for the plane in cylindrical coordinates: $ds^2 = d\rho^2 + \rho^2 d\phi^2$
5. Distance element in space-time $ds^2 = -d\tau^2 = -dt^2 + d\vec{x}^2$

6. Christofel symbols: $\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left[\partial_\alpha g_{\beta\nu} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\right]$.

7. Components of four-velocity: $u^0 = \gamma;\ u^i = \gamma v^i$.

8. Conversion factors: $1eV = 1.16 \times 10^4 \ oK$