Problem 1. Multipole Moments
Consider a system of charges with one charge of $+q$ at $z = a$ ($x = y = 0$), one of $-q$ at $z = -a$ ($x = y = 0$). Compute the first two non-vanishing moments, using
(a) Spherical harmonics

Solutions: For both parts of this problem, it is helpful to think about computing the field. The system, trivially, has azimuthal symmetry, so only $m = 1$ will contribute. Using the expansion of the Green’s function in spherical harmonics, we have

$$\Phi(r, \theta, \phi) = \sum_{\ell} \frac{4\pi}{2\ell + 1} \frac{1 \ell}{r} Y_{\ell 0}(\theta, \phi) q(Y_{\ell 0}(0, 0) - Y_{\ell 0}(\pi, 0)).$$  \hspace{2cm} (1)

Using the explicit forms of the $Y_{\ell 0}$’s provided,

$$\Phi(r, \theta, \phi) \approx \frac{2qa}{r^2} \cos(\theta) + \frac{2qa^3}{r^4} \left( \frac{5}{2} \cos^3 \theta - \frac{3}{2} \cos \theta \right).$$  \hspace{2cm} (2)

(b) Cartesian coordinates. Compare.

Solutions: Start with the basic expression:

$$\Phi(\vec{x}) = \int d^3x' \frac{1}{|\vec{x} - \vec{x'}|} \rho(\vec{x'})$$  \hspace{2cm} (3)

and expand

$$\frac{1}{|\vec{x} - \vec{x'}|} \approx \frac{1}{r} (1 - 2\epsilon_1 + \epsilon_2)^{-1/2}$$  \hspace{2cm} (4)

$$\approx \frac{1}{r} \left( 1 + \epsilon_1 - \frac{1}{2} \epsilon_2 + \frac{3}{8} \epsilon_1^2 - \frac{3}{2} \epsilon_1 \epsilon_2 + \frac{5}{2} \epsilon_1^3 + O(\epsilon^4) \right),$$

with $\epsilon_1 = \frac{\vec{x} \cdot \vec{x'}}{r^2} \quad \epsilon_2 = |\vec{x'}|^2/r^2$ So substituting back in the expression for $\Phi$, and noting that the $\epsilon^0$ and $\epsilon^2$ terms cancel, due to opposite signs of the charges, and that, for the first charge,

$$\epsilon_1 = \frac{a}{r} \cos \theta \quad \epsilon_2 = \frac{a^2}{r^2}$$  \hspace{2cm} (5)

while for the second charge, the sign of $\epsilon_1$ is reversed, we obtain exactly the expression we found using the spherical harmonics.

$$\Phi \approx \frac{1}{r^2}$$  \hspace{2cm} (6)

This should be expressed in terms of a tensor moment. Going back to the definition of $\epsilon_1$ and $\epsilon_2$, and including Cartesian indices, we can write the $1/r^4$ term in the potential as

$$\frac{1}{r^7} \left[ \frac{5}{2} (\vec{x} \cdot \vec{x'})^3 - \frac{3}{2} \vec{x}^{'2} \vec{x} \cdot \vec{x'} \vec{x}^2 \right]$$  \hspace{2cm} (7)
\[
= \frac{1}{v^2} x_i x_j x_k \left[ \frac{5}{2} x_i' x_j' x_k' \right. \\
\left. - \frac{3}{2} \delta_{ij} x^2 x_k' \right]
\]
or, in a more symmetric form
\[
\frac{1}{v^2} x_i x_j x_k \left[ \frac{5}{2} x_i' x_j' x_k' - \frac{1}{2} \delta_{ij} x^2 x_k' \right.
\left. - \frac{1}{2} \delta_{ik} x^2 x_j' + \delta_{jk} x_i' \right].
\]
(8)

So we identify what we might call the "quintopole moment:
\[
Q_{ijk} = \int d^3 x' \rho(x') \left[ \frac{5}{2} x_i' x_j' x_k' - \frac{1}{2} \delta_{ij} x^2 x_k' \right.
\left. + \frac{1}{2} \delta_{ik} x^2 x_j' + \delta_{jk} x_i' \right].
\]
(9)

Hints: \( 1/(1+\epsilon)^{3/2} \approx 1 - 1/2\epsilon + 3/8\epsilon^2 - 5/16\epsilon^3 \); relevant spherical harmonics may include
\[
Y_{10} = \sqrt{3/4\pi} \cos(\theta); \quad Y_{20} = \sqrt{5/4\pi} \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right); \quad Y_{30} = \sqrt{7/4\pi} \left( \frac{5}{2} \cos^3 \theta - \frac{3}{2} \cos \theta \right).
\]
(10)

Problem 2.
(a) For a constant magnetic field, \( \vec{B}_0 \), show that a suitable vector potential is
\[
\vec{A} = -\frac{1}{2} \vec{x} \times \vec{B}_0.
\]
(11)

Verify that \( \vec{A} \) satisfies the gauge condition \( \vec{\nabla} \cdot \vec{A} = 0 \).

Solutions:
This is easy. Using our standard formula in terms of \( \epsilon \)'s gives:
\[
B_i = \epsilon_{ijk} \partial_j A_k = -\frac{1}{2} \epsilon_{ijk} \epsilon_{klm} \partial_j x_l B_m
\]
(12)
\[
= -\frac{1}{2} \epsilon_{ijk} \epsilon_{klm} \delta_{jl} B_m = -\frac{1}{2} (1 - 3) B_i
\]
where in the last step we have used our identity (below) for the product of two \( \epsilon \) tensors. Similarly
\[
\partial_i A_i \propto \epsilon_{ijk} \partial_j x_l B_k = \epsilon_{iik} = 0.
\]
(13)

(b) For the vector potential of part (a), using the Hamiltonian we developed in class for a particle in an electromagnetic field:
\[
H = \frac{1}{2m} \left( \vec{p} - q\vec{A}/c \right)^2
\]
(14)
work out the term in the energy linear in the magnetic field. Express the result in terms of the magnetic moment of the particle. Does this look familiar?

Solutions: Just keeping the term linear in \( \vec{A} \):
\[
H_B = -\frac{1}{m} \vec{p} \cdot \left( -\frac{1}{2c} \vec{x} \times \vec{B}_0 \right)
\]
(15)
\[
= -\frac{q}{2mc} \vec{B}_0 \cdot (\vec{p} \times \vec{x})
\]
\[
= -\frac{q}{2mc} \vec{L} \cdot \vec{B}
\]
\[
= -\vec{m} \cdot \vec{B}.
\]
In the last line, we have used the standard result for the magnetic moment of a particle in terms of its angular momentum.

**Problem 3.**
(a) Write the wave equation for an electromagnetic wave propagating in a medium with dielectric constant $\epsilon$, and with $\mu = 1$. Show that the equation has plane wave solutions, and determine the velocity of the waves (assume no dispersion).

**Solutions:** This is already pretty standard. The equation for $\vec{E}$, say, is

$$\left[ \nabla^2 - \frac{\epsilon}{c^2} \frac{\partial^2}{\partial t^2} \right] \vec{E} = 0. \quad (16)$$

Plugging in

$$\vec{E} = \vec{E}_0 e^{i \vec{k} \cdot \vec{x} - i \omega t} \quad (17)$$

one has a solution provided $\omega^2 = c^2/\epsilon k^2$, corresponding to a phase velocity

$$v_p = c/\sqrt{\epsilon} \equiv \frac{c}{n}. \quad (18)$$

(b) Now suppose $\epsilon$ is complex, $\epsilon = \alpha + i \beta, \beta \ll \alpha$ (you can take both positive). Suppose the wave propagates along the $z$ axis, with frequency $\omega$. How does the wave behave with $z$? How do $u, \vec{S}$ behave? Is energy conserved? If not, offer an interpretation of where the energy is going.

**Solutions:** If $\epsilon$ is complex, writing

$$k = \sqrt{\epsilon \omega/c} = \sqrt{\alpha (1 + i \beta/\alpha)/c} \equiv k_r + i k_i, \quad (19)$$

we have that the wave behaves as:

$$e^{-i \omega t + ik_r z - k_i z} \quad (20)$$

i.e. the signal damps out as it traverses the medium. This can be understood as resulting from dissipation of energy in the medium, e.g. exciting transitions among the molecules of the material.

**Problem 4.**
Separate the wave equation, in spherical coordinates, assuming $e^{-i \omega t}$ time dependence. Find the solutions for large $r$ corresponding to incoming, outgoing spherical waves. Recall that in spherical coordinates, the Laplacian takes the form:

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \psi \right) + \text{angular pieces}. \quad (21)$$

(You may use facts you know about spherical harmonics here; you don’t need to re-derive them!).

**Solutions:** Here the idea is to take the solutions to be of the form

$$\psi(r, \theta, \phi) = \frac{e^{ikr-i\omega t}}{r} f(r) Y_{\ell m}(\theta, \phi) \quad (22)$$

where I have made the educated guess that the solution will look like an outgoing spherical wave, times a spherical harmonic, times a slowly varying function of $r$. It was not necessary to explore $f(r)$ carefully, but it is not hard to see what we would have to do. Plugging in the wave equation, using the last equation of the formula sheet:

$$\frac{1}{r} \frac{d}{dr} \frac{d^2}{dr^2} e^{ikr} f(r) - \frac{1}{r^2} \frac{\omega^2}{c^2} e^{ikr} f(r) - \frac{\ell (\ell + 1)}{r^3} e^{ikr} f(r) = 0. \quad (23)$$
Ignoring derivatives on \( f(r) \), and keeping only the \( 1/r \) terms, gives \( \omega = ck \). One can then write a term involving first derivatives of \( f \); these must give an extra \( 1/r^2 \) factor. So we might look for a solution of the form

\[
f(r) = a + \frac{b}{r}.
\]

(24)

When we discuss radiation, only the \( 1/r \) terms will correspond to radiated energy.

**Problem 5.**

(a) In general, the solution of the equations for the potentials has the form (in Lorentz gauge) has the form:

\[
\begin{cases}
\phi(\vec{x}, t) \\
\vec{A}(\vec{x}, t)
\end{cases} = \int d^3x' dt' G(\vec{x} - \vec{x}', t - t') \begin{cases}
\alpha \rho(\vec{x}', t') \\
\beta \vec{J}(\vec{x}', t')
\end{cases}
\]

(25)

where \( G \) is the Green’s function:

\[
G(\vec{x} - \vec{x}', t - t') = \frac{1}{|\vec{x} - \vec{x}'|} \delta(t - t' - \frac{1}{c}|\vec{x} - \vec{x}'|)
\]

(26)

What are the constants \( \alpha \) and \( \beta \) in (i) SI [electrostatic units]; (ii) Gaussian units. (Gaussian units are the units I have been using in my lectures; SI units are the units with \( \varepsilon_0, \mu_0 \).)

**Solutions:** SI units: \( \alpha = \frac{1}{\varepsilon_0} ; \beta = \mu_0 \); Gaussian: \( \alpha = 1 ; \beta = \frac{1}{c} \).

For a current density \( \vec{J}(\vec{x}) \)

\[
\vec{J}(\vec{x}, t) = \vec{J}(\vec{x}) e^{-i\omega t}
\]

(27)

solve for \( \phi, \vec{A} \) (in Lorentz gauge), in two ways:

(b). Use the Green’s function for the Helmholtz equation (\( \vec{R} = \vec{x} - \vec{x}' \), \( \omega = ck \)), to find the Fourier mode of \( \vec{A} \) with frequency \( \omega \),

\[
G(\vec{R}, \omega) = \frac{e^{i\vec{k}\vec{R}}}{\vec{R}}.
\]

(28)

Express your result as an integral over \( \vec{J}(\vec{x}) \).

**Solutions:** We can Fourier transform the wave equation for \( \vec{A} \):

\[
(k^2 + \nabla^2) \vec{A}(\vec{x}, \omega) = \vec{J}(\vec{x})
\]

(29)

where \( k = \frac{\omega}{c} \). Then, using the Green’s function for the Helmholtz equation:

\[
\vec{A}(\vec{x}, \omega) = \frac{1}{c} \int d^3x' \frac{1}{|\vec{x} - \vec{x}'|} e^{i\vec{x} - \vec{x}'|/c} \vec{J}(\vec{x}').
\]

(30)

You can leave the answer in this form, or simplify, by considering very large \( x \). Then

\[
|\vec{x} - \vec{x}'| \approx r - \frac{\vec{x} \cdot \vec{x}'}{r}.
\]

Expanding the exponent for small \( r \), we have

\[
\vec{A} = \frac{e^{-i\omega t + kr}}{r} \int d^3x'(1 + i \frac{\vec{x} \cdot \vec{x}'}{r}) \vec{J}(\vec{x}').
\]

(31)

When we encounter this in 214, we will see that the first term is zero, while the second is related to the time derivative of the dipole moment.

c. Use the complete Green’s function for the wave equation (eqn. 26):

**Solutions:**

It is easy to see that we obtain the same expression as in the previous part. We have

\[
\vec{A}(\vec{x}, t) = \int d^3x' \frac{1}{|\vec{x} - \vec{x}'|} e^{-i\omega t+i\omega/c|\vec{x} - \vec{x}'|} \vec{J}(\vec{x}').
\]

(32)
Useful formulae

1. $\frac{1}{1+\epsilon} \approx 1 - \epsilon$.

2. Maxwell’s Equations in a medium:
   \[
   \nabla \cdot \mathbf{D} = 4\pi \rho \quad \nabla \cdot \mathbf{B} = 0
   \]
   \[
   \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad \nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}
   \]

3. Maxwell’s Equations in vacuum (Gaussian):
   \[
   \nabla \cdot \mathbf{E} = 4\pi \rho \quad \nabla \cdot \mathbf{B} = 0
   \]
   \[
   \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \quad \nabla \times \mathbf{B} = 4\pi \mathbf{j} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}
   \]

4. Maxwell’s equations in vacuum in other units:
   \[
   \nabla \cdot \mathbf{E} = \rho/\epsilon_0 \quad \nabla \cdot \mathbf{B} = 0
   \]
   \[
   \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}
   \]

5. \[
   \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}).
   \]

6. \[
   \nabla \times (\nabla \times \mathbf{V}) = \nabla (\nabla \cdot \mathbf{V}) - (\nabla^2 \mathbf{V}).
   \]
   \[
   \epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}
   \]

7. Solution of Laplace equation in spherical coordinates, azimuthal symmetry:
   \[
   \Phi = \sum_\ell \left( a_\ell r^\ell + \frac{b_\ell}{r^{\ell+1}} \right) P_\ell(\cos \theta).
   \]

8. Green’s function in terms of spherical harmonics:
   \[
   \frac{1}{|\mathbf{x} - \mathbf{x}'|} = \sum_{\ell,m} \frac{4\pi}{2\ell + 1} \frac{1}{r_> r_<} \ell Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi').
   \]

9. Basic property of spherical harmonics:
   \[
   \nabla^2 Y_{\ell m} = -\frac{1}{r^2} \ell(\ell + 1) Y_{\ell m}
   \]