1. Fill in the details of the derivation of the effective Hamiltonian for an electron in a background electromagnetic field. Check that you get the correct $g$ value for the magnetic moment, and that you find the correct spin-orbit coupling in the field of an atomic nucleus.

**Solution:** It is remarkable how two facts about the electron emerge so easily from the Dirac equation. Below I repeat some of the material from the handout, with annotations relevant to this problem:

\[(p^0 - eA^0 - m)\phi - \vec{\sigma} \cdot (\vec{p} - e\vec{A})\chi = 0\]  
and \[ (p^0 - eA^0 + m)\chi - \vec{\sigma} \cdot (\vec{p} - e\vec{A})\phi = 0.\]  
for the upper and lower components of the Dirac spinor. We can solve for $\chi$ in terms of $\phi$.

We will first work to first order in fields. However, for the hydrogen atom problem, powers of $A_0$ are of order powers of $p^2$, so there we will set $\vec{A} = 0$ and work systematically order by order both in $p^2$ and $A_0$. In the present approximation we write:

\[
\chi = \frac{1}{p^0 - eA_0 + m} \vec{\sigma} \cdot (\vec{p} - e\vec{A})\phi
\]

Now substitute back in the equation for $\phi$:

\[
(p^0 - eA^0 - m)\phi + \frac{1}{p^0 + m} \vec{\sigma} \cdot (\vec{p} - e\vec{A})\vec{\sigma} \cdot (\vec{p} - e\vec{A})\phi + \vec{p} \cdot eA^0 \frac{1}{(p^0 + m)^2} \vec{\sigma} \cdot \vec{p}\phi
\]

Using the identity $\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk} \sigma_k$, we can rewrite this expression as:

\[
(p^0 + eA^0 - m)\phi + \frac{1}{p^0 + m} \left( \vec{p}^2 + e(\vec{p} \cdot \vec{A} + \vec{A} \cdot \vec{p}) + i\epsilon_{ijk}(p^i A^j + A^i p^j)\sigma^k \phi - \frac{1}{(p^0 + m)^2} \vec{\sigma} \cdot \vec{p} eA^0 \vec{\sigma} \cdot \vec{p}\phi \right).
\]

(It will be convenient to leave the last term in this form). Now the term $i\epsilon_{ijk}(p^i A^j + A^i p^j)$, would vanish, except that $p_i$ and $A_j$ don’t commute, and we obtain, from this term, $\epsilon_{ijk} \partial_i A^j \sigma^k = \vec{B} \cdot \vec{\sigma}$. The term involving $A^0$ can be rewritten as:

\[
\frac{1}{(p^0 + m)^2} \vec{\sigma} \cdot \vec{p} eA^0 \vec{\sigma} \cdot \vec{p}
\]

\[
= -\vec{p} \cdot \vec{\sigma} \cdot \vec{p} \frac{eA^0}{(p^0 + m)^2} - \frac{i\hbar \vec{\sigma} \cdot \vec{p} \partial_j A^0 \sigma^j}{(p^0 + m)^2}
\]

\[
= -\vec{p}^2 \frac{3A^0}{(p^0 + m)^2} + \frac{e\hbar}{(p^0 + m)^2} (i\vec{p} \cdot \vec{E} + \vec{\sigma} \cdot (\vec{E} \times \vec{p}))
\]
2. Peskin 2.1.

Solution:

a. It is convenient to rewrite:

\[ \mathcal{L} = -\frac{1}{2} F^{\mu \nu} \partial_\mu A_\nu \]  

(7)

where we have used the antisymmetry of \( F^{\mu \nu} \). So differentiating \( \partial_\mu A_\nu \) gives

\[ \partial_\mu F^{\mu \nu} = 0. \]  

(8)

(Note the symmetries among indices; differentiating the \( F^{\mu \nu} \) term gives the same result; in particular, if the lagrangian includes \( A^\mu j_\mu \), we obtain \( \partial_\mu F^{\mu \nu} = j^\nu \).)

b. The “naive” stress tensor is:

\[ T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu A^\rho)} \partial_\nu A^\rho - \mathcal{L} \delta^\mu_\nu \]  

(9)

\[ = -F^{\mu \rho} \partial_\nu A^\rho + \frac{1}{4} F^{2 \rho \sigma} \delta^\mu_\nu. \]

In the second line, we have used the Lagrangian in the form of eqn. 1, and noted that differentiating the \( F \) term gives the same result as the \( \partial A \) term (thus accounting for a factor of 2).

Now if we add \( \partial_\lambda K^{\lambda \mu \nu} \), we obtain an object which is conserved, if we differentiate \( \partial_\mu \), since

\[ \partial_\mu \partial_\lambda K^{\lambda \mu \nu} = 0. \]  

(10)

In particular, we can take \( K^{\lambda \mu \nu} = F^{\mu \lambda} A^\nu \). Then the term we add to \( T^\mu_\nu \) is

\[ \partial_\lambda (F^{\mu \lambda} A^\nu) = f^{\mu \lambda \rho} \partial_\lambda A^\rho + \partial_\lambda F^{\mu \lambda} A^\nu. \]  

(11)

The second term vanishes by the equations of motion. Rewriting the first term (just relabeling the indices) as \( F^{\mu \nu} \partial_\rho A^\nu \), we have

\[ T^\mu_\nu = F^{\mu \rho} F_{\rho \nu} + 1 \text{ over } 4F^{2 \rho \sigma} \delta^\mu_\nu. \]  

(12)

For example,

\[ T^0_0 = F^{0 \iota} F_{\iota 0} - \frac{1}{2}(\vec{E}^2 - \vec{B}^2) \]  

(13)

\[ = \frac{1}{2}(\vec{E}^2 + \vec{B}^2). \]

3. Peskin 2.2.

Solution:

a) Conjugate momenta:

\[ \Pi = \dot{\phi}^* \quad \Pi^* = \dot{\phi}. \]

Hamiltonian is constructed as usual. Heisenberg equation of motion for \( \phi \) follows almost by inspection from \([\phi(\bar{x}, t), \Pi(\bar{x}', t)] = i\delta(\bar{x} - \bar{x}')l:\)

\[ i\dot{\phi}(\bar{x}, t) = [H, \phi(\bar{x}, t)] = i\Pi^*. \]  

(14)

\[ i\Pi(\bar{x}, t) = [H, \Pi(\bar{x}, t)] = \int d^3 \bar{\chi}' [\Pi(\bar{x}, t), \left(-\nabla^2 \phi^*(\bar{\chi}', t)\phi(\bar{\chi}', t) + m^2 \phi^2(\bar{\chi}' t)\right)] \]  

(15)
where in the gradient term we have integrated by parts to get this into a convenient form. Evaluating the commutator:

\[
    i\Pi(\vec{x}, t) = [H, \Pi(\vec{x}, t)] = \int d^3x' i\delta(\vec{x} - \vec{x}') \left( -\nabla^2 \phi^*(\vec{x}', t) + m^2 \phi^* \right). \tag{16}
\]

With \( \Pi = i\dot{\phi}^* \), this is the Klein-Gordan equation for \( \phi^* \).

b) Because \( \phi \) is complex, we have to treat the field and its complex conjugate separately. As we have encountered for the Dirac field, we have

\[
    \phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{p})}} \left( a(\vec{p})e^{-ip\cdot x} + b^\dagger(\vec{p})e^{ip\cdot x} \right) \tag{17}
\]

\[
    \phi^*(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E(\vec{p})}} \left( a^\dagger(\vec{p})e^{ip\cdot x} + b(\vec{p})e^{-ip\cdot x} \right) \tag{18}
\]

The Hamiltonian is:

\[
    H = \int \frac{d^3p}{(2\pi)^3} \left( a^\dagger(\vec{p})a(\vec{p}) + b^\dagger(\vec{p})b(\vec{p}) \right) \omega(\vec{p}) \tag{19}
\]

up to a normal ordering constant.

c) We first need to understand the conserved current. The lagrangian is symmetric under \( \phi \to e^{i\alpha} \phi \). Taking \( \alpha \) infinitesimal, and a function of \( x \), we can implement the usual Noether procedure:

\[
    \delta L = i\partial_\mu \alpha (\partial^\mu \phi^* \phi - \phi^* \partial^\mu \phi) \tag{20}
\]

so that the conserved current is the coefficient of \( \partial^\mu \alpha \):

\[
    j^\mu = i(\partial^\mu \phi^* \phi - \phi^* \partial^\mu \phi). \tag{21}
\]

The charge is the integral of \( j^0 \). This is

\[
    Q = \int d^3x i(\Pi\phi - \phi^*\Pi) \tag{22}
\]

This differs by a sign and a factor of two from Peskin and Schroeder, but this can be absorbed into the definition of the current (the current is still conserved); it corresponds to making the replacement \( \alpha \to -\alpha/2 \). Plugging in our mode expansion above, gives, not surprisingly:

\[
    Q = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left( a^\dagger(\vec{p})a(\vec{p}) - b^\dagger(\vec{p})b(\vec{p}) \right) \tag{23}
\]

corresponding to particles of charge \( 1/2 \), anti-particles of charge \( -1/2 \).

d) This is a simple exercise with the commutation relations.