Problem 1. a. What are the components of the four velocity, $u^\mu = \frac{dx^\mu}{d\tau}$, written in terms of the ordinary velocity?

Solution:

$$u^0 = \gamma, \quad u^i = \gamma v^i.$$ 

b. Show that for $v \ll c$, the components of the four vector $p = mu^\mu$ reduce to the ordinary energy and momentum.

Solution: For small velocity, $\gamma \approx 1 + \frac{v^2}{2}$, so

$$p^0 \approx mc^2 + \frac{1}{2}mv^2, \quad p^i \approx mv^i.$$ 

c. For a particle of mass $m$, show that the four-velocity satisfies

$$u^\mu u_\mu = -1 \quad (1)$$

You can do this from the definition, $u^\mu = \frac{dx^\mu}{d\tau}$, or by thinking about the components.

Solution: Can use our expressions above, or just note $u^\mu u_\mu = \frac{dx^\mu dx_\mu}{d\tau^2} = -1$ from the definition of $d\tau$.

Problem 2. We know that in a weak gravitational field,

$$g_{00}(x) = -(1 + 2\Phi(x)/c^2)$$

where $\Phi$ is the gravitational potential. Estimate the size of the gravitational correction to $g_{00}$ at

a. The surface of the earth Here, in mks units, $\Phi \approx 9.8 \times 6 \times 10^6$, so we have $gR/c^2$, roughly one part in $3 \times 10^{10}$.

b. The surface of the sun (the mass of the sun is $M_\odot \approx 2 \times 10^{30}$ kg; its radius is $7 \times 10^8$ m; Newton’s constant is $6.7 \times 10^{-11}$ m$^3$ kg$^{-1}$ sec$^{-2}$). So $2\Phi/c^2 \approx 4 \times 10^{-6}$.

Problem 3.

a. Show that the action, $S = -mc^2 \int d\tau$, with $d\tau^2 = -dx^\mu dx_\mu = dt^2 - d\vec{x}^2$ reduces to the usual action for a particle in the non-relativistic limit.

Solution: Here we just expand

$$-m \int d\tau = -m \int dt \sqrt{1 - v^2} \approx \int dt(-m + \frac{m}{2}v^2).$$

b. Show that the action

$$S = \int \sqrt{-g_{\mu\nu} dx^\mu dx^\nu}$$

with

$$g_{00} = -(1 + 2\Phi); \quad g_{ij} = \delta_{ij}$$
reduces to the action for a particle in a gravitational field in Newton’s theory.

**Solution:** Here we just expand

\[-m \int d\tau = -m \int dt \sqrt{1 + 2\Phi - v^2} \approx \int dt(-m + \frac{m}{2}v^2 - m\Phi).\]

(Note that the sign is the correct one for the \(\Phi\) term, and the factor of 2.)

**Problem 4.** For the cylindrical coordinate system, in flat spacetime, starting with

\[ds^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2 + dz^2\]  \hspace{1cm} (2)

and requiring \(ds^2 = 0\), describe two simple trajectories for light rays (i.e. involving one space coordinate and time). **Hint:** things are simplest if you treat the polar angle as a constant.

**Solution:**

\[
\rho = t + c; \quad z = \pm t + c
\]

both satisfy \(ds^2 = 0\) (in the first, \(d\rho = dt\); in the second, \(dz = \pm dt\)).

**Problem 5** For the line element:

\[ds^2 = \frac{a^4}{r^4}(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2)\]

determine:

(a.) The area of the “sphere” of radius \(r = R\).

**Solution:** This is obtained from

\[
\int d\theta d\phi \sqrt{g_{\theta\theta}} \sqrt{g_{\phi\phi}} = \frac{a^4}{R^2} \int d\theta d\phi = 4\pi \frac{a^4}{R^2}.
\]

(b.) The distance from the surface of the sphere of radius \(R\) to infinity.

**Solution:**

\[
\int_{R}^{\infty} dr \sqrt{g_{rr}} = \frac{a^2}{R} \int_{R}^{\infty} \frac{1}{r^2} = \frac{a^2}{R}.
\]

(c.) The volume of the region \(r > R\), and the region with \(r < R\) (one of these is infinite).

**Solution:**

\[
V_{>} = \int_{R}^{\infty} dr \sqrt{g_{rr} g_{\theta\theta} g_{\phi\phi}} = a^2 \int_{R}^{\infty} \frac{r}{r^4} d\theta d\phi \sin \theta = \frac{4\pi a^6}{3 R^3}.
\]

\[
V_{<} = \infty
\]

**Problem 6.** For a massive particle moving in the plane, with

\[ds^2 = -dt^2 + d\rho^2 + \rho^2 d\theta^2\]

determine the Christoffel symbols, using either the general formula below or the action principle, and find solutions for radial motion (\(d\theta = 0\)).

**Solution:** Here \(g_{\rho\rho} = 1, \ g_{\theta\theta} = \rho^2\). It is helpful to note that \(g^{\rho\rho} = 1; \ g^{\theta\theta} = \rho^{-2}\). So we just need to plug into the formula for \(\Gamma\). It is not hard to see (noticing that the only non-trivial derivatives are derivatives with respect to \(\rho\) that the only non-vanishing components are

\[
\Gamma_{\theta\theta}^\rho = \frac{1}{2} g^{\rho\rho} \left( -\frac{\partial g_{\theta\theta}}{\partial \rho} \right) = -\rho
\]
and
\[ \Gamma^\theta_{\rho\rho} = \frac{1}{2} g^{\theta\theta} \frac{\partial g_{\theta\theta}}{\partial \rho} = \frac{1}{\rho}. \]

.. We can obtain the same result from the action approach. We have the "lagrangian"

\[ L = \sqrt{\dot{\rho}^2 + \rho^2 \dot{\theta}^2}. \]

So the "equation of motion" for \( \rho \) is

\[ \frac{d}{d\tau} \dot{\rho} = \rho \dot{\theta}^2 \]

and

\[ \frac{d}{d\tau} (\rho^2 \dot{\theta}) = 0. \]

In the second equation, one must differentiate carefully, i.e. one must differentiate both the \( \rho^2 \) term and the \( \dot{\theta} \) term; many people missed the first. We obtain the equation

\[ \ddot{\theta} + \frac{1}{\rho} \dot{\rho} \dot{\theta} = 0. \]

From these equations, we can read off our expressions above for the components of \( \Gamma \).

**Useful formulae**

1. \( \frac{1}{1+\epsilon} \approx 1 - \epsilon \quad (1 + \epsilon)^a \approx 1 + a\epsilon. \)
2. Lorentz transformation: \( x' = \gamma (x + vt) \quad t' = \gamma (t + vx) \)
3. Distance element in spherical coordinates: \( ds^2 = dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \)
4. Distance element for the plane in cylindrical coordinates: \( ds^2 = d\rho^2 + \rho^2 d\phi^2 \)
5. Distance element in space-time \( ds^2 = -d\tau^2 = -dt^2 + d\vec{x}^2 \)
6. Geodesic equation: \( \ddot{x}^\mu + \Gamma^\mu_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma = 0. \)
7. Christofel symbols: \( \Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left[ \frac{\partial g_{\nu\alpha}}{\partial x^\beta} + \frac{\partial g_{\nu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu} \right] \).
8. Components of four-velocity: \( u^0 = \gamma; \quad u^i = \gamma v^i. \)
9. Conversion factors: \( 1eV = 1.16 \times 10^4 \,^oK \)